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TRANSPORT EQUATION FOR GROWING BACTERIAL POPULATIONS (I)

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Abstract. This work deals with a mathematical study for growing a bacterial population. Each bacterium is distinguished by its degree of maturity and its maturation velocity. Here we study the limit case corresponding to infinite maturation velocities. We show that this model is governed by a strongly continuous semigroup. We also study the lattice and spectral properties of the generated semigroup and we compute its type.

1. Introduction

This article deals with a bacterial population in which each bacteria is distinguished by its own degree of maturity μ and its own maturation velocity v. At birth, the degree of maturity of a daughter bacteria is $\mu = 0$. At mitotic, the degree of maturity of a mother bacteria becomes $\mu = 1$. Between birth and division, the degree of maturity of each bacterium is $0 < \mu < 1$. As each bacterium may not become less mature, then its maturation velocity must be positive ($0 \leq a < v < b \leq \infty$). So, if $f = f(t, \mu, v)$ denotes the bacterial density with respect to the degree of maturity μ and the maturation velocity v, at time t, then

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \mu} - \sigma f, \tag{1.1}$$

where $\sigma = \sigma(\mu, v)$ denotes the rate of bacterial mortality or bacteria loss due to causes other than division.

In most bacterial populations observed, there is often a correlation $k = k(v, v')$ between the maturation velocity of a bacteria mother v' and that of its bacteria daughter v. The bacterial mitotic obeys then to the biological *transition law* mathematically described by the following transition boundary condition

$$vf(t, 0, v) = p\int_a^b k(v, v')f(t, 1, v')v'\,dv', \tag{1.2}$$

where $p \geq 0$ denotes the average number of bacteria daughter viable per mitotic. To ensure the continuity of the bacterial flux for $p = 1$, the kernel of correlation k

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must be positive and fulfils the normalization condition

$$\int_a^b k(v,v')dv = 1 \quad \text{for all } v \in (a,b). \tag{1.3}$$

The model (1.1)-(1.2) was introduced in [12], where only a numerical study has been made. The first theoretical studies of the model (1.1)-(1.2) were given in [5] and [6], where we have proved that the model (1.1)-(1.2) is governed by a strongly continuous semigroup provided that $b < \infty$. However, if $b = \infty$, all claims of [5] and [6] become no suitable which leads to a serious mathematical difficulty.

To show the extent of this difficulty (i.e., $b = \infty$), we have studied the model (1.1)-(1.2) in the particular case $k(v,v') = \delta_v(v')$ which corresponds to the *perfect memory* law mathematically described by the following boundary condition

$$f(t,0,v) = pf(t,1,v) \quad \text{for all } v \in (a,b). \tag{1.4}$$

We then proved, in [7], that the model (1.1)-(1.4) is well-posed if and only if $p \leq 1$. In other words, there are no solutions for the most interesting case $p > 1$ corresponding to an increasing number of bacteria.

In this work we study the case $b = \infty$. Furthermore, instead of using the biological transition law given by (1.2), we consider a general biological law mathematically described by the following boundary condition

$$f(t,0,v) = [Kf(t,1,\cdot)](v) \quad \text{for all } v \in (a,b), \tag{1.5}$$

where K denotes a linear operator on suitable spaces (see Section 3).

According to $0 < a < b = \infty$, we have recently proved in [3] that the general model (1.1), (1.5) is governed by a strongly continuous semigroup. However, all claims and computations unfortunately depend on $a > 0$. Therefore, we are naturally led to set the following question

What happens when $a = 0$ and $b = \infty$?

The purpose of this work is to answer the question above as follows

(2) Mathematical preliminaries
(3) The unperturbed model (i.e., $\sigma = 0$)
(4) Explicit form of the unperturbed semigroup
(5) Generation Theorem for the model (1.1), (1.5)
(6) Lattice property of the generated semigroup
(7) Spectral properties of the generated semigroup
(8) Application and comments

In the third Section, we consider the natural framework of the general model (1.1), (1.5) which is $L^1((0,1)\times(0,\infty))$ whose norm

$$\|f(t,\cdot,\cdot)\|_1 = \int_0^1 \int_0^\infty |f(t,\mu,v)|\, d\mu\, dv$$

denotes the bacteria number at time t. We show then that the unperturbed model (1.1), (1.5) (i.e., without bacterial mortality ($\sigma = 0$)) is governed by a strongly continuous semigroup whose explicit form is given in fourth Section. In fifth Section, we prove that the general model (1.1), (1.5) is governed by a strongly continuous semigroup whose Lattice and Spectral properties are studied in sixth and seventh Sections. The last section deals with an application to the model (1.1)–(1.2).

Finally, note the novelty of this work and for the used mathematical background, we refer the reader to [8] and [9].

2. Mathematical preliminaries

In this section, we are going to recall some useful mathematical tools about strongly continuous semigroups of linear operators in a Banach space X. So, the first one deals with the following known result

Lemma 2.1 ([9, Theorem III.1.3]). *Let T be the infinitesimal generator of a strongly continuous semigroup $U = (U(t))_{t\geq 0}$, on X, and let B be a bounded linear operator from X into itself. Then, the operator $C := T + B$ on the domain $D(C) := D(T)$ generates, on X, a strongly continuous semigroup $(V(t))_{t\geq 0}$ given, for all $x \in X$, by*

$$V(t)x = \lim_{n\to\infty} [e^{-\frac{t}{n}B} U(\tfrac{t}{n})]^n x \quad t \geq 0. \tag{2.1}$$

Let $U = (U(t))_{t\geq 0}$ be a strongly continuous semigroup, on X, whose generator is T. Following [9, Chapter IV], the *type* $\omega_0(U)$ is

$$\omega_0(U) = \lim_{t\to\infty} \frac{\ln \|U(t)\|_{\mathcal{L}(X)}}{t}. \tag{2.2}$$

The spectral bound $s(T)$ of the generator T is given by

$$s(T) = \begin{cases} \sup\{\mathrm{Re}(\lambda),\, \lambda \in \sigma(T)\} & \text{if } \sigma(T) \neq \emptyset, \\ -\infty & \text{if } \sigma(T) = \emptyset. \end{cases} \tag{2.3}$$

Generally, we have $\omega_0(U) \neq s(T)$. However, if X is an L_1 space then

$$\omega_0(U) = s(T) \tag{2.4}$$

because of [13]. Next, if X denotes a Banach lattice space then, the positivity and the irreducibility of the semigroup $U = (U(t))_{t\geq 0}$ are characterized as follows

Lemma 2.2 ([8, Proposition 7.1 and 7.6]).
1. *The semigroup $U = (U(t))_{t\geq 0}$ is positive if and only if the resolvent operator $(\lambda - T)^{-1}$ is positive for some great λ.*
2. *Suppose that the semigroup $U = (U(t))_{t\geq 0}$ is positive. It is irreducible if and only if the resolvent operator $(\lambda - T)^{-1}$ is irreducible for some great λ.*

3. The unperturbed model (i.e., $\sigma = 0$)

In this section, we are concerned with the unperturbed model (1.1), (1.5) (i.e., without bacterial mortality ($\sigma = 0$)). So, we are going to prove that this model is governed by a strongly continuous semigroup $U_K = (U_K(t))_{t\geq 0}$ which will be perturbed to infer the well posedness of the general model (1.1), (1.5) (see Section 5). Before we start, let us consider the functional framework $L^1(\Omega)$ whose norm is

$$\|\varphi\|_1 = \int_\Omega |\varphi(\mu, v)|\, d\mu\, dv, \tag{3.1}$$

where $\Omega = (0, 1) \times (0, \infty) := I \times J$. We also consider our regularity space

$$W_1 = \{\varphi \in L^1(\Omega)\ v\frac{\partial\varphi}{\partial\mu} \in L^1(\Omega) \quad \text{and} \quad v\varphi \in L^1(\Omega)\}$$

and the trace space $Y_1 := L^1(J, v\, dv)$ whose norms are

$$\|\varphi\|_{W_1} = \|v\frac{\partial\varphi}{\partial\mu}\|_1 + \|v\varphi\|_1 \quad \text{and} \quad \|\psi\|_{Y_1} = \int_0^\infty |\psi(v)|v\, dv.$$

Applying now [4, Theorem 2.2] to $\Omega = (0, 1) \times (0, \infty)$ we infer that

Lemma 3.1. *The trace mappings $\gamma_0\varphi := \varphi(0,\cdot)$ and $\gamma_1\varphi := \varphi(1,\cdot)$ are linear continuous from W_1 into Y_1.*

Next, let T_0 be the unbounded operator

$$\begin{gathered} T_0\varphi = -v\frac{\partial\varphi}{\partial\mu} \quad \text{on the domain,} \\ D(T_0) = \{\varphi \in W_1 \ \ \gamma_0\varphi = 0\} \end{gathered} \tag{3.2}$$

which makes sense because of Lemma 3.1. This operator corresponds to the model (1.1), (1.5) without bacterial mortality ($\sigma = 0$) and without bacterial division ($K = 0$). Some of its useful properties can be summarized as follows

Lemma 3.2.

(1) *The operator T_0 generates, on $L^1(\Omega)$, a strongly continuous positif semigroup $U_0 = (U_0(t))_{t\geq 0}$ of contractions given by*

$$U_0(t)\varphi(\mu, v) := \chi(\mu, v, t)\varphi(\mu - tv, v) \tag{3.3}$$

where

$$\chi(\mu, v, t) = \begin{cases} 1 & \text{if } \mu \geq tv; \\ 0 & \text{if } \mu < tv. \end{cases} \tag{3.4}$$

(2) *Let $\lambda > 0$. Then $(\lambda - T_0)^{-1}$ is a positive operator from $L^1(\Omega)$ into itself. Furthermore, for all $g \in L^1(\Omega)$ we have*

$$\|(\lambda - T_0)^{-1}g\|_1 \leq \frac{\|g\|_1}{\lambda} \tag{3.5}$$

$$\|v(\lambda - T_0)^{-1}g\|_1 \leq \|g\|_1. \tag{3.6}$$

(3) *Let $\lambda > 0$. Then $\gamma_1(\lambda - T_0)^{-1}$ is a strictly positive operator from $L^1(\Omega)$ into Y_1.*

(4) *For all $\varphi \in W_1$, the following mapping*

$$t \to \gamma_1(U_0(t)\varphi) \in Y_1 \tag{3.7}$$

is continuous with respect to $t \geq 0$.

Proof. For all $\lambda > 0$ and all $g \in L^1(\Omega)$, a simple computation shows that

$$(\lambda - T_0)^{-1}g(\mu, v) = \int_0^{\mu/v} e^{-\lambda s} g(\mu - sv, v)ds$$

which easily leads to the points (1) and (2) and (3).

(4) Let $t \geq 0$. Firstly, for all $\varphi \in W_1$, we have $\varphi = f + g$, where $f := \varepsilon(\gamma_0\varphi)$ and $g := \varphi - f$ with $\varepsilon(\mu, v) = e^{-\frac{\mu}{v}}$. Easy computations show that

$$\|v\frac{\partial f}{\partial\mu}\|_1 = \|f\|_1 \leq \|\gamma_0\varphi\|_{Y_1} \quad \text{and} \quad \|vf\|_1 \leq \|\gamma_0\varphi\|_{Y_1}$$

which leads to $f \in W_1$ because of Lemma 3.1. Furthermore

$$\|g\|_{W_1} \leq \|\varphi\|_{W_1} + \|f\|_{W_1} < \infty \quad \text{and} \quad \gamma_0 g = \gamma_0\varphi - \gamma_0 f = 0$$

and therefore $g \in D(T_0)$.

Next, as $U_0 = (U_0(t))_{t\geq 0}$ is also a strongly continuous semigroup on the domain $D(T_0)$ of its generator, then we can write

$$\lim_{h\to 0} \|U_0(t+h)g - U_0(t)g\|_{D(T_0)} = 0. \tag{3.8}$$

By virtue of Lemma 3.1 together with the fact that $D(T_0)$ is a closed subspace of W_1, it follows that

$$\|\gamma_1 U_0(t+h)g - \gamma_1 U_0(t)g\|_{Y_1} \leq \|\gamma_1\|\|U_0(t+h)g - U_0(t)g\|_{D(T_0)}$$

for all $h > 0$ and therefore

$$t \to \gamma_1 U_0(t)g \tag{3.9}$$

is a continuous mapping with respect to $t \geq 0$, because of (3.8).

On the other hand, due to (3.3) and (3.4), it is easy to check that

$$\begin{aligned}&\|\gamma_1 U_0(t+h)f - \gamma_1 U_0(t)f\|_{Y_1}\\&= \int_0^\infty |\chi(1,v,t+h)e^{-\frac{(1-(t+h)v)}{v}} - \chi(1,v,t)e^{-\frac{(1-tv)}{v}}||\gamma_0\varphi(v)|v\,dv\end{aligned}$$

for all $h > 0$ and therefore

$$t \to \gamma_1 U_0(t)f \tag{3.10}$$

is a continuous mapping with respect to $t \geq 0$.

Finally, writing (3.7) as follows

$$t \to \gamma_1 U_0(t)\varphi = \gamma_1 U_0(t)f + \gamma_1 U_0(t)g$$

we infer its continuity from those of the mappings (3.9) and (3.10). □

In the sequel we are going to study the model (1.1), (1.5), without bacterial mortality ($\sigma = 0$), modeled by the following unbounded operator

$$\begin{gathered}T_K\varphi = -v\frac{\partial\varphi}{\partial\mu} \text{ on the domain,}\\ D(T_K) = \{\varphi \in W_1;\, \gamma_0\varphi = K\gamma_1\varphi\}\end{gathered} \tag{3.11}$$

where K denotes a linear operator from Y_1 into itself. Note that (3.11) has a sense because of Lemma 3.1. So, in order to state the main goal of this section, we are going to prove some preparative results. The first one deals with the following operator

$$K_\lambda\psi := K(\theta_\lambda\psi), \quad \text{where } \theta_\lambda(v) = e^{-\lambda/v}, \quad v \in J = (0,\infty) \tag{3.12}$$

which is going to play an important role in the sequel. So we have

Lemma 3.3. *Let K be a linear operator from Y_1 into itself satisfying one of the following hypotheses*

- (Kb) *K is bounded and $\|K\|_{\mathcal{L}(Y_1)} < 1$;*
- (Kc) *K is compact and $\|K\|_{\mathcal{L}(Y_1)} \geq 1$.*

Then, for all $\lambda \geq 0$, the operator K_λ is bounded linear from Y_1 into itself. Furthermore, there exists a constant

$$\omega_K \begin{cases} = 0, & \textit{if } \text{(Kb)} \textit{ holds}\\ > 0, & \textit{if } \text{(Kc)} \textit{ holds,}\end{cases} \tag{3.13}$$

such that

$$\lambda > \omega_K(\ln k) \Longrightarrow \|K_\lambda\|_{\mathcal{L}(Y_1)} < 1, \tag{3.14}$$

where $k = \max\{1, \|K\|\}$.

Proof. Firstly, note that the boundedness of K_λ $(\lambda \geq 0)$ obviously follows from

$$\|K_\lambda \psi\|_{Y_1} \leq \|K\| \|\psi\|_{Y_1} \quad \text{for all } \psi \in Y_1.$$

Therefore, if (Kb) holds, then we clearly have

$$\lambda > 0 \Longrightarrow \|K_\lambda\| < 1. \tag{3.15}$$

Suppose now (Kc) holds and let $\omega \geq 0$ be given. So, the compactness of the operator K obviously leads to that of the operator $K\mathbb{I}_\omega$, where $\mathbb{I}_\omega \in \mathcal{L}(Y_1)$ is the following characteristic operator

$$\mathbb{I}_\omega \psi(v) = \begin{cases} \psi(v) & \text{if } v > \omega; \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

Hence, there exists a finite sequence $(\psi_i)_{i=1}^{N_K} \subset B(0,1) \subset Y_1$ such that

$$K\mathbb{I}_\omega\Big(B(0,1)\Big) \subset \cup_{i=1}^{N_K} B\Big(K\mathbb{I}_\omega \psi_i,\ \frac{1}{2}\Big), \tag{3.17}$$

where $B(0,1)$ is the closed unit ball into Y_1.

Now, for all $i \in \{1, \cdots, N_K\}$, we clearly have

$$\|K\mathbb{I}_\omega \psi_i\|_{Y_1} \leq \|K\| \int_0^\infty |\mathbb{I}_\omega \psi_i(v)| v\, dv$$

which implies that

$$\lim_{\omega \to \infty} \|K\mathbb{I}_\omega \psi_i\|_{Y_1} \leq \|K\| \lim_{\omega \to \infty} \int_0^\infty |\mathbb{I}_\omega \psi_i(v)| v\, dv = 0$$

and therefore, there exists $\delta_{K,i} > 0$ satisfying

$$\|K\mathbb{I}_\omega \psi_i\|_{Y_1} < \frac{1}{2} \quad \text{for all } \omega > \delta_{K,i}.$$

Furthermore, if we set

$$\delta_K := \max\{\delta_{K,i};\ i = 1, \cdots, N_K\} \tag{3.18}$$

it follows that

$$\delta_K > 0 \tag{3.19}$$

and

$$\max_{i \in \{1, \cdots, N_K\}} \|K\mathbb{I}_\omega \psi_i\|_{Y_1} < \frac{1}{2} \quad \text{for all } \omega > \delta_K. \tag{3.20}$$

Next, let $\omega > \delta_K$. For all $\psi \in B(0,1) \subset Y_1$, (3.17) implies that there exists $i_0 \in \{1, \cdots, N_K\}$ satisfying

$$K\mathbb{I}_\omega \psi \in B\Big(K\mathbb{I}_\omega \psi_{i_0},\ \frac{1}{2}\Big)$$

which implies that

$$\begin{aligned} \|K\mathbb{I}_\omega \psi\|_{Y_1} &\leq \|K\mathbb{I}_\omega \psi - K\mathbb{I}_\omega \psi_{i_0}\|_{Y_1} + \|K\mathbb{I}_\omega \psi_{i_0}\|_{Y_1} \\ &\leq \frac{1}{2} + \max_{i \in \{1, \cdots, N_K\}} \|K\mathbb{I}_\omega \psi_i\|_{Y_1} \end{aligned}$$

and therefore

$$\|K\mathbb{I}_\omega\| = \sup_{\psi \in B(0,1)} \|K\mathbb{I}_\omega \psi\|_{Y_1} \leq \frac{1}{2} + \max_{i \in \{1, \cdots, N_K\}} \|K\mathbb{I}_\omega \psi_i\|_{Y_1} < 1$$

because of (3.20). Hence, we can say that

$$\|K\mathbb{I}_\omega\| < 1 \quad \text{for all } \omega > \delta_K. \tag{3.21}$$

On the other hand, let $\omega > \delta_K$ and let $\overline{\mathbb{I}}_\omega \in \mathcal{L}(Y_1)$ be the following characteristic operator

$$\overline{\mathbb{I}}_\omega \psi(v) := \begin{cases} \psi(v) & \text{if } v \leq \omega; \\ 0 & \text{otherwise} \end{cases} \tag{3.22}$$

for which we clearly have

$$\psi = \mathbb{I}_\omega\psi + \overline{\mathbb{I}}_\omega\psi \quad \text{and} \quad \mathbb{I}_\omega^2\psi = \mathbb{I}_\omega\psi. \tag{3.23}$$

So, for all $\psi \in Y_1$ we have

$$K_\lambda\psi = K_\lambda(\mathbb{I}_\omega\psi + \overline{\mathbb{I}}_\omega\psi) = K_\lambda(\mathbb{I}_\omega^2\psi + \overline{\mathbb{I}}_\omega\psi) = K\mathbb{I}_\varepsilon(\mathbb{I}_\varepsilon\theta_\lambda\psi) + K\overline{\mathbb{I}}_\omega(\theta_\lambda\psi)$$

which implies that

$$\begin{aligned} \|K_\lambda\psi\|_{Y_1} &\leq \|K\mathbb{I}_\omega(\mathbb{I}_\omega\theta_\lambda\psi)\|_{Y_1} + \|K\overline{\mathbb{I}}_\omega(\theta_\lambda\psi)\|_{Y_1} \\ &\leq \|K\mathbb{I}_\omega\|\|\mathbb{I}_\omega\theta_\lambda\psi\|_{Y_1} + \|K\|\|\overline{\mathbb{I}}_\omega(\theta_\lambda\psi)\|_{Y_1} \\ &\leq \|K\mathbb{I}_\omega\|\|\mathbb{I}_\omega\psi\|_{Y_1} + e^{-\frac{\lambda}{\omega}}\|K\|\|\overline{\mathbb{I}}_\omega\psi\|_{Y_1} \\ &\leq \max\{\|K\mathbb{I}_\omega\|,\ e^{-\frac{\lambda}{\omega}}\|K\|\}\{\|\mathbb{I}_\omega\psi\| + \|\overline{\mathbb{I}}_\omega\psi\|\} \\ &= \max\{\|K\mathbb{I}_\omega\|,\ e^{-\frac{\lambda}{\omega}}\|K\|\}\|\psi\|_{Y_1} \end{aligned}$$

for all $\lambda \geq 0$, and therefore

$$\|K_\lambda\| \leq \max\big\{\|K\mathbb{I}_\omega\|,\ e^{-\frac{\lambda}{\omega}}\|K\|\big\}.$$

Now, (3.21) clearly leads to

$$\lambda > \omega \ln \|K\| \Longrightarrow \|K_\lambda\| < 1. \tag{3.24}$$

Let $\lambda > \delta_K \ln\|K\|$ be given. There exists ω such that $\frac{\lambda}{\ln\|K\|} > \omega > \delta_K$ which implies that $\lambda > \omega\ln\|K\|$ and therefore $\|K_\lambda\| < 1$ because of (3.24). Therefore, we can say that if (Kc) holds, then

$$\lambda > \delta_K \ln\|K\| \Longrightarrow \|K_\lambda\| < 1. \tag{3.25}$$

Finally, by (3.19) we can set that

$$\omega_K := \begin{cases} 0, & \text{if (Kb) holds;} \\ \delta_K, & \text{if (Kc) holds,} \end{cases}$$

which obviously leads to (3.13). Furthermore, (3.14) clearly holds because of (3.15) and (3.25). The proof is now achieved. □

Remark 3.4. In the sequel, any linear operator K from Y_1 into itself is said to be *admissible* if one of the following hypotheses holds

(Kb) K is bounded and $\|K\|_{\mathcal{L}(Y_1)} < 1$;
(Kc) K is compact and $\|K\|_{\mathcal{L}(Y_1)} \geq 1$,

The constant ω_K, given by (3.13), is called the *abscissa* of the admissible operator K. So, Lemma 3.3 means that (3.14) holds for any admissible operator K whose abscissa is ω_K.

Now, we compute the resolvent operator of T_K as follows

Proposition 3.5. *Let K be an admissible operator whose abscissa is ω_K. Then*

$$\Big(\omega_K \ln k;\, \infty\Big) \subset \rho(T_K). \tag{3.26}$$

Furthermore, if $\lambda > \omega_K(\ln k)$ $(k = \max\{1, \|K\|\})$ then we have

$$(\lambda - T_K)^{-1} g = \varepsilon_\lambda (I - K_\lambda)^{-1} K \gamma_1 (\lambda - T_0)^{-1} g + (\lambda - T_0)^{-1} g \tag{3.27}$$

for all $g \in L^1(\Omega)$, where $\varepsilon_\lambda(\mu, v) = e^{-\lambda \frac{\mu}{v}}$.

Proof. Let $\lambda > \omega_K(\ln k)$ and let $g \in L^1(\Omega)$. So, the general solution of the following equation

$$\lambda \varphi = -v \frac{\partial \varphi}{\partial \mu} + g \tag{3.28}$$

is given by

$$\varphi = \varepsilon_\lambda \psi + (\lambda - T_0)^{-1} g \tag{3.29}$$

where ψ is any function of the variable $v \in J = (0, \infty)$. When $\psi \in Y_1$, we claim that φ belongs to W_1. Indeed, integrating (3.29) and using (3.5) we infer that

$$\|\varphi\|_1 \le \frac{1}{\lambda} \|\psi\|_{Y_1} + \frac{1}{\lambda} \|g\|_1 < \infty$$

which leads, by virtue of (3.28), to

$$\|v \frac{\partial \varphi}{\partial \mu}\|_1 = \| - \lambda \varphi + g\|_1 \le \lambda \|\varphi\|_1 + \|g\|_1 < \infty.$$

Once more, integrating (3.29) and using (3.6) we obtain that

$$\begin{aligned} \|v\varphi\|_1 &= \|v \varepsilon_\lambda \psi\|_1 + \|v(\lambda - T_0)^{-1} g\|_1 \\ &\le \|\psi\|_{Y_1} + \|g\|_1 < \infty. \end{aligned}$$

Hence, $\varphi \in W_1$. Furthermore, φ belongs to $D(T_K)$ if $\gamma_0 \varphi = K \gamma_1 \varphi$. Namely, ψ satisfies

$$\psi = K_\lambda \psi + K \gamma_1 (\lambda - T_0)^{-1} g$$

which admits, by virtue of (3.14), the following unique solution

$$\psi = (I - K_\lambda)^{-1} K \gamma_1 (\lambda - T_0)^{-1} g \in Y_1. \tag{3.30}$$

In order to achieve the proof, it suffices to put (3.30) in (3.29). □

Now we are able to state the main result of this section as follows.

Theorem 3.6. *Let K be an admissible operator whose abscissa is ω_K. Then, the operator T_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $U_K = (U_K(t))_{t \ge 0}$ satisfying*

$$\|U_K(t)\varphi\|_1 \le k e^{\omega_K(\ln k) t} \|\varphi\|_1 \quad t \ge 0 \tag{3.31}$$

for all $\varphi \in L^1(\Omega)$, where $k = \max\{1; \|K\|\}$.

Remark 3.7. Actually, the admissibility concept that we have gave in Remark 3.4, is a particular case of a general and theoretical concept already defined in [1]. Accordingly, Theorem 3.6 can be inferred from [1, Theorem 3.1]. However, taking into account to the practical and biological aspect of this work, we prefer to give a slightly different proof for the reader.

Proof of Theorem 3.6. Firstly, let $\omega > \omega_K$ be given and let

$$|||\varphi|||_1 = \int_0^\infty \int_0^1 |\varphi(\mu, v)| h_\omega(\mu, v)\, d\mu\, dv \tag{3.32}$$

be another norm on $L^1(\Omega)$ where

$$h_\omega(\mu, v) = k^{\min\{\omega \frac{\mu}{v}\,;\,1\}}.$$

The norms (3.1) and (3.32) are equivalent because, for all $\varphi \in L^1(\Omega)$ we have

$$\|\varphi\|_1 \le |||\varphi|||_1 \le k\|\varphi\|_1. \tag{3.33}$$

Next, let $\lambda > \omega(\ln k)$ and let $g \in L^1(\Omega)$. Proposition 3.5 means that $\varphi = (\lambda - T_K)^{-1} g \in D(T_K)$ is the unique solution of the following system

$$\lambda\varphi = -v\frac{\partial\varphi}{\partial\mu} + g \tag{3.34}$$

$$\gamma_0\varphi = K\gamma_1\varphi. \tag{3.35}$$

So, multiplying (3.34) by $(\operatorname{sgn}\varphi)h_\omega$ and integrating it over Ω, we obtain that

$$\begin{aligned}
&\lambda|||\varphi|||_1 \\
&= -\int_\Omega \Big(v\frac{\partial|\varphi|}{\partial\mu}\Big)(\mu, v) h_\omega(\mu, v)\, d\mu\, dv + \int_\Omega \operatorname{sgn}\varphi(\mu, v)(h_\omega g)(\mu, v)\, d\mu\, dv \\
&\le -\int_\Omega \Big(v\frac{\partial|\varphi|}{\partial\mu}\Big)(\mu, v) h_\omega(\mu, v)\, d\mu\, dv + |||g|||_1 \\
&:= A + |||g|||_1.
\end{aligned} \tag{3.36}$$

Integrating A by parts and using (3.35) we infer that

$$\begin{aligned}
A &= \int_0^\infty |\gamma_0(h_\omega\varphi)(v)| v\, dv - \int_0^\infty |\gamma_1(h_\omega\varphi)(v)| v\, dv \\
&\quad + \int_\Omega \Big(v\frac{\partial h_\omega}{\partial\mu}\Big)(\mu, v)|\varphi(\mu, v)|\, d\mu\, dv \\
&= \int_0^\infty |\gamma_0\varphi(v)| v\, dv - \int_0^\infty |\gamma_1(h_\omega\varphi)(v)| v\, dv \\
&\quad + \int_\Omega \Big(v\frac{\partial h_\omega}{\partial\mu}\Big)(\mu, v)|\varphi(\mu, v)|\, d\mu\, dv \\
&= \int_0^\infty |K\gamma_1\varphi(v)| v\, dv - \int_0^\infty |\gamma_1(h_\omega\varphi)(v)| v\, dv \\
&\quad + \int_\Omega \Big(v\frac{\partial h_\omega}{\partial\mu}\Big)(\mu, v)|\varphi(\mu, v)|\, d\mu\, dv \\
&:= A_1 - A_2 + A_3.
\end{aligned} \tag{3.37}$$

Applying (3.23) together with (3.16) and (3.22) for $\psi = \gamma_1\varphi \in Y_1$, it follows that

$$\begin{aligned}
A_1 &= \int_0^\infty \Big|K\Big(\mathbb{I}_\omega(\gamma_1\varphi) + \overline{\mathbb{I}}_\omega(\gamma_1\varphi)\Big)(v)\Big| v\, dv \\
&\le \int_0^\infty \Big|K\Big(\mathbb{I}_\omega(\gamma_1\varphi)\Big)(v)\Big| v\, dv + \int_0^\infty \Big|K\Big(\overline{\mathbb{I}}_\omega(\gamma_1\varphi)\Big)(v)\Big| v\, dv \\
&\le \int_0^\infty \Big|K\Big(\mathbb{I}_\omega(\gamma_1\varphi)\Big)(v)\Big| v\, dv + \|K\| \int_0^\infty \Big|\Big(\overline{\mathbb{I}}_\omega(\gamma_1\varphi)\Big)(v)\Big| v\, dv
\end{aligned}$$

$$= \|K\Big(\mathbb{I}_\omega\Big(\theta_{\lambda'}(\gamma_1 h_\omega)\Big)(\gamma_1\varphi)\Big)\|_{Y_1} + \|K\|\|\bar{\mathbb{I}}_\omega(\gamma_1\varphi)\|_{Y_1}$$

because of $\Big(\theta_{\lambda'}(\gamma_1 h_\omega)\Big)(v) = 1$ for all $v \in (\omega, \infty)$, where $\theta_{\lambda'}$ is given by (3.12) for $\lambda' = \omega(\ln k)$. Hence

$$\begin{aligned} A_1 &\le \|K\theta_{\lambda'}\Big(\mathbb{I}_\omega\gamma_1(h_\omega\varphi)\Big)\|_{Y_1} + \|K\|\|\bar{\mathbb{I}}_\omega(\gamma_1\varphi)\|_{Y_1} \\ &= \|K_{\lambda'}\Big(\mathbb{I}_\omega\gamma_1(h_\omega\varphi)\Big)\|_{Y_1} + \|K\|\|\bar{\mathbb{I}}_\omega(\gamma_1\varphi)\|_{Y_1} \\ &\le \|K_{\lambda'}\|\|\mathbb{I}_\omega\gamma_1(h_\omega\varphi)\|_{Y_1} + \|K\|\|\bar{\mathbb{I}}_\omega(\gamma_1\varphi)\|_{Y_1} \end{aligned}$$

and therefore

$$A_1 \le \|\mathbb{I}_\omega\gamma_1(h_\omega\varphi)\|_{Y_1} + \|K\|\|\bar{\mathbb{I}}_\omega(\gamma_1\varphi)\|_{Y_1} \tag{3.38}$$

because of (3.14). Once more, applying (3.23) together with (3.16) and (3.22) for $\psi = |\gamma_1(h_\omega\varphi)| \in Y_1$, we infer that

$$\begin{aligned} A_2 &= \int_0^\infty \Big(\mathbb{I}_\omega|\gamma_1(h_\omega\varphi)| + \bar{\mathbb{I}}_\omega|\gamma_1(h_\omega\varphi)|\Big)(v)v\,dv \\ &= \int_0^\infty |\mathbb{I}_\omega\gamma_1(h_\omega\varphi)(v)|v\,dv + \int_0^\infty |\bar{\mathbb{I}}_\omega(\gamma_1 h_\omega)(\gamma_1\varphi)(v)|v\,dv \\ &= \int_0^\infty |\mathbb{I}_\omega\gamma_1(h_\omega\varphi)(v)|v\,dv + k\int_0^\infty |\bar{\mathbb{I}}_\omega(\gamma_1\varphi)(v)|v\,dv \end{aligned}$$

and therefore

$$A_2 = \|\mathbb{I}_\omega\gamma_1(h_\omega\varphi)\|_{Y_1} + k\|\bar{\mathbb{I}}_\omega(\gamma_1\varphi)\|_{Y_1}. \tag{3.39}$$

Next, for almost all $(\mu, v) \in \Omega$ we have

$$\begin{aligned} (v\frac{\partial h_\omega}{\partial\mu})(\mu, v) &= v\frac{\partial}{\partial\mu}(\begin{cases} k^{\omega\frac{\mu}{v}} & \text{if } \omega\frac{\mu}{v} \le 1 \\ k & \text{if } \omega\frac{\mu}{v} > 1 \end{cases}) \\ &= \begin{cases} \omega(\ln k)k^{\omega\frac{\mu}{v}} & \text{if } \omega\frac{\mu}{v} \le 1 \\ 0 & \text{if } \omega\frac{\mu}{v} > 1 \end{cases} \\ &\le \omega(\ln k)\begin{cases} k^{\omega\frac{\mu}{v}} & \text{if } \omega\frac{\mu}{v} \le 1 \\ k & \text{if } \omega\frac{\mu}{v} > 1 \end{cases} \end{aligned}$$

which leads to

$$(v\frac{\partial h_\omega}{\partial\mu}) \le \omega(\ln k)h_\omega.$$

Hence

$$A_3 \le \omega(\ln k)|||\varphi|||_1. \tag{3.40}$$

Replacing now (3.38) and (3.39) and (3.40) into (3.37) we infer that

$$A \le \omega(\ln k)|||\varphi|||_1$$

which we put into (3.36) to finally get that

$$|||\varphi|||_1 = |||(\lambda - T_K)^{-1}g|||_1 \le \frac{|||g|||_1}{(\lambda - \omega(\ln k))}.$$

On the other hand, (3.26) obviously leads to $\rho(T_K) \ne \emptyset$ and therefore T_K is a closed operator. Furthermore, T_K is densely defined because of $\mathcal{C}_c(\Omega) \subset D(T_K) \subset L^1(\Omega)$.

Now, thanks to Hille-Yosida Theorem, the operator T_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $U_K = (U_K(t))_{t\geq 0}$ satisfying

$$|||U_K(t)\varphi|||_1 \leq e^{\omega(\ln k)t}|||\varphi|||_1 \quad t \geq 0 \tag{3.41}$$

for all $\varphi \in L^1(\Omega)$. As ω ($\omega > \omega_K$) is arbitrary chosen, then passing at the limit $\omega \to \omega_K$ in (3.41) we obtain that

$$|||U_K(t)\varphi|||_1 \leq e^{\omega_K(\ln k)t}|||\varphi|||_1 \quad t \geq 0. \tag{3.42}$$

Finally, in order to archives the proof, it suffices to infer (3.31) from (3.42) together with (3.33). □

Now, let us infer some interesting Corollaries.

Corollary 3.8. *Let K be a bounded linear operator from Y_1 into itself such that $\|K\| < 1$. Then the operator T_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $U_K = (U_K(t))_{t\geq 0}$ of contractions; i.e.,*

$$\|U_K(t)\varphi\|_1 \leq \|\varphi\|_1 \quad t \geq 0$$

for all $\varphi \in L^1(\Omega)$.

Proof. Thanks to Remark 3.4, the hypothesis (Kb) holds and therefore K is an admissible operator whose abscissa $\omega_K = 0$ because of (3.13). Now, it suffices to apply Theorem 3.6 for $\omega_K = 0$ and $k = 1$. □

Remark 3.9. According to Corollary 3.8 we infer that

$$\|U_K(t)\varphi\|_1 = \|U_K(t-s)U_K(s)\varphi\|_1 \leq \|U_K(s)\varphi\|_1$$

for all initial data $\varphi \in L^1(\Omega)$, where t and s ($t > s$) are two arbitrary times. Namely, the unperturbed model (1.1), (1.5) (without bacterial mortality ($\sigma = 0$)), corresponding to Corollary 3.8 is biologically uninteresting because the bacteria number is obviously decreasing.

In contrary to Remark 3.9, we can say that $\|K\| > 1$ corresponds to an increasing bacteria number during each mitotic. Hence we have

Corollary 3.10. *Let K be a linear compact operator from Y_1 into itself such that $\|K\| > 1$. Then, the operator T_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $U_K = (U_K(t))_{t\geq 0}$ satisfying*

$$\|U_K(t)\varphi\|_1 \leq \|K\|\|K\|^{t\omega_K}\|\varphi\|_1 \quad t \geq 0$$

for all $\varphi \in L^1(\Omega)$, where $\omega_K > 0$ is abscissa of the operator K.

Proof. By virtue of Remark 3.4, we infer that the hypothesis (Kc) holds and therefore K is an admissible operator. Furthermore, its abscissa $\omega_K > 0$ because of (3.13). Now, Theorem 3.6 together with $k = \|K\|$ achieve the proof. □

4. Explicit form of the unperturbed semigroup

The purpose of this section is to find the explicit form of the semigroup $U_K = (U_K(t))_{t\geq 0}$ which will be very useful to describe the asynchronous exponential growth related to the model (1.1), (1.5).

Theorem 4.1. *Let K be an admissible operator whose abscissa is ω_K. Then, for all $\varphi \in L^1(\Omega)$, we have*

$$U_K(t)\varphi = U_0(t)\varphi + A_K(t)\varphi \quad t \geq 0, \tag{4.1}$$

where the operator $A_K(t)$ is defined by

$$A_K(t)\varphi(\mu, v) := \xi(\mu, v, t)K(\gamma_1 U_K(t - \frac{\mu}{v})\varphi)(v) \tag{4.2}$$

with

$$\xi(\mu, v, t) = \begin{cases} 0 & \text{if } \mu \geq tv; \\ 1 & \text{if } \mu < tv. \end{cases} \tag{4.3}$$

Proof. Let $\lambda > \omega_K(\ln k)$ be fixed, where $k = \max\{1, \|K\|\}$. In the sequel, we are going to divide the proof in several steps.

Step I. Let $L^1_\lambda := L^1\Big(\Delta,\ e^{\lambda x/v}\Big)$ be the weighted Banach space whose norm is

$$\|f\|_{L^1_\lambda} = \int_\Delta |f(x, v)| e^{\lambda x/v}\, dx\, dv,$$

where $\Delta = (-\infty, 0) \times (0, \infty)$. Let H_K and V_K be the following linear operators

$$H_K f(x, v) := K\Big(\xi(1, \cdot, -xv^{-1}) f(1 + xv^{-1}\cdot, \cdot)\Big)(v)$$
$$V_K\varphi(x, v) := K\Big(\gamma_1 U_0(-xv^{-1})\varphi\Big)(v).$$

So, for all $f \in L^1_\lambda$,

$$\begin{aligned}
\|H_K f\|_{L^1_\lambda} &= \int_{-\infty}^0 \int_0^\infty |K\Big(\xi(1, \cdot, -xv^{-1}) f(1 + xv^{-1}\cdot, \cdot)\Big)(v)| e^{\lambda x/v}\, dx\, dv \\
&= \int_0^\infty \Big[\int_0^\infty |K\Big(\xi(1, \cdot, t) f(1 - t\cdot, \cdot)\Big)(v)| v\, dv\Big] e^{-\lambda t} dt \\
&= \int_0^\infty \Big[\int_0^\infty |K\theta_\lambda\Big(\theta_{-\lambda}\xi(1, \cdot, t) f(1 - t\cdot, \cdot)\Big)(v)| v\, dv\Big] e^{-\lambda t} dt \\
&= \int_0^\infty \Big[\int_0^\infty |K_\lambda\Big(\theta_{-\lambda}\xi(1, \cdot, t) f(1 - t\cdot, \cdot)\Big)(v)| v\, dv\Big] e^{-\lambda t} dt
\end{aligned}$$

because of (3.12). Due to the boundedness of the operator K_λ (see Lemma 3.3) we infer that

$$\begin{aligned}
\|H_K f\|_{L^1_\lambda} &\leq \|K_\lambda\|_{\mathcal{L}(Y_1)} \int_0^\infty \int_0^\infty e^{\frac{\lambda}{v}} \xi(1, v, t) |f(1 - tv, v)| e^{-\lambda t} v\, dv dt \\
&= \|K_\lambda\|_{\mathcal{L}(Y_1)} \int_0^\infty \int_{-\infty}^1 e^{\frac{\lambda}{v}} \xi(1, v, \tfrac{(1-x)}{v}) |f(x, v)| e^{-\lambda \frac{(1-x)}{v}}\, dx\, dv \\
&= \|K_\lambda\|_{\mathcal{L}(Y_1)} \int_0^\infty \int_{-\infty}^0 |f(x, v)| e^{\lambda x/v} dx dv \\
&= \|K_\lambda\|_{\mathcal{L}(Y_1)} \|f\|_{L^1_\lambda}
\end{aligned}$$

and therefore

$$\|H_K\|_{\mathcal{L}(L^1_\lambda)} \leq \|K_\lambda\|_{\mathcal{L}(Y_1)}. \tag{4.4}$$

On the other hand, for all $\varphi \in L^1(\Omega)$ we have

$$\begin{aligned}
\|V_K\varphi\|_{L^1_\lambda} &= \int_0^\infty \int_{-\infty}^0 |K\Big(\gamma_1 U_0(-xv^{-1})\varphi\Big)(v)|e^{\lambda x/v}\,dx\,dv \\
&\le \int_0^\infty [\int_0^\infty |K\Big(\gamma_1 U_0(t)\varphi\Big)(v)|v\,dv]dt \\
&\le \|K\|_{\mathcal{L}(Y_1)} \int_0^\infty \int_0^\infty |\Big(\gamma_1 U_0(t)\varphi\Big)(v)|vdtdv
\end{aligned}$$

which leads, by virtue of (3.3) and (3.4), to

$$\begin{aligned}
\|V_K\varphi\|_{L^1_\lambda} &\le \|K\|_{\mathcal{L}(Y_1)} \int_0^\infty \int_0^\infty |\chi(1,v,t)\varphi(1-tv,v)|vdtdv \\
&= \|K\|_{\mathcal{L}(Y_1)} \int_0^\infty \int_{-\infty}^1 |\chi(1,v,\tfrac{(1-\mu)}{v})\varphi(\mu,v)|\,d\mu\,dv \\
&= \|K\|_{\mathcal{L}(Y_1)} \int_0^\infty \int_0^1 |\varphi(\mu,v)|\,d\mu\,dv
\end{aligned}$$

and therefore

$$\|V_K\varphi\|_{L^1_\lambda} \le \|K\|_{\mathcal{L}(Y_1)}\|\varphi\|_1. \tag{4.5}$$

Now, (3.14) together with (4.4) and (4.5) imply that the problem

$$f = H_K f + V_K\varphi \tag{4.6}$$

admits, for all $\varphi \in L^1(\Omega)$, the unique solution

$$f_\varphi^K = (I-H_K)^{-1}V_K\varphi \in L^1_\lambda \tag{4.7}$$

satisfying

$$\|f_\varphi^K\|_{L^1_\lambda} \le \frac{\|K\|_{\mathcal{L}(Y_1)}}{1-\|K_\lambda\|_{\mathcal{L}(Y_1)}}\|\varphi\|_1. \tag{4.8}$$

Furthermore

$$\varphi \in L^1(\Omega) \to f_\varphi^K \in L^1_\lambda \tag{4.9}$$

is a linear mapping because of those of the operators H_K and V_K.

Now we can say that : *If $\lambda > \omega_K(\ln k)$ then, for all $\varphi \in L^1(\Omega)$ the problem* (4.6) *admits the unique solution* (4.7) *satisfying* (4.8). *Moreover* (4.9) *is a linear continuous mapping from $L^1(\Omega)$ into L^1_λ.*

Step II. Thanks to the step I, we can define the following operator

$$B_K(t)\varphi(\mu,v) := \xi(\mu,v,t)f_\varphi^K(\mu-tv,v) \quad t \ge 0.$$

Note that the linearity of the operator $B_K(t)$ follows from that of (4.9).

First, let $t \ge 0$. For all $\varphi \in L^1(\Omega)$ we have

$$\begin{aligned}
\|B_K(t)\varphi\|_1 &= \int_\Omega \xi(\mu,v,t)|f_\varphi^K(\mu-tv,v)|\,d\mu\,dv \\
&\le \int_0^\infty \int_0^{tv} |f_\varphi^K(\mu-tv,v)|e^{\lambda\frac{\mu}{v}}\,d\mu\,dv \\
&= \int_0^\infty \int_{-tv}^0 |f_\varphi^K(x,v)|e^{\lambda(\frac{x}{v}+t)}\,dx\,dv
\end{aligned}$$

which implies that

$$\|B_K(t)\varphi\|_1 \le e^{\lambda t} \int_0^\infty \int_{-tv}^0 |f_\varphi^K(x,v)| e^{\lambda x/v}\, dx\, dv \tag{4.10}$$

and therefore

$$\|B_K(t)\varphi\|_1 \le e^{\lambda t} \|f_\varphi^K\|_{L^1_\lambda} \le e^{\lambda t} \frac{\|K\|_{\mathcal{L}(Y_1)}}{1 - \|K_\lambda\|_{\mathcal{L}(Y_1)}} \|\varphi\|_1$$

because of (4.8). Hence, $B_K(t)$ is a bounded operator from $L^1(\Omega)$ into itself. Furthermore, (4.10) obviously leads to

$$\lim_{t\to 0_+} \|B_K(t)\varphi\|_1 = 0 \quad \text{and} \quad B_K(0) = 0. \tag{4.11}$$

Next. For all $\varphi \in W_1$ we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty |\gamma_0 B_K(t)\varphi(v) - K\gamma_1 B_K(t)\varphi(v) - K\gamma_1 U_0(t)\varphi(v)| v dt dv \\
&= \int_0^\infty \int_0^\infty |f_\varphi^K(-tv,v) - H_K f_\varphi^K(-tv,v) - V_K\varphi(-tv,v)| v dt dv \\
&= \int_\Delta |f_\varphi^K(x,v) - H_K f_\varphi^K(x,v) - V_K \varphi(x,v)|\, dx\, dv = 0
\end{aligned}$$

because f_φ^K is the unique solution of the problem (4.6) and therefore

$$\gamma_0 B_K(t)\varphi - K\gamma_1 B_K(t)\varphi = K\gamma_1 U_0(t)\varphi \quad \text{a.e. } t \in \mathbb{R}_+. \tag{4.12}$$

Due to the continuity of the mapping (3.7), it follows that (4.12) holds for all $t \ge 0$; i.e.,

$$\gamma_0 B_K(t)\varphi - K\gamma_1 B_K(t)\varphi = K\gamma_1 U_0(t)\varphi \quad \text{for all } t \in \mathbb{R}_+. \tag{4.13}$$

Now we can say that : *for all $t \ge 0$, $B_K(t)$ is a bounded linear operator from $L^1(\Omega)$ into itself satisfying* (4.11) *and* (4.13).

Step III. Thanks to the step II together with Lemma 3.2, we can define the following operator

$$S_K(t) := U_0(t) + B_K(t) \quad t \ge 0 \tag{4.14}$$

which is clearly linear and bounded from $L^1(\Omega)$ into itself.

First, let $\varphi \in L^1(\Omega)$. By virtue Lemma 3.2(1) together with (4.11) we infer that

$$S_K(0) = U_0(0) + B_K(0) = U_0(0) = I, \tag{4.15}$$

where I is the identity operator into $L^1(\Omega)$, and

$$\lim_{t\to 0_+} \|S_K(t)\varphi - \varphi\|_1 \le \lim_{t\to 0_+} \|U_0(t)\varphi - \varphi\|_1 + \lim_{t\to 0_+} \|B_K(t)\varphi\|_1 = 0. \tag{4.16}$$

Next, let $t \ge 0$ and let $\varphi \in W_1$. Due to (4.14) we obtain

$$\begin{aligned}
&\xi(\mu,v,t)\gamma_0 \Big(S_K(t - \frac{\mu}{v})\varphi\Big)(v) \\
&= \xi(\mu,v,t)\gamma_0 \Big(U_0(t - \frac{\mu}{v})\varphi\Big)(v) + \xi(\mu,v,t)\gamma_0 \Big(B_K(t - \frac{\mu}{v})\varphi\Big)(v)
\end{aligned} \tag{4.17}$$

for almost all $(\mu, v) \in \Omega$. However, (3.3) together with (3.4) and (4.3) lead to

$$\xi(\mu,v,t)\gamma_0 \Big(U_0(t - \frac{\mu}{v})\varphi\Big)(v) = 0$$

for almost all $(\mu, v) \in \Omega$, and therefore (4.17) becomes

$$\begin{aligned}\xi(\mu,v,t)\gamma_0\Big(S_K(t-\frac{\mu}{v})\varphi\Big)(v) &= \xi(\mu,v,t)\gamma_0\Big(B_K(t-\frac{\mu}{v})\varphi\Big)(v)\\ &= \xi(\mu,v,t)B_K(t-\frac{\mu}{v})\varphi(0,v)\\ &= \xi(\mu,v,t)f_\varphi^K(\mu-tv,v).\end{aligned}$$

Hence

$$\xi(\mu,v,t)\gamma_0\Big(S_K(t-\frac{\mu}{v})\varphi\Big)(v) = B_K(t)\varphi(\mu,v) \tag{4.18}$$

for almost all $(\mu, v) \in \Omega$.

On the other hand, (4.13) and (4.14) imply that

$$\begin{aligned}\gamma_0 S_K(t)\varphi - K\gamma_1 S_K(t)\varphi &= \gamma_0 U_0(t)\varphi + \gamma_0 B_K(t)\varphi - K\gamma_1 U_0(t)\varphi - K\gamma_1 B_K(t)\varphi\\ &= \gamma_0 B_K(t)\varphi - K\gamma_1 U_0(t)\varphi - K\gamma_1 B_K(t)\varphi = 0\end{aligned}$$

for $t \geq 0$ and therefore

$$\xi(\mu,v,t)\gamma_0\big(S_K(t-\tfrac{\mu}{v})\varphi\big)(v) = \xi(\mu,v,t)\big(K\gamma_1 S_K(t-\tfrac{\mu}{v})\varphi\big)(v) \tag{4.19}$$

for almost all $(\mu, v) \in \Omega$. Hence

$$B_K(t)\varphi(\mu,v) = \xi(\mu,v,t)\big(K\gamma_1 S_K(t-\tfrac{\mu}{v})\varphi\big)(v) \quad t \geq 0 \tag{4.20}$$

because of (4.18) and (4.19). Moreover, the density of W_1 in $L^1(\Omega)$ implies that (4.20) holds too for all $\varphi \in L^1(\Omega)$.

Now we can say that : *for all $t \geq 0$, $S_K(t)$ given by* (4.14)*, is a bounded linear operator from $L^1(\Omega)$ into itself satisfying* (4.15) *and* (4.16)*. Furthermore,* (4.20) *holds for all $\varphi \in L^1(\Omega)$.*

Step IV. In order to prove that $(S_K(t))_{t\geq 0}$ is a strongly continuous semigroup it remains, by virtue of the step III, to show only that

$$G(t,s) := S_K(t)S_K(s) - S_K(t+s) = 0 \quad \text{for all } t \geq 0 \text{ and all } s \geq 0.$$

So, let $t \geq 0$ and $s \geq 0$ and let $\varphi \in L^1(\Omega)$. By virtue of (3.3) and (4.14) and (4.20), a simple computation leads to

$$G(t,s)\varphi(\mu,v) = \xi(\mu,v,t)\big(K\gamma_1 G(t-\tfrac{\mu}{v},s)\varphi\big)(v) \tag{4.21}$$

for almost all $(\mu, v) \in \Omega$.

First, applying the trace mapping γ_1 to (4.21) and integrating it, we obtain that

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^{\lambda(\frac{1}{v}-t)}|\gamma_1 G(t,s)\varphi(v)|v\,dt\,dv\\ &= \int_0^\infty\int_0^\infty e^{\lambda(\frac{1}{v}-t)}\xi(1,v,t)|\Big(K\gamma_1 G(t-\tfrac{1}{v},s)\varphi\Big)(v)|v\,dt\,dv\\ &\leq \int_0^\infty e^{-\lambda x}\Big[\int_0^\infty |K\Big(\gamma_1 G(x,s)\varphi\Big)(v)|v\,dv\Big]dx\\ &= \int_0^\infty e^{-\lambda x}\Big[\int_0^\infty |K_\lambda\Big(\theta_{-\lambda}\gamma_1 G(x,s)\varphi\Big)(v)|v\,dv\Big]dx\end{aligned}$$

because of (3.12). Due to the boundedness of the operator K_λ (see Lemma 3.3), it follows that

$$\int_0^\infty\int_0^\infty e^{\lambda(\frac{1}{v}-t)}|\gamma_1 G(t,s)\varphi(v)|v\,dt\,dv$$

$$\begin{aligned}
&\le \|K_\lambda\| \int_0^\infty e^{-\lambda x}\Big[\int_0^\infty \big|\theta_{-\lambda}\big(\gamma_1 G(x,s)\varphi\big)(v)\big| v\,dv\Big]dx \\
&= \|K_\lambda\| \int_0^\infty \int_0^\infty e^{\lambda(\frac{1}{v}-x)}\big|\gamma_1 G(x,s)\varphi(v)\big| v\,dx\,dv
\end{aligned}$$

which leads, by virtue of (3.14), to

$$\gamma_1 G(t,s) = 0 \quad \text{for all } t \ge 0 \text{ and all } s \ge 0. \tag{4.22}$$

On the other hand, integrating (4.21) we obtain that

$$\begin{aligned}
\int_\Omega |G(t,s)\varphi(\mu,v)|\,d\mu\,dv &= \int_\Omega \xi(\mu,v,t)|K\Big(\gamma_1 G(t-\tfrac{\mu}{v},s)\varphi\Big)(v)|\,d\mu\,dv \\
&= \int_0^\infty \int_0^{tv} |K\Big(\gamma_1 G(t-\tfrac{\mu}{v},s)\varphi\Big)(v)|\,d\mu\,dv \\
&= \int_0^\infty \int_0^{t} |K\Big(\gamma_1 G(x,s)\varphi\Big)(v)|v\,dx\,dv \\
&= 0
\end{aligned}$$

because of (4.22) and therefore $G(t,s) = 0$ for all $t \ge 0$ and all $s \ge 0$.

Now we can say that: *the family operators $(S_K(t))_{t\ge0}$ is a strongly continuous semigroup on $L^1(\Omega)$.*

Step V. To achieve the proof, it suffices to show that the semigroups $(S_K(t))_{t\ge0}$ and $(U_K(t))_{t\ge0}$ are equal. So, let us suppose that B denotes the generator of the semigroup $(S_K(t))_{t\ge0}$.

First, let $\varphi \in L^1(\Omega)$. Due to (4.14) and (4.20) we infer that

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} S_K(t)\varphi(\mu,v)dt \\
&= \int_0^\infty e^{-\lambda t} U_0(t)\varphi(\mu,v)dt + \int_0^\infty e^{-\lambda t}\xi(\mu,v,t)\left[K\gamma_1 S_K(t-\tfrac{\mu}{v})\varphi\right](v)dt \\
&= \int_0^\infty e^{-\lambda t} U_0(t)\varphi(\mu,v)dt + e^{-\lambda\frac{\mu}{v}}\int_0^\infty e^{-\lambda t}\left[K\gamma_1 S_K(t)\varphi\right](v)dt \\
&= \int_0^\infty e^{-\lambda t} U_0(t)\varphi(\mu,v)dt + \varepsilon_\lambda(\mu,v)K\gamma_1\Big[\int_0^\infty e^{-\lambda t}S_K(t)\varphi dt\Big](v)
\end{aligned}$$

for all almost $(\mu,v)\in\Omega$, and therefore

$$(\lambda - B)^{-1}\varphi = \varepsilon_\lambda K\gamma_1(\lambda-B)^{-1}\varphi + (\lambda - T_0)^{-1}\varphi. \tag{4.23}$$

Applying $\theta_\lambda^{-1}\gamma_1$ to both hand side of (4.23) we obtain

$$\begin{aligned}
\theta_\lambda^{-1}\gamma_1(\lambda-B)^{-1}\varphi &= \theta_\lambda^{-1}\gamma_1\Big(\varepsilon_\lambda K\gamma_1(\lambda-B)^{-1}\varphi\Big) + \theta_\lambda^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi \\
&= \theta_\lambda^{-1}\theta_\lambda K\gamma_1(\lambda-B)^{-1}\varphi + \theta_\lambda^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi \\
&= K\gamma_1(\lambda-B)^{-1}\varphi + \theta_\lambda^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi \\
&= K\theta_\lambda\theta_\lambda^{-1}\gamma_1(\lambda-B)^{-1}\varphi + \theta_\lambda^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi
\end{aligned}$$

which leads, by (3.12), to

$$\theta_\lambda^{-1}\gamma_1(\lambda-B)^{-1}\varphi = K_\lambda\theta_\lambda^{-1}\gamma_1(\lambda-B)^{-1}\varphi + \theta_\lambda^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi$$

and therefore

$$\gamma_1(\lambda-B)^{-1}\varphi = \theta_\lambda(I-K_\lambda)^{-1}\theta_\lambda^{-1}\gamma_1(\lambda-T_0)^{-1}\varphi \tag{4.24}$$

because of (3.14). Putting now (4.24) into (4.23) we infer that

$$\begin{aligned}(\lambda - B)^{-1}\varphi &= \varepsilon_\lambda K\theta_\lambda(I - K_\lambda)^{-1}\theta_\lambda^{-1}\gamma_1(\lambda - T_0)^{-1}\varphi + (\lambda - T_0)^{-1}\varphi\\ &= \varepsilon_\lambda K_\lambda(I - K_\lambda)^{-1}\theta_\lambda^{-1}\gamma_1(\lambda - T_0)^{-1}\varphi + (\lambda - T_0)^{-1}\varphi\\ &= \varepsilon_\lambda(I - K_\lambda)^{-1}K_\lambda\theta_\lambda^{-1}\gamma_1(\lambda - T_0)^{-1}\varphi + (\lambda - T_0)^{-1}\varphi.\end{aligned}$$

Hence

$$(\lambda - B)^{-1}\varphi = \varepsilon_\lambda(I - K_\lambda)^{-1}K\gamma_1(\lambda - T_0)^{-1}\varphi + (\lambda - T_0)^{-1}\varphi. \tag{4.25}$$

Finally, (3.27) and (4.25) obviously imply that $(\lambda - T_K)^{-1} = (\lambda - B)^{-1}$ and therefore

$$U_K(t)\varphi = S_K(t)\varphi \quad t \geq 0 \tag{4.26}$$

because of the uniqueness of the generated semigroup. Now, in order to achieves the proof, it suffices to infer (4.1) and (4.2) from (4.14) and (4.20) together with (4.26). □

5. Generation theorem for the model (1.1), (1.5)

The main goal of this section is to prove that the general model (1.1), (1.5) is governed by a strongly continuous semigroup $V_K = (V_K(t))_{t\geq 0}$ as a linear perturbation of the unperturbed semigroup $U_K = (U_K(t))_{t\geq 0}$ already studied. To this end, we suppose that the rate of bacterial mortality fulfills the hypothesis

(H_σ) $\sigma \in \big(L^\infty(\Omega)\big)_+$

and we denote

$$\underline{\sigma} := \operatorname{ess\,inf}_{(\mu,v)\in\Omega} \sigma(\mu, v) \quad \text{and} \quad \overline{\sigma} := \operatorname{ess\,sup}_{(\mu,v)\in\Omega} \sigma(\mu, v). \tag{5.1}$$

Thanks to the hypothesis (H_σ), the perturbation operator

$$S\varphi(\mu, v) := -\sigma(\mu, v)\varphi(\mu, v) \quad (\mu, v) \in \Omega$$

is obviously linear and bounded from $L^1(\Omega)$ into itself. So, let L_K be the unbounded operator

$$\begin{gathered}L_K := T_K + S\\ D(L_K) = D(T_K)\end{gathered}$$

closely related to the model (1.1), (1.5), and for which we finally have

Theorem 5.1. *Let K be an admissible operator whose abscissa is ω_K. If the hypothesis (H_σ) holds, then the operator L_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $V_K = (V_K(t))_{t\geq 0}$ satisfying*

$$\|V_K(t)\varphi\|_1 \leq ke^{t(\omega_K \ln k - \underline{\sigma})}\|\varphi\|_1 \quad t \geq 0$$

for all $\varphi \in L^1(\Omega)$, where $k = \max\{1;\ \|K\|\}$.

Proof. As $L_K = T_K + S$ is a bounded linear perturbation of the generator T_K, it follows by virtue of Lemma 2.1 that L_K is a generator of a strongly continuous semigroup denoted $V_K = (V_K(t))_{t\geq 0}$ satisfying

$$V_K(t)\varphi = \lim_{n\to\infty}[e^{-\sigma\frac{t}{n}}U_K(\frac{t}{n})]^n\varphi \quad t \geq 0$$

for all $\varphi \in L^1(\Omega)$. Using the norm (3.32) together with (3.42) and the hypothesis (H_σ) we infer

$$|||V_K(t)\varphi|||_1 \leq \lim_{n\to\infty}\Big[e^{-\underline{\sigma}\frac{t}{n}}e^{\frac{t}{n}\omega_K(\ln k)}\Big]^n|||\varphi|||_1 = e^{t(\omega_K \ln k-\underline{\sigma})}|||\varphi|||_1.$$

Now (3.33) completes the proof. □

Let us end this section with some interesting Corollaries.

Corollary 5.2. *Let K be a bounded linear operator from Y_1 into itself such that $\|K\| < 1$. If the hypothesis (H_σ) holds, then the operator L_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $V_K = (V_K(t))_{t\geq 0}$ satisfying*

$$\|V_K(t)\varphi\|_1 \leq e^{-t\underline{\sigma}}\|\varphi\|_1 \quad t \geq 0$$

for all $\varphi \in L^1(\Omega)$.

The proof of the above corollary is similar to that of Corollary 3.8, and is omitted.

Remark 5.3. Corollary 5.2 means that the general model (1.1), (1.5) corresponding to the case $\|K\| < 1$ is biologically uninteresting because the bacteria number is decreasing. Indeed, if t and s $(t > s)$ are two arbitrary times, then we have

$$\|V_K(t)\varphi\|_1 = \|V_K(t-s)V_K(s)\varphi\|_1 \leq e^{-(t-s)\underline{\sigma}}\|V_K(s)\varphi\|_1 < \|V_K(s)\varphi\|_1$$

for all initial data $\varphi \in L^1(\Omega)$.

In contrary to Remark 5.3, we understand that $\|K\| > 1$ is closely related to an increasing bacteria number during each mitotic. This is the most observed and biologically interesting case for which we have

Corollary 5.4. *Let K be a linear compact operator from Y_1 into itself such that $\|K\| > 1$. If the hypothesis (H_σ) holds, then the operator L_K generates, on $L^1(\Omega)$, a strongly continuous semigroup $V_K = (V_K(t))_{t\geq 0}$ satisfying*

$$\|V_K(t)\varphi\|_1 \leq \|K\|\|K\|^{t(\omega_K-\underline{\sigma})}\|\varphi\|_1 \quad t \geq 0$$

for all $\varphi \in L^1(\Omega)$, where $\omega_K > 0$ is abscissa of the operator K.

The proof of the above is similar to that of Corollary 3.10, and it is omitted. PAGE 18

6. Lattice property of the generated semigroup

In this section we are concerned with the lattice properties of the generated semigroup $V_K = (V_K(t))_{t\geq 0}$. These properties can be inferred from those of the linear operator

$$\overline{K}_\lambda\psi := \theta_\lambda K\psi, \quad \text{where } \theta_\lambda(v) = e^{-\lambda/v}, \quad v \in J = (0,\infty) \tag{6.1}$$

which is going to play an important role in the sequel. So, before we start, let us note that $\overline{K}_\lambda$ $(\lambda \geq 0)$ is clearly a bounded operator from Y_1 into itself because, for all $\psi \in Y_1$ we have

$$\|\overline{K}_\lambda\psi\|_{Y_1} \leq \|K\psi\|_{Y_1} \leq \|K\|\|\psi\|_{Y_1}.$$

Despite the obvious difference between (6.1) and (3.12), both operators are related by the following result.

Lemma 6.1. *Let K be an admissible operator whose abscissa is ω_K. If K is positive, then*

(1) $\overline{K}_\lambda$ $(\lambda \geq 0)$ *is positive too. Moreover, if* $\lambda > \omega_K(\ln k)$ *then*

$$(I - K_\lambda)^{-1}K \geq \overline{K}_\lambda^n \quad \textit{for all integers } n \geq 1, \tag{6.2}$$

where $k = \max\{1, \|K\|\}$.

(2) *Furthermore, if* K *is irreducible, then* $\overline{K}_\lambda$ *is also irreducible.*

Proof. (1). Let $\lambda \geq 0$. Firstly, it is easy to see that the positivity of $\overline{K}_\lambda$ follows from that K. Furthermore, we obviously have

$$K \geq \overline{K}_\lambda. \tag{6.3}$$

Next, as we clearly have $K_\lambda K = K\overline{K}_\lambda$, it follows by induction that

$$K_\lambda^n K = K\overline{K}_\lambda^n \quad \text{for all integers} \quad n \geq 1. \tag{6.4}$$

Therefore, if $\lambda > \omega_K(\ln k)$, then (3.14), (6.4) and (6.3) lead to

$$(I - K_\lambda)^{-1}K = \sum_{m\geq 0} K_\lambda^m K = \sum_{m\geq 0} K\overline{K}_\lambda^m \geq K\overline{K}_\lambda^{n-1} \geq \overline{K}_\lambda^n$$

for all integers $n \geq 1$.

(2). Firstly, let M be a closed ideal in Y_1 such that

$$K_\lambda(M) \subset M. \tag{6.5}$$

By virtue of the characterization of closed ideals in L_1-spaces (see [9, pp.309]), there exists $\Delta \subset J$ such that

$$M = \{\psi \in Y_1;\ \psi(v) = 0 \text{ a.e. } v \in \Delta\}.$$

So, for all $\varphi \in K(M)$, there exists $\psi \in M$ such that $\varphi = K\psi$. This implies that

$$\theta_\lambda\varphi = \theta_\lambda K\psi = \overline{K}_\lambda\psi \in M$$

which leads to $\varphi \in M$ and therefore $K(M) \subset M$. Now, by virtue of the irreducibility of K, we obviously infer that

$$M = \emptyset \quad \text{or} \quad M = Y_1 \tag{6.6}$$

and therefore, $\overline{K}_\lambda$ is irreducible because (6.5) holds only for (6.6). $\square$

Now, the lattice properties of the semigroup $U_K = (U_K(t))_{t\geq 0}$ are given as follows.

Proposition 6.2. *Let* K *be an admissible operator whose abscissa is* ω_K*. If* K *is positive, then*

(1) *The semigroup* $U_K = (U_K(t))_{t\geq 0}$ *is also positive.*

(2) *Furthermore, if* K *is irreducible then* $U_K = (U_K(t))_{t\geq 0}$ *is also irreducible.*

Proof. (1) Let $\lambda > \omega_K(\ln k)$ and let $\varphi \in (L^1(\Omega))_+$. Thanks to the second and the third point of Lemma 3.2, we infer that $(\lambda - T_0)^{-1}\varphi \geq 0$ and $\gamma_1(\lambda - T_0)^{-1}\varphi \geq 0$. This together with (3.27) and (6.2) imply

$$(\lambda - T_K)^{-1}\varphi \geq \varepsilon_\lambda \overline{K}_\lambda^n \gamma_1(\lambda - T_0)^{-1}\varphi \quad \text{for all integers } n \geq 1 \tag{6.7}$$

and therefore $(\lambda - T_K)^{-1}\varphi \geq 0$ because of the positivity of the operator $\overline{K}_\lambda$ (see Lemma 6.1). Now, the positivity of $U_K = (U_K(t))_{t\geq 0}$ follows from the first point of Lemma 2.2.

(2) Let $\lambda > \omega_K(\ln k)$ and let $\varphi \in (L^1(\Omega))_+$ be such that $\varphi \neq 0$. Thanks to the third point of Lemma 3.2, we infer that $\gamma_1(\lambda - T_0)^{-1}\varphi$ is a strictly positive

function. As $\overline{K}_\lambda$ is an irreducible operator (see Lemma 6.1) then, there exists an integer $m \geq 1$ such that

$$\overline{K}_\lambda^m \gamma_1 (\lambda - T_0)^{-1} \varphi(v) > 0 \quad \text{a.e. } v \in (0, \infty). \tag{6.8}$$

Putting now $n = m$ into (6.7) we infer that

$$(\lambda - T_K)^{-1} \varphi(\mu, v) \geq \varepsilon_\lambda(\mu, v) \overline{K}_\lambda^m \gamma_1 (\lambda - T_0)^{-1} \varphi(v) \quad \text{a.e. } (\mu, v) \in \Omega$$

which leads, by virtue of (6.8), to

$$(\lambda - T_K)^{-1} \varphi(\mu, v) > 0 \quad \text{a.e. } (\mu, v) \in \Omega$$

and therefore the irreducibility of $(\lambda - T_K)^{-1}$ follows. Finally, the second point of Lemma 2.2 leads to the irreducibility of $U_K = (U_K(t))_{t \geq 0}$. □

Now, the main resutl of this section is as follows.

Theorem 6.3. *Let K be an admissible operator whose abscissa is ω_K and suppose that* (H_σ) *holds. If K is positive, then*

(1) *The semigroup $V_K = (V_K(t))_{t \geq 0}$ is positive satisfying*

$$e^{-t\overline{\sigma}} U_K(t) \leq V_K(t), \quad t \geq 0 \tag{6.9}$$

where $\overline{\sigma}$ is given by (5.1).

(2) *Furthermore, if K is irreducible, then $V_K = (V_K(t))_{t \geq 0}$ is also irreducible.*

Proof. (1) Let $t > 0$ and let $\varphi \in (L^1(\Omega))_+$. By the first point of Proposition 6.2 we obtain the positivity of the semigroup $U_K = (U_K(t))_{t \geq 0}$ which leads to

$$\Big[e^{-\frac{t}{n}\sigma} U_K(t)\Big]^n \varphi \in (L^1(\Omega))_+ \quad \text{for all integers } n \in \mathbb{N},$$

and therefore

$$e^{-t\overline{\sigma}} U_K(t) \varphi \leq \Big[e^{-\frac{t}{n}\sigma} U_K(\tfrac{t}{n})\Big]^n \varphi \quad \text{for all integers } n \in \mathbb{N}.$$

Passing at the limit $n \to \infty$ and using (2.1), we infer that

$$e^{-t\overline{\sigma}} U_K(t) \varphi \leq V_K(t) \varphi$$

and therefore the positivity $V_K = (V_K(t))_{t \geq 0}$ and (6.9) follow.

(2) The irreducibility of the semigroup $V_K = (V_K(t))_{t \geq 0}$ obviously follows from that of the semigroup $U_K = (U_K(t))_{t \geq 0}$ (Proposition 6.2) together with (6.9). □

7. Spectral properties of the generated semigroup

The purpose of this section is to compute the type $\omega_0(V_K)$ of the semigroup $V_K = (V_K(t))_{t \geq 0}$. This will be obtained through spectral properties of the operator $\overline{K}_\lambda$ given by (6.1).

In the sequel, we suppose that the operator K is compact from Y_1 into itself. Thanks to Remark 3.4, K is then an admissible operator whose abscissa is denoted by ω_K. Therefore, all results of this work hold. So, let us start by the following preparative result.

Lemma 7.1. *Let K be a compact operator from Y_1 into itself and let $\lambda \in \mathbb{C}$ be such that* $\operatorname{Re} \lambda \geq 0$. *Then we have*

$$\lambda \in \sigma(T_K) \Longrightarrow 1 \in \sigma_p\left(\overline{K}_\lambda\right).$$

Proof. Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda \geq 0$ and let $g \in L^1(\Omega)$. If $1 \in \rho(\overline{K}_\lambda)$, then the equation
$$\psi = \overline{K}_\lambda \psi + \gamma_1(\lambda - T_0)^{-1} g \tag{7.1}$$
admits a unique solution $\psi \in Y_1$. So, let φ be the function
$$\varphi = \varepsilon_\lambda K\psi + (\lambda - T_0)^{-1} g. \tag{7.2}$$
On one hand, simple computations together with (3.5) and (3.6) infer that
$$\|\varphi\|_1 \leq \frac{1}{\lambda}\|K\|\|\psi\|_{Y_1} + \frac{1}{\lambda}\|g\|_1 < \infty \tag{7.3}$$
and
$$\begin{aligned} \|v\varphi\|_1 &= \|K\|\|\psi\|_{Y_1} + \|v(\lambda - T_0)^{-1} g\|_1 \\ &\leq \|\psi\|_{Y_1} + \|g\|_1 < \infty. \end{aligned} \tag{7.4}$$
Moreover, we have
$$\begin{aligned} \lambda\varphi + v\frac{\partial\varphi}{\partial\mu} &= \lambda\varphi + v\frac{\partial}{\partial\mu}(\varepsilon_\lambda K\psi + (\lambda - T_0)^{-1} g) \\ &= \lambda\varphi - \lambda\varepsilon_\lambda K\psi - \lambda(\lambda - T_0)^{-1} g + g = g \end{aligned}$$
and therefore
$$\|v\frac{\partial\varphi}{\partial\mu}\|_1 = \| - \lambda\varphi + g\|_1 \leq \lambda\|\varphi\|_1 + \|g\|_1 < \infty. \tag{7.5}$$
So, we have $\varphi \in W_1$ because of (7.3), (7.4) and (7.5).

On the other hand, (7.1) and (7.2) lead to
$$\begin{aligned} \gamma_0\varphi = K\psi &= K\left[\overline{K}_\lambda\psi + \gamma_1(\lambda - T_0)^{-1} g\right] \\ &= K\gamma_1\left[\varepsilon_\lambda K\psi + (\lambda - T_0)^{-1} g\right] \\ &= K\gamma_1\varphi \end{aligned}$$
which implies that $\varphi \in D(T_K)$ and therefore $(\lambda - T_K)$ is an operator with bounded inverse. Hence, we have $\lambda \in \rho(T_K)$. □

Now we are ready to state the main result of this section.

Theorem 7.2. *Let K be a positive, irreducible and compact operator from Y_1 into itself such that*
$$r(\overline{K}_{\overline{\sigma}-\underline{\sigma}}) > 1. \tag{7.6}$$
If (H_σ) *holds, then the type $\omega_0(V_K)$ of the semigroup $V_K = (V_K(t))_{t\geq 0}$ satisfies to*
$$\omega_0(V_K) > -\underline{\sigma}, \tag{7.7}$$
where $\underline{\sigma}$ and $\overline{\sigma}$ are given by (5.1).

Proof. We divide this proof into several steps.

Step I. Let $\lambda \geq 0$. Due to Lemma 6.1, we infer the positivity and the irreducibility of the operator $\overline{K}_\lambda$. Furthermore, its compactness follows from that of the operator K. So, thanks to [10] we infer that $r(\overline{K}_\lambda) > 0$ and there exists a quasi-interior vector ψ_λ of $(Y_1)_+$ and a strictly positive functional $\psi^*_\lambda \in (Y^*_1)_+$ such that
$$\overline{K}_\lambda\psi_\lambda = r(\overline{K}_\lambda)\psi_\lambda \quad \text{and} \quad \overline{K}^*_\lambda\psi^*_\lambda = r(\overline{K}_\lambda)\psi^*_\lambda \tag{7.8}$$
with $\|\psi_\lambda\|_{Y_1} = \|\psi^*_\lambda\|_{Y^*_1} = 1$, where $\overline{K}^*_\lambda$ is the adjoint operator of $\overline{K}_\lambda$. Now, we claim that
$$\lambda \geq 0 \to r(\overline{K}_\lambda) \tag{7.9}$$

is a continuous and strictly decreasing mapping. So, let $\lambda > \eta \geq 0$. First, for all $\psi \in (Y_1)_+$ we have

$$\overline{K}_\lambda \psi = \theta_\lambda K\psi = \theta_{\lambda-\eta}\theta_\eta K\psi < \theta_\eta K\psi = \overline{K}_\eta \psi$$

and therefore

$$\overline{K}_\eta \psi - \overline{K}_\lambda \psi > 0. \tag{7.10}$$

Using (7.8) for λ and for η we obtain that

$$\begin{aligned}
r(\overline{K}_\eta) - r(\overline{K}_\lambda) &= \frac{\langle \overline{K}_\eta^* \psi_\eta^*, \psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} - r(\overline{K}_\lambda)\\
&= \frac{\langle \psi_\eta^*, \overline{K}_\eta \psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} - r(\overline{K}_\lambda)\\
&= \frac{\langle \psi_\eta^*, \overline{K}_\lambda \psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} + \frac{\langle \psi_\eta^*, (\overline{K}_\eta - \overline{K}_\lambda)\psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} - r(\overline{K}_\lambda)\\
&= r(\overline{K}_\lambda) + \frac{\langle \psi_\eta^*, (\overline{K}_\eta - \overline{K}_\lambda)\psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} - r(\overline{K}_\lambda)
\end{aligned}$$

which leads, by (7.10), to

$$r(\overline{K}_\eta) - r(\overline{K}_\lambda) = \frac{\langle \psi_\eta^*, (\overline{K}_\eta - \overline{K}_\lambda)\psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} > 0 \tag{7.11}$$

because ψ_η^* is a strictly positive functional on $(Y_1)_+$ and therefore (7.9) is a strictly decreasing mapping. In particular, we infer that $r(\overline{K}_0) \geq r(\overline{K}_{\overline{\sigma}-\underline{\sigma}})$ which leads to

$$r(\overline{K}_0) > 1 \tag{7.12}$$

because (7.6). On the other hand, (7.11) implies that

$$\begin{aligned}
|r(\overline{K}_\eta) - r(\overline{K}_\lambda)| &\leq \frac{\|\psi_\eta^*\|_{Y_1^*}}{\langle \psi_\eta^*, \psi_\lambda\rangle} \|(\overline{K}_\eta - \overline{K}_\lambda)\psi_\lambda\|_{Y_1}\\
&\leq \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \sup_{\psi\in B} \|(\overline{K}_\eta - \overline{K}_\lambda)\psi\|_{Y_1}\\
&= \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \sup_{\psi\in B} \|(\theta_\eta - \theta_\lambda)K\psi\|_{Y_1}\\
&\leq \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \sup_{\varphi\in K(B)} \|(\theta_\eta - \theta_\lambda)\varphi\|_{Y_1}\\
&\leq \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \sup_{\varphi\in \overline{K(B)}} \|(\theta_\eta - \theta_\lambda)\varphi\|_{Y_1},
\end{aligned}$$

where B is the unit ball into Y_1. Since $\overline{K(B)}$ is a compact set, then there exists $\varphi_0 \in \overline{K(B)}$ such that

$$\big|r(\overline{K}_\eta) - r(\overline{K}_\lambda)\big| \leq \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \|(\theta_\eta - \theta_\lambda)\varphi_0\|_{Y_1}$$

which leads to

$$\lim_{\mu\to\lambda} \big|r(\overline{K}_\mu) - r(\overline{K}_\lambda)\big| \leq \lim_{\mu\to\lambda} \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \lim_{\mu\to\lambda} \|(\theta_\eta - \theta_\lambda)\varphi_0\|_{Y_1} = 0$$

and therefore (7.9) is a continuous mapping. Note that a similar computation leads to

$$r(\overline{K}_\lambda) \le \|\overline{K}_\lambda\|_{\mathcal{L}(Y_1)} \le \sup_{\varphi \in \overline{K(B)}} \|\theta_\lambda \varphi\|_{Y_1} = \|\theta_\lambda \varphi_0\|_{Y_1}$$

and therefore

$$\lim_{\lambda \to \infty} r(\overline{K}_\lambda) = 0. \tag{7.13}$$

Finally, as (7.9) is a continuous and strictly decreasing mapping, then by (7.12) and (7.13), there exists a unique λ_0 such that

$$\lambda_0 > 0 \quad \text{and} \quad r(\overline{K}_{\lambda_0}) = 1. \tag{7.14}$$

Step II. In this step we prove that $\lambda_0 = \omega_0(U_K)$, where $\omega_0(U_K)$ is the type of the semigroup $U_K = (U_K(t))_{t\ge0}$. So, let $\lambda \in \sigma(T_K)$ such that $\mathrm{Re}(\lambda) \ge 0$. By virtue of Lemma 7.1, there exists $\psi \neq 0$ such that $\overline{K}_\lambda \psi = \psi$. This clearly leads to

$$|\psi| = |\overline{K}_\lambda \psi| \le |\theta_\lambda| K|\psi| = \theta_{\mathrm{Re}\,\lambda} K|\psi| = \overline{K}_{\mathrm{Re}\,\lambda}|\psi|$$

which implies that $(\overline{K}_{\mathrm{Re}\,\lambda})^n|\psi| \ge |\psi|$ for all integers n and therefore $r(\overline{K}_{\mathrm{Re}\,\lambda}) \ge 1$. This together with (7.14) lead to $\mathrm{Re}\,\lambda \le \lambda_0$ because (7.9) is a strictly decreasing mapping and therefore (2.3) leads to

$$s(T_K) \le \lambda_0. \tag{7.15}$$

Conversely. Applying (7.8) to λ_0, it follows that $\overline{K}_{\lambda_0}\psi_{\lambda_0} = \psi_{\lambda_0}$ with $\psi_{\lambda_0} \neq 0$. Following the proof of Lemma 7.1 (put $g = 0$ in (7.1) and (7.2)) we easily infer that $\varphi := \varepsilon_{\lambda_0} K \psi_{\lambda_0}$ satisfies to

$$\varphi \in W_1 \quad \text{and} \quad -v\frac{\partial \varphi}{\partial \mu} = \lambda_0 \varphi \quad \text{and} \quad \gamma_0\varphi = K\gamma_1\varphi$$

which implies that $T_K\varphi = \lambda_0\varphi$ and therefore $\lambda_0 \in \sigma_p(T_K) \subset \sigma(T_K)$. Now, (2.3) leads to

$$\lambda_0 \le s(T_K). \tag{7.16}$$

Finally, (7.14) together with (7.15) and (7.16) and (2.4) imply that

$$\omega_0(U_K) = \lambda_0 > 0 \quad \text{and} \quad r(\overline{K}_{\lambda_0}) = 1. \tag{7.17}$$

Step III. On one hand, (7.6) and (7.17) lead to

$$\overline{\sigma} - \underline{\sigma} < \omega_0(U_K) \tag{7.18}$$

because (7.9) is a strictly decreasing mapping. On the other hand, Proposition 6.2 and Theorem 6.3 imply the positivity of the semigroups $U_K = (U_K(t))_{t\ge0}$ and $V_K = (V_K(t))_{t\ge0}$ which leads, by virtue of (6.9), to

$$e^{-t\overline{\sigma}}\|U_K(t)\|_{\mathcal{L}(L^1(\Omega))} \le \|V_K(t)\|_{\mathcal{L}(L^1(\Omega))}$$

for all $t \ge 0$ and therefore

$$-\overline{\sigma} + \lim_{t\to\infty} \frac{\ln\|U_K(t)\|_{\mathcal{L}(L^1(\Omega))}}{t} \le \lim_{t\to\infty} \frac{\ln\|V_K(t)\|_{\mathcal{L}(L^1(\Omega))}}{t}.$$

Hence, we have

$$-\overline{\sigma} + \omega_0(U_K) \le \omega_0(V_K) \tag{7.19}$$

because of (2.2). Now, (7.18) and (7.19) achieve the proof. □

Remark 7.3. Note that the choice of the functional framework $L^1(\Omega)$ was natural because $\|V_K(t)\varphi\|_1$ denotes the bacteria number at time t; nevertheless, according to a lot of modification, all the results of this work still hold into $L^p(\Omega)$ $(p > 1)$.

8. Application and comments

Taking now the particular model (1.1)-(1.2) that is

$$\begin{gathered}\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \mu} - \sigma f \quad t \geq 0\\ vf(t,0,v) = p\int_0^\infty k(v,v')f(t,1,v')v'dv' \quad t \geq 0\\ f(0,\cdot,\cdot) = \varphi \in L^1(\Omega),\end{gathered} \tag{8.1}$$

where $p \geq 1$ denotes the average number of daughter bacteria viable per mitotic. T ensure the continuity of the bacterial flux for $p = 1$, the kernel of correlation k must be positive and fulfils the normalization condition

$$\int_0^\infty k(v,v')dv = 1 \quad \text{for all } v \in (0,\infty).$$

If K denotes the transition operator

$$K\psi(v) = \frac{p}{v}\int_0^\infty k(v,v')\psi(v')v'dv'$$

then for all $\psi \in (Y_1)_+$ we have

$$\|K\psi\|_{Y_1} = \int_0^\infty vK\psi(v)dv = p\int_0^\infty (\int_0^\infty k(v,v')dv)\psi(v')v'dv' = p\|\psi\|_{Y_1}$$

which leads to $\|K\|_{\mathcal{L}(Y_1)} = p$ and therefore K is a bounded linear operator from Y_1 into itself. Furthermore, if k is a continuous kernel, then K becomes compact which leads to its admissibility because of Remark 3.4. Now, thanks to Theorems 5.1 and 6.3, we can say that the model (R) is well posed and admits, for all initial data $\varphi \in (L^1(\Omega))_+$, the following positive solution

$$f(t,\cdot,\cdot) = V_K(t)\varphi \quad t \geq 0.$$

Remark 8.1. In [11, p. 475], there is an incorrect study of the model (8.1). Indeed, the authors claim that there exists a unique solution f belonging to $L^1((0,1) \times (0,\infty))$ when $1 < p < 2$ and they have proceeded as follows:

In order to use the easy case $0 < p < 1$, the authors consider the change

$$\widetilde{f} = q^\mu f$$

and say that the model (8.1) becomes

$$\begin{gathered}\frac{\partial \widetilde{f}}{\partial t} = -v\frac{\partial \widetilde{f}}{\partial \mu} - \sigma f + (\ln q)\widetilde{f}\\ \widetilde{f}(t,0,v) = \frac{p}{qv}\int_0^\infty k(v,v')\widetilde{f}(t,1,v')v'dv'\\ \widetilde{f}(0,\cdot,\cdot) = q^\mu\varphi \in L^1(\Omega),\end{gathered} \tag{8.2}$$

where $q > p$ is fixed. So, after some computations, the authors infer that the well posedness of the model (8.2) follows from the boundedness of the multiplicative operator

$$\widetilde{f} \to (\ln q)\widetilde{f}$$

from $L^1(\Omega)$ into itself. Actually, the previous model (8.2) is incorrectly computed and the correct model is

$$\begin{gathered}
\frac{\partial \widetilde{f}}{\partial t} = -v\frac{\partial \widetilde{f}}{\partial \mu} - \sigma f + (v \ln q)\widetilde{f} \\
\widetilde{f}(t,0,v) = \frac{p}{qv}\int_0^\infty k(v,v')\widetilde{f}(t,1,v')v'dv' \\
\widetilde{f}(0,\cdot,\cdot) = q^\mu \varphi \in L^1((0,1)\times(0,\infty)).
\end{gathered} \tag{8.3}$$

Unfortunately, the operator

$$\widetilde{f} \to (v \ln q)\widetilde{f}$$

is obviously not bounded or dissipative into $L^1((0,1)\times(0,\infty))$ and therefore we cannot, by this way, infer any well posedness of the model (8.1).

References

[1] M. Boulanouar; *New Results in Abstract Time Dependent Transport Equations.* Trans. Theor. Stat. Phys., 40, 85-125, 2011.

[2] M. Boulanouar; *New results for neutronic equations (II).* C.R.A.S. Tome 348, Série I, 348, 549–552, 2010.

[3] M. Boulanouar; *Transport equation in cell population dynamics (I).* Elec. J. Diff. Equa., N.144, 1–20, 2010.

[4] M. Boulanouar; *New trace theorem for neutronic function spaces.* Trans. Theor. Stat. Phys., Vol., 38, 228-242, 2009.

[5] M. Boulanouar; *A transport equation in cell population dynamics.* Diff. Inte. Equa. Vol. 13., 125–144, 2000.

[6] M. Boulanouar; *Asymptotic behavior of transport equation in cell population dynamics with a null maturation velocity.* J. Math. Anal. Appl. N. 243., 47–63, 2000.

[7] M. Boulanouar; *Un modèle de Rotenberg avec la loi à mémoire parfaite.* C.R.A.S. Tome 327. Série I, 955–958, 1998.

[8] Ph. Clement, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, B. de Pagter; *One-parameter semigroups.* CWI Monographs, 5. North-Holland Publishing Co., Amsterdam, 1987.

[9] K. Engel, R. Nagel; *One-Parameter Semigroups for Linear Evolution Equations.* Graduate texts in mathematics, 194. Springer-Verlag, New York, Berlin, Heidelberg, 1999.

[10] B. Pagter; *Irreducible Compact Operators.* Math. Z. 192, 149–153. 1986.

[11] W. Greenberg, C. V. M. van der Mee; V. Protopopescu; *Bondary Value Problem in Abstract Kinetic Theory.* Birkhäuser. Basel. 1987.

[12] M. Rotenberg; *Transport theory for growing cell populations*, J. theor. Biol. 103., 181–199, 1983.

[13] L. Weis; *The Stability of Positive Semigroups on L_1-spaces.* J. Proc. Amer. Soci., 123, pp.3089–3094, 1995.

Addendum posted on June 24, 2013.

The author would like to make the following changes:
(1) Lemma 3.3 and Remark 3.4: (Kc) must be replaced by: "$\|K\mathbb{I}_\omega\| < 1$ *for some* $\omega > 0$ *and* $\|K\| \geq 1$. $\mathbb{I}_\omega$ *denotes the characteristic operator of the set* (ω,∞)."

(2) Relation (3.13) must be replaced by

$$\omega_K = \begin{cases} 0 & \text{if (Kb) holds;} \\ \delta_K := \inf\{\omega > 0 : \|K\mathbb{I}_\omega\| < 1\} & \text{if (Kc) holds.} \end{cases} \tag{3.13}$$

(3) Proof of Lemma 3.3: from "*So, the compactness...*"(Page 6 line 5) to "*because of (3.20)*" (Page 7 line 1) and from "*Finally, by (3.19)...*" (Page 7 line 24) to the end the proof, must be deleted.

(4) Corollaries 3.10 and 5.4: "*Let K be a linear compact*" must be replaced by "*Let K be a linear admissible*". The proof must be replaced by : "*Proof: Obvious.*"

(5) Page 20: from "*In the sequel...to...result* " (Lines -7 to -4) must be deleted.

(6) Theorem 7.2: "*K admissible*" must be inserted in the preamble.

(7) Page 24: "*T ensure...$v \in (0, \infty)$*" (Lines 10 to 12) must be deleted. "*Furthermore...Ramark 3.4*" (Lines 18 and 19) must be replaced by "*If $p \sup_{v' \geq \omega} \int_0^\infty |k(v, v')| dv < 1$ for some $\omega > 0$, then K is an admissible operator.*"

End of addendum.

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