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INFINITELY MANY LARGE ENERGY SOLUTIONS OF SUPERLINEAR SCHRÖDINGER-MAXWELL EQUATIONS

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Abstract. In this article we study the existence of infinitely many large energy solutions for the superlinear Schrödinger-Maxwell equations

$$\begin{gathered} -\Delta u+V(x)u+\phi u=f(x,u) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi=u^2, \quad \text{in } \mathbb{R}^3, \end{gathered}$$

via the Fountain Theorem in critical point theory. In particular, we do not use the classical Ambrosetti-Rabinowitz condition.

1. Introduction and main results

In this article, we study the system of Schrödinger-Maxwell equations

$$\begin{gathered} -\Delta u+V(x)u+\phi u=f(x,u) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi=u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.1}$$

Such a system, also called Schrödinger-Poisson equations, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charge particle with an electro-magnetic field can be described by coupling the nonlinear Schrödinger's and Maxwell's equations (we refer the reader to [8] and the references therein for more details on the physical aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve (1.1).

In recent years, system (1.1) with $V(x)\equiv 1$ or being radially symmetric, has been widely studied under various conditions on f, see for example [4, 13, 12, 15, 17, 21, 22, 28]. Specially, in [13, 12] it is proved the existence of a sequence of radial solutions for system (1.1) by the Symmetric Mountain Pass Theorem in [5]. The case of nonradial potential $V(x)$ has been considered in [24], when f is asymptotically linear at infinity, and in [4, 27], when f is superlinear at infinity. Moreover, in [27], the authors considered system (1.1) with periodic potential $V(x)$, and the existence of infinitely many geometrically distinct solutions has been proved by the nonlinear superposition principle established in [1]. By the way, we would like to point out that nonexistence results for (1.1) can be found in [4, 14, 17, 21, 24].

The problem of finding infinitely many large energy solutions is a very classical problem: there is an extensive literature concerning the existence of infinitely many

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large energy solutions of a plethora of problems via the Symmetric Mountain Pass Theorem and Fountain Theorem (cf. Ambrosetti and Rabinowitz [3], Rabinowitz [20], Bartsch [6], Bartsch and Willem [7], Struwe [23], Willem [25], etc). The infinitely many large energy solutions for system (1.1) are obtained in [11] with the following variant "Ambrosetti-Rabinowitz" type condition (AR for short),

(AR) There exist $\mu > 4$ such that for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^3$,
$$\mu F(x,s) := \mu \int_0^s f(x,t)\,\mathrm{d}t \le s f(x,s).$$

After that, Li et al. [18] study (1.1) without the (AR) condition. They use variant Fountain Theorem establish by Zou [29]. Later, some authors also study this problem without the (AR) condition, see Alves et al. [2] and Yang and Han [26].

In this article, we use the Fountain Theorem (see Theorem 2.4) to find infinitely many large energy solutions to system (1.1). We can see that (1.1) can be proved directly with the Fountain Theorem under Cerami condition. We assume the following assumptions:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \ge a_1 > 0$, where $a_1 > 0$ is a constant. Moreover, for every $M > 0$, $\mathrm{meas}(\{x \in \mathbb{R}^3 : V(x) \le M\}) < \infty$, where meas denote the Lebesgue measure in $\mathbb{R}^3$.

(F1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and for some $2 < p < 2^* = 6$, $a_2 > 0$,
$$|f(x,z)| \le a_2(|z| + |z|^{p-1}),$$
for a.e. $x \in \mathbb{R}^3$ and all $z \in \mathbb{R}$.
$$\lim_{z\to 0} \frac{f(x,z)}{z} = 0$$
uniformly for $x \in \mathbb{R}^3$.

(F2) $\lim_{|z|\to\infty} \frac{F(x,z)}{|z|^4} = +\infty$, uniformly in $x \in \mathbf{R}^3$ and $F(x,0) \equiv 0$, $F(x,z) \ge 0$ for all $(x,z) \in \mathbb{R}^3 \times \mathbb{R}$.

(F3) There exits a constant $\theta \ge 1$ such that
$$\theta H(x,z) \ge H(x,sz)$$
for all $x \in \mathbb{R}^3$, $z \in \mathbb{R}$ and $s \in [0,1]$, where $H(x,z) = zf(x,z) - 4F(x,z)$.

(F4) $f(x,-z) = -f(x,z)$ for any $x \in \mathbb{R}^3$ and all $z \in \mathbb{R}$.

The main results of the present article are as follows.

Theorem 1.1. *Assume that* (V1), (F1)–(F4) *hold, then system* (1.1) *has infinitely many solutions* $\{(u_k, \phi_k)\}$ *in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *satisfying*
$$\frac{1}{2}\int_{\mathbb{R}^3}\Big(|\nabla u_k|^2 + V(x)u_k^2\Big)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_k|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\phi_k u_k^2\,\mathrm{d}x - \int_{\mathbb{R}^3}F(x,u_k)\,\mathrm{d}x \to +\infty.$$

Remark 1.2. Obviously, (F2) can be derived from (AR). Under (AR), any (PS) sequence of the corresponding energy functional is bounded, which plays an important role of the application of variational methods. Indeed, there are many superlinear functions which do not satisfy the (AR) condition. For instance the function
$$f(x,z) = z^3 \ln(1 + |z|) \tag{1.2}$$
does not satisfy the (AR) condition. But it is easy to see this function satisfies (F2) and (F3). There are many functions which satisfy (F3), but do not satisfy

condition (AR) for any $\mu > 4$. However, we can not deduce condition $(F3)$ from condition (AR). For example, let

$$f(x,u) = 5|u|^4 \int_0^u |t|^{1+\sin t} t \, dt + |u|^{6+\sin u} u,$$

then

$$F(x,z) = |z|^5 \int_0^z |t|^{1+\sin t} t \, dt\,.$$

It is easy to see that $f(x,u)$ satisfies condition (AR) for $\mu = 5$, but it does not satisfy $(F3)$. Thus, $(F3)$ is also superlinear conditions and complement with (AR).

Remark 1.3. In [26], Yang and Han used

(F3') $\frac{f(x,u)}{u^3}$ is increasing for $u > 0$ and decreasing for $u < 0$, for all $x \in \mathbb{R}^3$.

to obtain a bounded Cerami sequence. Li et al. [18], used

(F3") $H(x,s) \le H(x,t)$ for all $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, $s \le t$ and a.e. $x \in \mathbb{R}^3$

to solve the problem (1.1). (F3') implies that (F3"), as we can see in [19, Lemma 2.2]. We see that our condition (F3) is more general than (F3"). If $\theta = 1$ we can get that $H(x,z)$ is increasing in $\mathbb{R}^+$ with respect to z. Moreover, (F3) gives some general sense of monotony when $\theta > 1$ and we can find some examples that satisfy (F3) but do not satisfy (F3"). For example, let

$$f(x,z) = 4z^3 \ln(1+z^4) + 2\sin z,$$

it follows that

$$H(x,z) = 4z^4 - 4\ln(1+z^4) + 2z\sin z + 8\cos z.$$

Let $\theta = 100$, we can prove by some simple computation that f satisfies (F3) but does not satisfy (F3") any more.

2. Variational settings and preliminary results

Before stating our main results, we give several notations. Define the function space

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with the usual norm

$$\|u\|_{H^1} := \Big(\int_{\mathbb{R}^3} \Big(|\nabla u|^2 + u^2\Big)\, dx\Big)^{1/2},$$

and define the function space

$$D^{1,2}(\mathbb{R}^3) := \{u \in L^{2^*}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with the norm

$$\|u\|_{D^{1,2}} := \Big(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\Big)^{1/2}.$$

Let

$$E := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx < \infty\}.$$

Then E is a Hilbert space with the inner product

$$(u,v)_E = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

and the norm $\|u\|_E = (u,u)_E^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for any $s \in [2, 2^*]$.

Lemma 2.1 ([30, Lemma 3.4])**.** *Under assumption* (V1)*, the embedding*

$$E \hookrightarrow L^s(\mathbb{R}^3)$$

is compact for any $s \in [2, 2^*)$.

It is clear that system (1.1) is the Euler-Lagrange equations of the functional $J : E \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u,\phi) = \frac{1}{2}\|u\|_E^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x.$$

Evidently, the action functional J belongs to $C^1(E \times D^{1,2}(\mathbb{R}^3), \mathbb{R})$ and its critical points are the solutions of (1.1). It is easy to know that J exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [9], by which we are led to study a one variable functional that does not present such a strongly indefinite nature.

Now, we recall this method.

For any $u \in E$, the Lax-Milgram theorem (see [16]) implies there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta\phi_u = u^2$$

in a weak sense. We can write an integral expression for ϕ_u in the form:

$$\phi_u = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\,\mathrm{d}y, \tag{2.1}$$

for any $u \in E$ (for detail, see section 2 of [11]). The functions ϕ_u possess the following properties:

Lemma 2.2 ([11, Lemma 2.2])**.** *For any $u \in E$, we have:*

(1) $\|\phi_u\|_{D^{1,2}} \le a_3\|u\|_{L^{12/5}}^2$*, where $a_3 > 0$ does not depend on u. As a consequence there exists $a_4 > 0$ such that*

$$\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x \le a_4\|u\|_E^4;$$

(2) $\phi_u \ge 0$.

So, we can consider the functional $I : E \to \mathbb{R}$ defined by $I(u) = J(u, \phi_u)$. After multiplying $-\Delta\phi_u = u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x.$$

Therefore, the reduced functional takes the form

$$I(u) = \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x.$$

From Lemma 2.2, I is well defined. Furthermore, it is well known that I is C^1 functional with derivative given by

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} \Big(\nabla u \cdot \nabla v + V(x)uv + \phi_u uv - f(x,u)v\Big)\,\mathrm{d}x. \tag{2.2}$$

Now, we can apply Theorem 2.3 of [9] to our functional J and obtain:

Proposition 2.3. *The following statements are equivalent:*

(1) $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ *is a critical point of* J *(i.e.* (u, ϕ) *is a solution of* (1.1)*);*
(2) u *is a critical point of* I *and* $\phi = \phi_u$.

For reader's convenience, we introduce the Cerami condition, which was established by Cerami [10].

Definition 2.4. Assume functional Φ is C^1 and $c \in \mathbb{R}$, if any sequence $\{u_n\}$ satisfying $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\|\Phi'(u_n)\| \to 0$ has a convergence subsequence, we say Φ satisfies Cerami condition at the level c.

To complete the proof of our theorems, we need the following critical point theorem.

Theorem 2.5 (Fountain Theorem under Cerami conditon)**.** *Let* X *be a Banach space with the norm* $\|\cdot\|$ *and let* X_j *be a sequence of subspace of* X *with* $\dim X_j < \infty$ *for each* $j \in \mathbf{N}$*. Further,* $X = \overline{\oplus_{j\in\mathbf{N}} X_j}$*, the closure of the direct sum of all* X_j*. Set* $W_k = \oplus_{j=0}^{k} X_j$*,* $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$*. Consider an even functional* $\Phi \in C^1(X, \mathbb{R})$ *(i.e.* $\Phi(-u) = \Phi(u)$ *for all* $u \in E$*). If, for every* $k \in \mathbf{N}$*, there exist* $\rho_k > r_k > 0$ *such that*

(Φ1) $a_k := \max_{u\in W_k,\|u\|=\rho_k} \Phi(u) \le 0$,
(Φ2) $b_k := \inf_{u\in Z_k,\|u\|=r_k} \Phi(u) \to +\infty$, *as* $k \to \infty$,
(Φ3) *the Cerami condition holds at any level* $c > 0$.

Then Φ *has an unbounded sequence of critical values.*

Remark 2.6. Cerami condition is weaker than the (PS) condition. However, it was shown in [5] that from Cerami condition a deformation lemma follows and, as a consequence, we can also get minimax theorems.

3. Proof of Theorem 1.1

We choose an orthogonal basis $\{e_j\}$ of $X := E$ and define $W_k := \operatorname{span}\{e_1, \cdots, e_k\}$, $Z_k := W_{k-1}^{\perp}$. To complete the proof of our theorems, we need the following lemma.

Lemma 3.1 ([11, Lemma 2.5])**.** *For any* $2 \le p < 2^*$*, we have that*

$$\beta_k := \sup_{u\in Z_k,\|u\|_E=1} \|u\|_{L^p} \to 0, \quad k \to \infty.$$

Now, we show that the functional I satisfies the Cerami condition.

Lemma 3.2. *Under the assumptions* (F1)–(F3)*, the functional* $I(u)$ *satisfies the Cerami condition at any positive level.*

Proof. We suppose that $\{u_n\}$ is the Cerami sequence, that is for some $c \in \mathbb{R}^+$

$$I(u_n) = \frac{1}{2}\|u_n\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x, u_n)\,\mathrm{d}x \to c \quad (n \to \infty) \tag{3.1}$$

and

$$(1 + \|u_n\|_E) I'(u_n) \to 0 \quad (n \to \infty). \tag{3.2}$$

From (3.1) and (3.2), for n large enough, we have

$$\begin{aligned} 1+c &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} f(x,u_n)u_n \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u_n)\,\mathrm{d}x. \end{aligned} \tag{3.3}$$

We claim that $\{u_n\}$ is bounded. Otherwise there should exist a subsequence of $\{u_n\}$ satisfying $\|u_n\|_E \to \infty$ as $n \to \infty$. Denote $w_n = \frac{u_n}{\|u_n\|_E}$, then $\{w_n\}$ is bounded. Up to a subsequence, for some $w \in E$, we obtain

$$\begin{gathered} w_n \rightharpoonup w \quad \text{in } E, \\ w_n \to w \quad \text{in } L^t(\mathbb{R}^3),\ 2 \leq t < 2^*, \\ w_n(x) \to w(x) \quad \text{a.e. in } \mathbb{R}^3. \end{gathered} \tag{3.4}$$

Suppose, $w \neq 0$ in E. Dividing by $\|u_n\|_E^4$ in both sides of (3.1), by (1) of lemma 2.2 we obtain

$$\int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|_E^4}\,\mathrm{d}x = \frac{1}{2\|u_n\|_E^2} + \frac{\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,\mathrm{d}x - c}{4\|u_n\|_E^4} + o(\|u_n\|_E^{-4}) \leq a_5 < \infty, \tag{3.5}$$

where a_5 is a positive constant. We consider this situation, $\Omega := \{x \in \mathbb{R}^3 | w(x) \neq 0\}$, by $(F2)$, for all $x \in \Omega$,

$$\frac{F(x,u_n)}{\|u_n\|_E^4} = \frac{F(x,u_n)}{|u_n|^4} w_n^4(x) \to +\infty \quad (n \to \infty).$$

Since $|\Omega| > 0$, using Fatou's Lemma, we obtain

$$\int_{\mathbb{R}^3} \frac{F(x,u(x)_n)}{\|u(x)_n\|_E^4}\,\mathrm{d}x \to +\infty \quad (n \to \infty).$$

This contradicts (3.5).

On the another hand, if $w(x) = 0$, we can define a sequence $\{t_n\} \subset \mathbb{R}$:

$$I(t_n u_n) = \max_{t \in [0,1]} I(tu_n).$$

Fix any $m > 0$, let $\overline{w}_n = \sqrt{4m}\frac{u_n}{\|u_n\|_E} = \sqrt{4m}w_n$. By $(F1)$,

$$|f(x,z)| \leq a_2|z| + a_2|z|^{p-1},$$

for a.e. $x \in \mathbb{R}^3$ and all $z \in \mathbb{R}$. By the equality $F(x,z) = \int_0^1 f(x,tz)z\,\mathrm{d}t$ we obtain

$$F(x,z) \leq \frac{a_2}{2}|z|^2 + a_6|z|^p \tag{3.6}$$

for any $x \in \mathbb{R}^3$ and all $z \in \mathbb{R}$, where $a_6 = \frac{a_2}{p}$. Due to (3.5), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(x,\overline{w}_n)\,\mathrm{d}x \leq \lim_{n\to\infty} \left(\frac{a_2}{2}\int_{\mathbb{R}^3} |\overline{w}_n|^2\,\mathrm{d}x + a_6\int_{\mathbb{R}^3} |\overline{w}_n|^p\,\mathrm{d}x\right) = 0.$$

Then for n large enough,

$$\begin{aligned} I(t_n u_n) &\geq I(\overline{w}_n) \\ &= 2m + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{\overline{w}_n}\overline{w}_n^2\,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,\overline{w}_n)\,\mathrm{d}x \geq m. \end{aligned} \tag{3.7}$$

Due to (3.7), $\lim_{n\to\infty} I(t_n u_n) = +\infty$. Since $I(0) = 0$, and $I(u_n) \to c$, then $0 < t_n < 1$ if n large enough, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\nabla t_n u_n \nabla t_n u_n + V(x) t_n u_n t_n u_n + \phi_{t_n u_n} t_n u_n t_n u_n - f(x, t_n u_n) t_n u_n)\,\mathrm{d}x \\
&= \langle I'(t_n u_n), t_n u_n\rangle \\
&= t_n \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_n} I(t u_n) = 0.
\end{aligned}$$

Thus, by (F3) we obtain

$$\begin{aligned}
&I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{4}\|u_n\|_E^2 + \int_{\mathbb{R}^3} \Big[\frac{1}{4} f(x, u_n) u_n - F(x, u_n)\Big]\,\mathrm{d}x \\
&= \frac{1}{4}\|u_n\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} H(x, u_n)\,\mathrm{d}x \\
&\geq \frac{1}{4\theta}\|t_n u_n\|_E^2 + \frac{1}{4\theta}\int_{\mathbb{R}^3} H(x, t_n u_n)\,\mathrm{d}x \\
&= \frac{1}{4\theta}\|t_n u_n\|_E^2 + \frac{1}{\theta}\int_{\mathbb{R}^3} \Big[\frac{1}{4} f(x, t_n u_n) t_n u_n - F(x, t_n u_n)\Big]\,\mathrm{d}x \\
&= \frac{1}{\theta} I(t_n u_n) - \frac{1}{4\theta}\langle I'(t_n u_n), t_n u_n\rangle \to +\infty.
\end{aligned}$$

This contradicts (3.3). So $\{u_n\}$ is bounded. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in E. In view of Lemma 2.1, $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 2^*)$. By (2.2), we easily get

$$\begin{aligned}
\|u_n - u\|_E^2 = \langle I'(u_n) - I'(u), u_n - u\rangle &+ \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)\,\mathrm{d}x \\
&- \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)\,\mathrm{d}x.
\end{aligned}$$

It is clear that

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0.$$

According to assumptions $(F1)$, there exists $a_6 > 0$ such that

$$f(x, u) \leq \frac{a_2}{2}|u| + a_6|u|^{p-1}$$

for a.e. $x \in \mathbb{R}^3$, and all $z \in \mathbb{R}$. Using the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \Big[\frac{a_2}{2}(|u_n| + |u|) + a_6\left(|u_n|^{p-1} + |u|^{p-1}\right)\Big]|u_n - u|\,\mathrm{d}x \\
&\leq \frac{a_2}{2}\left(\|u_n\|_{L^2}^2 + \|u\|_{L^2}^2\right)\|u_n - u\|_{L^2}^2 + a_6\Big(\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1}\Big)\|u_n - u\|_{L^p}
\end{aligned}$$

Since $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 2^*)$, we have

$$\int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)\,\mathrm{d}x \to 0, \quad \text{as } n \to \infty.$$

By the Hölder inequality, Sobolev inequality and Lemma 2.2, we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u)\,\mathrm{d}x\Big| &\le \|\phi_{u_n}u_n\|_{L^2}\|u_n-u\|_{L^2}\\
&\le \|\phi_{u_n}\|_{L^6}\|u_n\|_{L^3}\|u_n-u\|_{L^2}\\
&\le a_8\|\phi_{u_n}\|_{D^{1,2}}\|u_n\|_{L^3}\|u_n-u\|_{L^2}\\
&\le a_4a_8\|u_n\|^2_{L^{12/5}}\|u_n\|_{L^3}\|u_n-u\|_{L^2},
\end{aligned}$$

where $a_8>0$ is a constant. Again using $u_n\to u$ in $L^s(\mathbb{R}^3)$ for any $s\in[2,2^*)$, we have

$$\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u)\,\mathrm{d}x\to 0,\quad \text{as } n\to\infty.$$

Similarly, we obtain

$$\int_{\mathbb{R}^3}\phi_u u(u_n-u)\,\mathrm{d}x\to 0,\quad \text{as } n\to\infty.$$

Thus,

$$\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,\mathrm{d}x\to 0,\quad \text{as } n\to\infty,$$

so that $\|u_n-u\|_E\to 0$. We get that $I(u)$ satisfies Cerami condition. $\square$

Proof of Theorem 1.1. Due to Lemma 3.2, $I(u)$ satisfies Cerami condition. Next, we verify that $I(u)$ satisfies the rest conditions of Theorem 2.5.

First, we verify that $I(u)$ satisfies ($\Phi1$). It follows from (F2) that for any $M>0$, there exists $\delta(M)>0$, such that for all $x\in\mathbb{R}^3$, $|z|\ge\delta$, we have

$$F(x,z)\ge\frac14 M|z|^4. \tag{3.8}$$

Taking $\widetilde{M}:=\sup_{|z|<\delta}\big(\frac14 M|z|^4-\frac{F(x,z)}{|z|^2}\big)$, then by (3.8) we obtain

$$F(x,z)\ge\frac14 M|z|^4-\widetilde{M}|z|^2$$

for a.e. $x\in\mathbb{R}^3$, and all $z\in\mathbb{R}$. Hence we have

$$I(u)\le\frac12\|u\|_E^2+\frac{a_4}{4}\|u\|_E^4-\frac14 M\|u\|^4_{L^4}+\widetilde{M}\|u\|^2_{L^2}.$$

Since, on the finitely dimensional space W_k all norms are equivalent, we have that

$$I(u)\le\frac12\|u\|_E^2+\frac{a_4}{4}\|u\|_E^4-\frac14 Ma_{10}\|u\|^4_E+\widetilde{M}a_{10}\|u\|^2_E,$$

where a_{10} is a constant. Now since $\frac{a_4}{4}-\frac14 Ma_{10}<0$, when M is large enough, it follows that

$$a_k:=\max_{u\in W_k,\|u\|_E=\rho_k}I(u)\le 0$$

for some $\rho_k>0$ large enough.

Secondly, we prove that $I(u)$ satisfies ($\Phi2$). Due to (3.6), we have

$$\begin{aligned}
I(u)&\ge\frac12\|u\|_E^2-\varepsilon\|u\|^2_{L^2}-a_6\|u\|^p_{L^p}\\
&\ge\Big(\frac12-\frac{\varepsilon}{a_1}\Big)\|u\|_E^2-a_6{\beta_k}^p\|u\|^p_E,
\end{aligned}$$

where a_1 is a lower bound of $V(x)$ from (V1) and β_k are defined in Lemma 3.1. Choosing $r_k := (a_6 p \beta_k^p)^{1/(2-p)}$, we obtain

$$\begin{aligned} b_k &= \inf_{u\in Z_k, \|u\|_E = r_k} I(u) \\ &\geq \inf_{u\in Z_k, \|u\|_E = r_k} \big[\big(\frac{1}{2} - \frac{\varepsilon}{a_1}\big)\|u\|_E^2 - a_6 {\beta_k}^p \|u\|_E^p\big] \\ &\geq \Big(\frac{1}{2} - \frac{\varepsilon}{a_1} - \frac{1}{p}\Big)(a_6 p \beta_k^p)^{\frac{2}{2-p}}. \end{aligned}$$

Because $\beta_k \to 0$ as $k \to 0$ and $p > 2$, we have

$$b_k \geq \Big(\frac{1}{2} - \frac{\varepsilon}{a_1} - \frac{1}{p}\Big)(a_6 p \beta_k^p)^{\frac{2}{2-p}} \to +\infty$$

for enough small ε. This proves (Φ2). Now, we apply Theorem 2.5 to complete the proof o Theorem 1.1. □

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