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MONOTONE ITERATIVE METHOD FOR OBTAINING POSITIVE SOLUTIONS OF INTEGRAL BOUNDARY-VALUE PROBLEMS WITH ϕ-LAPLACIAN OPERATOR

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Abstract. This article concerns the existence, multiplicity of positive solutions for the integral boundary-value problem with ϕ-Laplacian,

$$\begin{gathered}\big(\phi(u'(t))\big)' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\ u(0) = \int_0^1 u(r)g(r)\,\mathrm{d}r, \quad u(1) = \int_0^1 u(r)h(r)\,\mathrm{d}r,\end{gathered}$$

where ϕ is an odd, increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Using a monotone iterative technique, we obtain the existence of positive solutions for this problem, and present iterative schemes for approximating the solutions.

1. Introduction

Consider the integral boundary-value problem

$$\begin{gathered}\big(\phi(u'(t))\big)' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\ u(0) = \int_0^1 u(r)g(r)\,\mathrm{d}r, \quad u(1) = \int_0^1 u(r)h(r)\,\mathrm{d}r,\end{gathered} \tag{1.1}$$

where ϕ, f, g and h satisfy

(H1) ϕ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0,\infty)$ onto $(0,\infty)$ such that

$$\psi_1(u)\phi(v) \le \phi(uv) \le \psi_2(u)\phi(v), u, v > 0.$$

Moreover, $\phi, \phi^{-1} \in C^1(R)$, where ϕ^{-1} denotes the inverse of ϕ.

(H2) $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to (0,+\infty)$ is continuous. $g, h \in L^1[0,1]$ are nonnegative, and $0 < \int_0^1 g(t)\,\mathrm{d}t < 1, 0 < \int_0^1 h(t)\,\mathrm{d}t < 1$.

The assumption (H1) on the function ϕ was first introduced by Wang [1, 2]. It covers two special cases: $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u, p > 1$. Many authors have studied the positive solutions of two-point and multi-point boundary value problems when $\phi(u) = u$ or $\phi(u) = |u|^{p-2}u, p > 1$. For details and references see [1, 3, 4, 5, 8, 9].

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In a recent paper [2], the author proved the existence and multiplicity of positive solutions of (1.1) by applying the Krasnoselskii fixed point theorem and Avery-Peterson fixed point theorem. However there is an interesting question which is showing how to find these solutions, since they exist. Motivated by the question and all the works above, in this article, by applying classical monotone iterative techniques, we not only obtain the existence of positive solutions of (1.1), but also give iterative schemes for approximating the solutions. It is worth stating that we will not require the existence of lower and upper solutions, and the first term of iterative scheme is a constant function or a simple function. Therefore, the iterative scheme is significant and feasible.

2. Preliminaries

The basic space used in this article is the real Banach space $C^1[0,1]$ with norm $\|u\|_1 = \max\{\|u\|_c, \|u'\|_c\}$, where $\|u\|_c = \max_{0\le t\le 1}|u(t)|$. Let

$$P = \big\{u \in C^1[0,1] : u(t) \ge 0, u \text{ is concave on } [0,1]\big\}.$$

It is obvious that P is a cone in $C^1[0,1]$.

For any $x \in C^1[0,1]$, $x(t) \ge 0$, $t \in [0,1]$, we suppose that u is a solution of the boundary-value problem

$$\begin{gathered} \big(\phi(u'(t))\big)' + f(t,x(t),x'(t)) = 0, \quad t\in[0,1], \\ u(0) = \int_0^1 u(r)g(r)\,\mathrm{d}r, \quad u(1) = \int_0^1 u(r)h(r)\,\mathrm{d}r. \end{gathered} \tag{2.1}$$

By integrating (2.1) on $[0,t]$, we have

$$\phi(u'(t)) = A_x - \int_0^t f(s,x(s),x'(s))\,\mathrm{d}s,$$

then

$$u'(t) = \phi^{-1}\Big(A_x - \int_0^t f(s,x(s),x'(s))\,\mathrm{d}s\Big). \tag{2.2}$$

Thus

$$u(t) = u(0) + \int_0^t \phi^{-1}\Big(A_x - \int_0^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \tag{2.3}$$

or

$$u(t) = u(1) - \int_t^1 \phi^{-1}\Big(A_x - \int_0^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{2.4}$$

According to the boundary condition, we have

$$u(0) = \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(A_x - \int_0^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r$$

and

$$u(1) = -\frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\Big(A_x - \int_0^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r,$$

where A_x satisfies the equation

$$\begin{aligned} H_x(c) = {} & \frac{1-\int_0^1 h(r)\,\mathrm{d}r}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(c-\int_0^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r \\ & + \Big(1-\int_0^1 h(r)\,\mathrm{d}r\Big)\int_0^1 \phi^{-1}\Big(c-\int_0^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ & + \int_0^1 h(r)\int_r^1 \phi^{-1}\Big(c-\int_0^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r = 0. \end{aligned} \tag{2.5}$$

Lemma 2.1. *Assume* (H1) *and* (H2) *hold, for any $x \in C^1[0,1]$ with $x(t) \ge 0, t \in [0,1]$, there exists a unique $A_x \in (-\infty,+\infty)$ satisfying* (2.5). *Moreover, there is a unique $\delta_x \in (0,1)$ such that*

$$A_x = \int_0^{\delta_x} f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau.$$

Proof. From the expression of $H_x(c)$ it is easy to see that $H_x : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and

$$H_x(0) < 0, \quad H_x\Big(\int_0^1 f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big) > 0.$$

Hence there exists a unique $A_x \in (0, \int_0^1 f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau) \subset (-\infty,+\infty)$ satisfying (2.5). Let

$$F(t) = \int_0^t f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau,$$

then $F(t)$ is continuous and strictly increasing on $[0,1]$, and $F(0) = 0$, $F(1) = \int_0^1 f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau$. So

$$0 = F(0) < A_x < F(1) = \int_0^1 f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau.$$

Therefore, there exists a unique $\delta_x \in (0,1)$ such that

$$A_x = \int_0^{\delta_x} f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau.$$

The proof is complete. □

By (2.3), (2.4) and Lemma 2.1, if u is a solution of (2.1), then $u(t)$ can be expressed as

$$\begin{aligned} u(t) = {} & \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_s^{\delta_x} f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r \\ & + \int_0^t \phi^{-1}\Big(\int_s^{\delta_x} f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \end{aligned}$$

or

$$\begin{aligned} u(t) = {} & \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\Big(\int_{\delta_x}^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r \\ & + \int_t^1 \phi^{-1}\Big(\int_{\delta_x}^s f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \end{aligned}$$

Lemma 2.2. *Suppose* (H1) *and* (H2) *hold. If* $u(t)$ *is a solution of* (2.1)*, then*

(i) $u(t)$ *is concave and* $u(t) \geq 0$ *on* $[0,1]$*;*
(ii) *there exists a unique* $t_0 \in (0,1)$ *such that* $u(t_0) = \max_{0\leq t\leq 1} u(t)$ *and* $u'(t_0) = 0$*;*
(iii) $\delta_x = t_0$.

Proof. (i) Let $u(t)$ is a solution of (2.1). Then

$$(\phi(u'(t)))' = -f(t, x(t), x'(t)) < 0, \quad t \in [0,1].$$

Therefore, $\phi(u'(t))$ is strictly decreasing. It follows that $u'(t)$ is also strictly decreasing. Thus, $u(t)$ is strictly concave on $[0,1]$. Without loss of generality, we assume that $u(0) = \min\{u(0), u(1)\}$. By the concavity of u, we know that $u(t) \geq u(0)$, $t \in [0,1]$. So we obtain

$$u(0) = \int_0^1 u(t)g(t)\,\mathrm{d}t \geq u(0)\int_0^1 g(t)\,\mathrm{d}t.$$

By $0 < \int_0^1 g(t)\,\mathrm{d}t < 1$, it is obvious that $u(0) \geq 0$. Hence, $u(t) \geq 0$, $t \in [0,1]$.

(ii) Since $u(t)$ is strictly concave on $[0,1]$, there exist a unique $t_0 \in [0,1]$ such that $u(t_0) = \max_{0\leq t\leq 1} u(t)$. By the boundary conditions and $u(t) \geq 0$, we know that $t_0 \neq 0$ or 1, that is, $t_0 \in (0,\ 1)$ such that $u(t_0) = \max_{0\leq t\leq 1} u(t)$.

(iii) By (2.2) and Lemma 2.1, it is easy to see that

$$u'(t) = \phi^{-1}\Big(\int_t^{\delta_x} f(s, x(s), x'(s))\,\mathrm{d}s\Big),$$

therefore, $u'(\delta_x) = 0$, this implies $\delta_x = t_0$. The proof is complete. □

Now we define an operator T by

$$Tx(t) = \begin{cases} \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_s^{\delta_x} f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s\,\mathrm{d}r \\ +\int_0^t \phi^{-1}\Big(\int_s^{\delta_x} f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s, & 0 \leq t \leq \delta_x, \\ \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\Big(\int_{\delta_x}^s f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s\,\mathrm{d}r \\ +\int_t^1 \phi^{-1}\Big(\int_{\delta_x}^s f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s, & \delta_x \leq t \leq 1. \end{cases} \tag{2.6}$$

It is easy to prove that each fixed point of T is a solution for (1.1).

Lemma 2.3. *the operator* $T : P \to P$ *is completely continuous.*

Proof. Let $x \in P$, then from the definition of T, we have

$$(Tx)'(t) = \begin{cases} \phi^{-1}\Big(\int_t^{\delta_x} f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\Big) \geq 0, & 0 \leq t \leq \delta_x, \\ -\phi^{-1}\Big(\int_{\delta_x}^t f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau\Big) \leq 0, & \delta_x \leq t \leq 1. \end{cases} \tag{2.7}$$

So $(Tx)'(t)$ is monotone decreasing continuous and $(Tx)'(\delta_x) = 0$. Hence, $(Tx)(t)$ is nonnegative and concave on $[0,1]$. This shows that $T(P) \subset P$. Next, we prove T is compact on $C^1[0,1]$.

Let D be a bounded subset of P and $m > 0$ is a constant such that

$$\int_0^1 f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau < m, \quad \forall\ x \in D.$$

From the definition of T, for any $x \in D$, we obtain

$$|Tx(t)| < \begin{cases} \frac{\phi^{-1}(m)}{1-\int_0^1 g(r)\,\mathrm{d}r}, & 0 \le t \le \delta_x, \\ \frac{\phi^{-1}(m)}{1-\int_0^1 h(r)\,\mathrm{d}r}, & \delta_x \le t \le 1, \end{cases}$$

$$|(Tx)'(t)| < \phi^{-1}(m), \quad 0 \le t \le 1.$$

Hence, TD is uniformly bounded and equicontinuous. So we have TD is compact on $C[0,1]$. From (2.7) we know that for all $\varepsilon > 0$ there exists $\kappa > 0$, such that when $|t_1 - t_2| < \kappa$, we have

$$|\phi(Tx)'(t_1) - \phi(Tx)'(t_2)| < \varepsilon.$$

So $\phi(TD)'$ is compact on $C[0,1]$, it follows that $(TD)'$ is compact on $C[0,1]$. Therefore, TD is compact on $C^1[0,1]$. Thus, $T : P \to P$ is completely continuous. The proof is complete. □

3. Existence of positive solutions

For convenience, we denote

$$A = \max\Big\{\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}, \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\Big\}.$$

Our result is as follows.

Theorem 3.1. *Assume* (H1) *and* (H2) *hold. If there exists* $a > 0$ *such that*

(i) $f(t, x_1, y_1) \le f(t, x_2, y_2)$ *for any* $0 \le t \le 1$, $0 \le x_1 \le x_2 \le a$, $0 \le |y_1| \le |y_2| \le a$;

(ii) $\max_{0 \le t \le 1} f(t, a, a) \le \phi(\frac{a}{A})$.

Then (1.1) *has at least two positive, concave solutions* w^* *and* v^* *satisfying*

$$0 < w^* \le a, \quad 0 < |(w^*)'| \le a,$$
$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*,$$
$$\lim_{n\to\infty} (w_n)' = \lim_{n\to\infty} (T^n w_0)' = (w^*)',$$

where

$$w_0(t) = a\frac{\min\big\{\frac{\int_0^1 g(r)\,\mathrm{d}r}{1-\int_0^1 g(r)\,\mathrm{d}r} + t, \frac{\int_0^1 h(r)\,\mathrm{d}r}{1-\int_0^1 h(r)\,\mathrm{d}r} + 1 - t\big\}}{\max\big\{\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}, \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\big\}}, \quad 0 \le t \le 1,$$

and

$$0 < v^* \le a, \quad 0 < |(v^*)'| \le a,$$
$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*,$$
$$\lim_{n\to\infty} (v_n)' = \lim_{n\to\infty} (T^n v_0)' = (v^*)',$$

where $v_0(t) = 0$, $0 \le t \le 1$.

Proof. Let $P_a = \{u \in P|\ \|u\|_1 < a\}$, $\overline{P}_a = \{u \in P|\ \|u\|_1 \le a\}$. Next, we show that $T(\overline{P}_a) \subset \overline{P}_a$. If $u \in \overline{P}_a$, then $\|u\|_1 \le a$. Hence,

$$0 \le u(t) \le \|u\|_c \le a, \quad 0 \le |u'(t)| \le \|u'\|_c \le a.$$

From (i) and (ii), we have that

$$0 < f(t, u(t), u'(t)) \le f(t, a, a) \le \max_{0\le t\le 1} f(t, a, a) \le \phi(\frac{a}{A}). \tag{3.1}$$

By this inequality and the definition of T, we have

$$\begin{aligned}
\|Tu\|_c &= (Tu)(\delta_u)\\
&= \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^r \phi^{-1}\Big(\int_s^{\delta_u} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad + \int_0^{\delta_u} \phi^{-1}\Big(\int_s^{\delta_u} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\int_0^1 h(r)\int_r^1 \phi^{-1}\Big(\int_{\delta_u}^{s} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad + \int_{\delta_u}^1 \phi^{-1}\Big(\int_{\delta_u}^{s} f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\le \max\Big\{\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}\int_0^1 g(r)\int_0^1 \phi^{-1}\Big(\int_0^1 f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad + \int_0^1 \phi^{-1}\Big(\int_0^1 f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s,\\
&\quad \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\int_0^1 h(r)\int_0^1 \phi^{-1}\Big(\int_0^1 f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad + \int_0^1 \phi^{-1}\Big(\int_0^1 f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big\}\\
&\le \frac{a}{A}\max\Big\{\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}, \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\Big\}\\
&= a.
\end{aligned}$$

Furthermore, by (2.7) it is easy to verify that

$$\|(Tu)'\|_c \le \frac{a}{A} < a.$$

Hence, $\|Tu\|_1 \le a$. So we have $T(\overline{P}_a) \subset \overline{P}_a$. Let

$$w_0(t) = a\frac{\min\big\{\frac{\int_0^1 g(r)\,\mathrm{d}r}{1-\int_0^1 g(r)\,\mathrm{d}r} + t,\ \frac{\int_0^1 h(r)\,\mathrm{d}r}{1-\int_0^1 h(r)\,\mathrm{d}r} + 1 - t\big\}}{\max\big\{\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}, \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\big\}}, \quad 0 \le t \le 1.$$

Now we define a sequence $\{w_n\}$ by the iterative scheme

$$w_{n+1} = Tw_n = T^n w_0, \quad n = 0, 1, 2, \dots \tag{3.2}$$

Since $T(\overline{P}_a) \subset \overline{P}_a$ and $w_0(t) \in \overline{P}_a$, we have $w_n \in \overline{P}_a, n = 0, 1, 2, \dots$. From Lemma 2.3, T is compact, we assert that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in \overline{P}_a$, such that $w_{n_k} \to w^*$.

On the other hand, since

$$w_1(t) = Tw_0(t)$$

$$
\begin{aligned}
&= \begin{cases}
\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r} \int_0^1 g(r) \int_0^r \phi^{-1}\Big(\int_s^{\delta_{w_0}} f(\tau, w_0(\tau), w_0'(\tau))\,\mathrm{d}\tau \Big)\,\mathrm{d}s\,\mathrm{d}r \\
+ \int_0^t \phi^{-1}\Big(\int_s^{\delta_{w_0}} f(\tau, w_0(\tau), w_0'(\tau))\,\mathrm{d}\tau \Big)\,\mathrm{d}s, \\
\quad \text{if } 0 \le t \le \delta_{w_0}, \\
\frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r} \int_0^1 h(r) \int_r^1 \phi^{-1}\Big(\int_{\delta_{w_0}}^s f(\tau, w_0(\tau), w_0'(\tau))\,\mathrm{d}\tau \Big)\,\mathrm{d}s\,\mathrm{d}r \\
+ \int_t^1 \phi^{-1}\Big(\int_{\delta_{w_0}}^s f(\tau, w_0(\tau), w_0'(\tau))\,\mathrm{d}\tau \Big)\,\mathrm{d}s, \\
\quad \text{if } \delta_{w_0} \le t \le 1
\end{cases} \\
&\le \min\Big\{ \frac{a}{A} \frac{\int_0^1 g(r)\,\mathrm{d}r}{1-\int_0^1 g(r)\,\mathrm{d}r} + \frac{a}{A} t, \frac{a}{A} \frac{\int_0^1 h(r)\,\mathrm{d}r}{1-\int_0^1 h(r)\,\mathrm{d}r} + \frac{a}{A}(1-t) \Big\} \\
&= a \frac{\min\big\{ \frac{\int_0^1 g(r)\,\mathrm{d}r}{1-\int_0^1 g(r)\,\mathrm{d}r} + t, \frac{\int_0^1 h(r)\,\mathrm{d}r}{1-\int_0^1 h(r)\,\mathrm{d}r} + 1 - t \big\}}{\max\big\{ \frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}, \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r} \big\}} \\
&= w_0(t), \quad 0 \le t \le 1,
\end{aligned}
$$

and

$$
\begin{aligned}
|w_1'(t)| &= |(Tw_0)'(t)| \\
&= \begin{cases}
\phi^{-1}\Big(\int_t^{\delta_{w_0}} f(\tau, w_0(\tau), w_0'(\tau))\,\mathrm{d}\tau \Big), & 0 \le t \le \delta_{w_0}, \\
\phi^{-1}\Big(\int_{\delta_{w_0}}^t f(\tau, w_0(\tau), w_0'(\tau))\,\mathrm{d}\tau \Big), & \delta_{w_0} \le t \le 1
\end{cases} \\
&\le \phi^{-1}\Big(\int_0^1 f(\tau, w_0(\tau), w_0'(\tau))\,\mathrm{d}\tau \Big) \\
&\le \frac{a}{A} = |w_0'(t)|, \quad 0 \le t \le 1,
\end{aligned}
$$

we have

$$
w_1(t) \le w_0(t), \quad |w_1'(t)| \le |w_0'(t)|, \quad 0 \le t \le 1.
$$

By (i) and (2.6), we easily see that T is increasing, it follows that

$$
\begin{gathered}
w_2(t) = Tw_1(t) \le Tw_0(t) = w_1(t), \quad 0 \le t \le 1, \\
|w_2'(t)| = |(Tw_1)'(t)| \le |(Tw_0)'(t)| = |w_1'(t)|, \quad 0 \le t \le 1.
\end{gathered}
$$

Moreover, we have

$$
w_{n+1}(t) \le w_n(t), \ |w_{n+1}'(t)| \le |w_n'(t)|, \quad 0 \le t \le 1, \quad n = 0, 1, 2, \dots
$$

Therefore, $w_n \to w^*$. Let $n \to \infty$ in (3.2) to obtain $Tw^* = w^*$.

Next, Let $v_0(t) = 0$, $0 \le t \le 1$, then $v_0(t) \in \overline{P}_a$. Let $v_1 = Tv_0$, then $v_1 \in \overline{P}_a$. We denote

$$
v_{n+1} = Tv_n = T^n v_0, \quad n = 0, 1, 2, \dots \tag{3.3}
$$

Similar to $\{w_n\}_{n=1}^{\infty}$, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in \overline{P}_a$, such that $v_{n_k} \to v^*$. Since

$$
\begin{gathered}
v_1(t) = (Tv_0)(t) = (T0)(t) \ge 0, \quad 0 \le t \le 1, \\
|v_1'(t)| = |(Tv_0)'(t)| = |(T0)'(t)| \ge 0, \quad 0 \le t \le 1,
\end{gathered}
$$

we have

$$
\begin{gathered}
v_2(t) = (Tv_1)(t) \ge (T0)(t) = v_1(t), \quad 0 \le t \le 1, \\
|v_2'(t)| = |(Tv_1)'(t)| \le |(T0)'(t)| = |v_1'(t)|, \quad 0 \le t \le 1.
\end{gathered}
$$

Moreover, we have

$$v_{n+1}(t) \le v_n(t), \quad |v'_{n+1}(t)| \le |v'_n(t)|, \quad 0 \le t \le 1, \quad n = 0, 1, 2, \dots$$

Thus $v_n \to v^*$ and $Tv^* = v^*$.

It is well know that the fixed point of T is the solution of (1.1). Therefore, w^* and v^* are positive, concave solutions of (1.1). The proof is complete. □

Remark 3.2. we can see easily that w^* and v^* are the maximal and minimal solutions of (1.1). If $w^* \equiv v^*$, then (1.1) has a unique positive solution in $\overline{P}_a$.

From Theorem 3.1 we immediately obtain the following result.

Corollary 3.3. *Assume* (H1), (H2) *and Theorem 3.1(i) hold. If there exists* $a > 0$ *such that*

(iii) $\lim_{\ell\to\infty} \max_{0\le t\le 1} f(t, \ell, a) \le \phi(\frac{1}{A})$.

Then (1.1) *has at least two positive, concave solutions* w^* *and* v^* *such that the conclusion of Theorem 3.1 hold.*

Corollary 3.4. *Assume* (H1), (H2) *and Theorem 3.1(i) hold. If there exists* $0 < a_1 < a_2 < \dots < a_n$ *such that*

(iv) $\max_{0\le t\le 1} f(t, a_k, a_k) \le \phi(\frac{a_k}{A})$, $k = 1, 2, \dots, n$.

Then (1.1) *has at least* $2n$ *positive, concave solutions* w_k^* *and* v_k^* *satisfying*

$$\begin{gathered} 0 < w_k^* \le a_k,\ 0 < |(w_k^*)'| \le a_k, \\ \lim_{n\to\infty} w_{k_n} = \lim_{n\to\infty} T^n w_{k_0} = w_k^*, \\ \lim_{n\to\infty} (w_{k_n})' = \lim_{n\to\infty} (T^n w_{k_0})' = (w_k^*)', \end{gathered}$$

where

$$w_{k_0}(t) = a_k \frac{\min\big\{\frac{\int_0^1 g(r)\,\mathrm{d}r}{1-\int_0^1 g(r)\,\mathrm{d}r} + t, \frac{\int_0^1 h(r)\,\mathrm{d}r}{1-\int_0^1 h(r)\,\mathrm{d}r} + 1 - t\big\}}{\max\big\{\frac{1}{1-\int_0^1 g(r)\,\mathrm{d}r}, \frac{1}{1-\int_0^1 h(r)\,\mathrm{d}r}\big\}}, \quad 0 \le t \le 1,$$

and

$$\begin{gathered} 0 < v_k^* \le a_k, \quad 0 < |(v_k^*)'| \le a_k, \\ \lim_{n\to\infty} v_{k_n} = \lim_{n\to\infty} T^n v_{k_0} = v_k^*, \\ \lim_{n\to\infty} (v_{k_n})' = \lim_{n\to\infty} (T^n v_{k_0})' = (v_k^*)', \end{gathered}$$

where $v_{k_0}(t) = 0, 0 \le t \le 1$.

4. Example

In this section, we give a example as an application of the main results.

Let $\phi(u) = |u|u, \quad g(t) = h(t) = 1/2$. We consider the boundary-value problem

$$\begin{gathered} \big(\phi(u')\big)' + f(t, u(t), u'(t)) = 0, \quad t \in [0, 1], \\ u(0) = \frac{1}{2}\int_0^1 u(t)\,\mathrm{d}t, \quad u(1) = \frac{1}{2}\int_0^1 u(t)\,\mathrm{d}t, \end{gathered} \tag{4.1}$$

where

$$f(t, u, v) = -t^2 + t + \frac{1}{8}u + \frac{1}{16}v^2 + 2, \quad (t, u, v) \in [0, 1] \times [0, \infty) \times (-\infty, \infty).$$

Let $\psi_1(u) = \psi_2(u) = u^2$, $u > 0$. Choosing $a = 4$. By calculations we obtain $A = 2$. It is easy to verify that $f(t, u, v)$ satisfies

(1) $f(t, x_1, y_1) \le f(t, x_2, y_2)$ for any $0 \le t \le 1, 0 \le x_1 \le x_2 \le 4, 0 \le |y_1| \le |y_2| \le 4$;

(2) $\max_{0\le t\le 1} f(t, a, a) = f(\frac{1}{2}, 4, 4) \le \phi(\frac{a}{A}) = 4$.

Hence, by Theorem 3.1, (4.1) has two positive solutions w^* and v^*. For $n = 0, 1, 2, \dots$, the two iterative schemes are:

$$w_0(t) = \begin{cases} 2 + 2t, & 0 \le t \le \frac{1}{2}, \\ 4 - 2t, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$w_{n+1}(t) = \begin{cases} \int_0^1 \int_0^r \phi^{-1}\Big(\int_s^{\delta_n} (-\tau^2 + \tau + \frac{1}{8} w_n(\tau) + \frac{1}{16} w_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s\, \mathrm{d}r \\ + \int_0^t \phi^{-1}\Big(\int_s^{\delta_n} (-\tau^2 + \tau + \frac{1}{8} w_n(\tau) + \frac{1}{16} w_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s, & 0 \le t \le \delta_n, \\ \int_0^1 \int_r^1 \phi^{-1}\Big(\int_{\delta_n}^s (-\tau^2 + \tau + \frac{1}{8} w_n(\tau) + \frac{1}{16} w_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s\, \mathrm{d}r \\ + \int_t^1 \phi^{-1}\Big(\int_{\delta_n}^s (-\tau^2 + \tau + \frac{1}{8} w_n(\tau) + \frac{1}{16} w_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s, & \delta_n \le t \le 1. \end{cases}$$

$$v_0(t) = 0,$$

$$v_{n+1}(t) = \begin{cases} \int_0^1 \int_0^r \phi^{-1}\Big(\int_s^{\delta_n} (-\tau^2 + \tau + \frac{1}{8} v_n(\tau) + \frac{1}{16} v_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s\, \mathrm{d}r \\ + \int_0^t \phi^{-1}\Big(\int_s^{\delta_n} (-\tau^2 + \tau + \frac{1}{8} v_n(\tau) + \frac{1}{16} v_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s, & 0 \le t \le \delta_n, \\ \int_0^1 \int_r^1 \phi^{-1}\Big(\int_{\delta_n}^s (-\tau^2 + \tau + \frac{1}{8} v_n(\tau) + \frac{1}{16} v_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s\, \mathrm{d}r \\ + \int_t^1 \phi^{-1}\Big(\int_{\delta_n}^s (-\tau^2 + \tau + \frac{1}{8} v_n(\tau) + \frac{1}{16} v_n'^2(\tau))\, \mathrm{d}\tau\Big)\, \mathrm{d}s, & \delta_n \le t \le 1. \end{cases}$$

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