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DIRECT AND INVERSE PROBLEMS FOR SYSTEMS OF SINGULAR DIFFERENTIAL BOUNDARY-VALUE PROBLEMS

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Abstract. Real interpolation spaces are used for solving some direct and inverse linear evolution problems in Banach spaces, on the ground of space regularity assumptions.

1. Introduction

Several articles are devoted to studying identification problems of the type

$$\begin{gathered} y'(t) + Ay(t) = f(t)z + h(t), \quad 0 \le t \le \tau, \\ y(0) = y_0, \\ \Phi[y(t)] = g(t), \quad 0 \le t \le \tau, \end{gathered} \tag{1.1}$$

Here $-A$ is a linear closed operator generating a C_0-semigroup in a Banach space X or C^∞-semigroup in X. Moreover, z is a fixed element in X, $y_0 \in X$, $\Phi \in X^*$, $g \in C^1([0,\tau];\mathbb{C})$, $h \in C^1([0,\tau];X)$.

Roughly speaking, we look for solutions (y, f) in $[C^1([0,\tau];X) \cap C([0,\tau];\mathcal{D}(A)] \times C([0,\tau];\mathbb{C})$. More precisely, we recall that in [1, 2, 3, 13, 4, 10, 11, 12, 18], all concerned with the parabolic case, the *scalar* function f is sought for in the more regular space $C^\theta([0,\tau];\mathbb{C})$, for some $\theta \in (0,1)$, so that known results of maximal Hölder regularity in time can be applied. For this purpose, A is assumed (cfr. [16]) to satisfy the estimate

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \le c(1 + |\lambda|)^{-\beta} \tag{1.2}$$

for all λ in the sector

$$\Sigma_\alpha := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge -c(1 + |\operatorname{Im}\lambda|)^\alpha\}, \quad 0 < \beta \le \alpha \le 1. \tag{1.3}$$

Taira [20] deals with the case $\alpha = 1$ and introduces the power A^γ for $\gamma > 1 - \beta$. He proves that $D(A^\gamma) \supseteq D(A)$ if $\beta > 1/2$ and $1 - \beta < \gamma < \beta$.

The results on the Cauchy problem

$$y' + Ay = f(t), \quad 0 \le t \le \tau, \tag{1.4}$$

$$y(0) = y_0, \tag{1.5}$$

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$f \in C([0,\tau];X), y_0 \in D(A)$, corresponding to the case $\alpha = \beta = 1$ are by now classical after the works by Da Prato, Lunardi, Sinestrari, and their followers, concerning maximal in time and/or spatial regularity of the strict solutions.

Further, when $0 < \beta \le 1$, $\alpha = 1$, the existence (and also time regularity) of the solution was considered by Wild in [23].

In the case of a multi-valued operator A, for which (1.4) takes the form of an inclusion, in [16] the authors introduced the spaces

$$X_A^{\theta,\infty} := \{u \in X : \sup_{t>0} t^\theta \|A^\circ (t+A)^{-1}u\|_X < \infty\}, \quad 0 < \theta < 1,$$

where $A^\circ(t+A)^{-1}$ is the linear section of $A(t+A)^{-1}$ defined in Theorem 2.7 of the quoted monograph. If A is not multivalued and $\alpha = \beta = 1$, then $X_A^{\theta,\infty} = (X, D(A))_{\theta,\infty}$, the latter being a real interpolation space between X and $D(A)$. In general, the following inclusions hold true (cfr. [16, p .26])

$$X_A^{\theta,\infty} \subseteq (X, D(A))_{\theta,\infty}, \quad 0 < \theta < 1, \tag{1.6}$$

$$(X, D(A))_{\theta,\infty} \subseteq X_A^{\theta+\beta-1,\infty}, \quad 1 - \beta < \theta < 1. \tag{1.7}$$

Whence it follows that $D(A) \subseteq X_A^{\theta,\infty}$ provided $0 < \theta < \beta$. Therefore $X_A^{\theta,\infty}$ is not intermediate between $D(A)$ and X. However $D(A^2) \subseteq X_A^{\theta,\infty}$.

Restricting ourselves to the *univalent* case, if $u \in D(A^2)$, $t \ge 1$,

$$t^\theta A(A+t)^{-1}u = t^\theta (A+t)^{-1}A^{-1}A^2u = t^{\theta-1}(A^{-1} - (A+t)^{-1})A^2u, \quad 0 < \theta < 1.$$

Consequently, for $t \ge 1$, we obtain

$$t^\theta \|A(A+t)^{-1}u\|_X \le t^{\theta-1}\|Au\|_X + ct^{\theta-1-\beta}\|A^2u\|_X.$$

Since A is assumed to be invertible, the inclusion follows.

Let us introduce some spaces $\tilde{X}_A^{\theta,\infty}$ which are intermediate between X and $D(A)$ (and which reduce to $X_A^{\theta,\infty}$ if $\alpha = \beta = 1$). Such spaces seem more appropriate to solve (1.4), (1.5) and to deduce the spatial regularity of the related solution. For the sake of brevity, we drop out "∞" from $X_A^{\theta,\infty}$ and write X_A^θ and $\tilde{X}_A^\theta$, respectively.

Section 2 is devoted to the intermediate spaces, while in Section 3 the spatial and temporal regularity of solutions to (1.4), (1.5) is studied. Section 4 deals with the identification problem (1.2), (1.3), under suitable spatial regularity assumptions. Section 5 is devoted to the new identification problem

$$y'(t) + Ay(t) = f_1(t)z_1 + f_2(t)z_2 + h(t),$$
$$\Phi_j[y(t)] = g_j(t), \quad t \in [0,\tau], \quad j = 0, 1,$$

In Section 6 the results of Section 5 will be applied to solve an inverse problem for systems of evolution differential equations. Section 7 is devoted to general weakly coupled identification problems. Finally, in Sections 8 and 9 the previous abstract results will be applied to a few systems of PDE's, both regular and degenerate.

2. Interpolation spaces

Let A be a closed linear operator acting in the complex space X with

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \le C(1 + |\lambda|)^{-\beta}, \quad \lambda \in \Sigma_\alpha, \tag{2.1}$$

for some

$$0 < \beta \le \alpha \le 1, \quad \alpha + \beta > 1. \tag{2.2}$$

Denote now (cf. [16, p. 26])

$$X_A^\theta = \{u \in X : [x]_{X_A^\theta} = \sup_{t>0} t^\theta \|A(t+A)^{-1}u\|_X < +\infty\},\\ \|x\|_{X_A^\theta} = \|x\| + [x]_\theta. \tag{2.3}$$

It is known that [16, Theorem 1.12, p. 26],

$$X_A^\theta \subset (X, \mathcal{D}(A))_{\theta,\infty}, \quad \theta \in (0,1), \tag{2.4}$$

$$(X, \mathcal{D}(A))_{\theta,\infty} \subset X_A^{\theta+\beta-1}, \quad \theta \in (1-\beta, 1). \tag{2.5}$$

According to [16, Proposition 3.4], if $1-\beta < \theta < 1$ we obtain

$$t^{(2-\beta-\theta)/\alpha} \|Ae^{-tA}x\|_X \le C\|x\|_{X_A^\theta}. \tag{2.6}$$

Moreover, from [16, Theorem 3.5] with $\theta \in (2-\alpha-\beta, 1)$, we obtain

$$\|(e^{-tA} - I)x\|_X \le Ct^{(\alpha+\beta+\theta-2)/\alpha}\|x\|_{X_A^\theta}. \tag{2.7}$$

This implies that, for any $x \in X_A^\theta$ and $\theta \in (2-\alpha-\beta, 1)$, $e^{-tA}x \to x$ in X as $t \to 0+$. Grounding on (2.2), let us now introduce the intermediate space

$$\widetilde{X}_A^\theta = \{u \in X : \sup_{t>0} t^{(2-\beta-\theta)/\alpha}\|Ae^{-tA}u\|_X < +\infty\}, \quad 0 < \theta < 1, \tag{2.8}$$

endowed with the norm

$$\|u\|_{\widetilde{X}_A^\theta} = \|u\|_X + \sup_{t>0} t^{(2-\beta-\theta)/\alpha}\|Ae^{-tA}u\|_X < +\infty. \tag{2.9}$$

From the known semigroup estimate (cf. [15, Proposition 3.2])

$$\|A^\theta e^{-tA}x\|_X \le Ct^{(\beta-\theta-1)/\alpha}\|x\|_X, \quad \theta \in [0, +\infty), \tag{2.10}$$

in particular we deduce

$$\|Ae^{-tA}u\|_X \le Ct^{(\beta-2)/\alpha}\|u\|_X, \quad u \in X, \tag{2.11}$$

$$\|Ae^{-tA}u\|_X = \|e^{-tA}Au\|_X \le Ct^{(\beta-1)/\alpha}\|Au\|_X\\ \le Ct^{(\beta-1)/\alpha}\|u\|_{\mathcal{D}(A)}, \quad u \in \mathcal{D}(A), \tag{2.12}$$

where $\|u\|_{\mathcal{D}(A)} = \|u\|_X + \|Au\|_X$. By interpolation we obtain

$$\|Ae^{-tA}u\|_X \le Ct^{(1-\theta)(\beta-2)/\alpha}t^{\theta(\beta-1)/\alpha}\|u\|_{(X,\mathcal{D}(A))_{\theta,\infty}}\\ = t^{(\beta+\theta-2)/\alpha}\|u\|_{(X,\mathcal{D}(A))_{\theta,\infty}}. \tag{2.13}$$

This implies

$$\sup_{t>0} t^{(2-\beta-\theta)/\alpha}\|Ae^{-tA}u\|_X \le C\|u\|_{(X,\mathcal{D}(A))_{\theta,\infty}}, \quad \theta \in (0,1). \tag{2.14}$$

Therefore, we deduce the continuous embeddings

$$X_A^\theta \hookrightarrow (X, \mathcal{D}(A))_{\theta,\infty} \hookrightarrow \widetilde{X}_A^\theta, \quad \theta \in (0,1). \tag{2.15}$$

Lemma 2.1. *If $u \in \widetilde{X}_A^\theta$ and $\theta \in (2-\alpha-\beta, 1)$, one has*

$$\lim_{t\to+0} e^{-tA}u = u.$$

Proof. If $u \in \widetilde{X}_A^\theta$ and $0 < s < t$, we have

$$\begin{aligned}(e^{-tA} - e^{-sA})u &= \int_s^t D_r(e^{-rA})u\,dr = \int_s^t (-A)e^{-rA}u\,dr \\ &= \int_s^t r^{(2-\beta-\theta)/\alpha} r^{(\beta+\theta-2)/\alpha}(-A)e^{-rA}u\,dr,\end{aligned} \tag{2.16}$$

so that

$$\begin{aligned}\|(e^{-tA} - e^{-sA})u\|_X &\le C\|u\|_{\widetilde{X}_A^\theta} \int_s^t r^{(\beta+\theta-2)/\alpha}\,dr \\ &\le C\|u\|_{\widetilde{X}_A^\theta}(t-s)^{[\theta-(2-\alpha-\beta)]/\alpha}, \quad \theta \in (2-\alpha-\beta, 1).\end{aligned} \tag{2.17}$$

It follows that there exists $\lim_{t\to 0+} e^{-tA}u =: \xi$ for all $u \in \widetilde{X}_A^\theta$. This implies $A^{-1}e^{-tA}u \to A^{-1}\xi$ as $t \to 0+$.

Denote now by Γ the path parameterized by $\operatorname{Re} z = a - c(1+|\operatorname{Im} z|)^\alpha$, $a > c > 0$, oriented from $\operatorname{Im} z = -\infty$ to $\operatorname{Im} z = +\infty$. Note then that

$$\begin{aligned}A^{-1}e^{-tA}u &= (2\pi i)^{-1}\int_\Gamma e^{t\lambda}A^{-1}(\lambda + A)^{-1}u\,d\lambda \\ &= (2\pi i)^{-1}\int_\Gamma e^{t\lambda}\lambda^{-1}[A^{-1} - (\lambda + A)^{-1}]u\,d\lambda \\ &= A^{-1}u - (2\pi i)^{-1}\int_\Gamma e^{t\lambda}\lambda^{-1}(\lambda + A)^{-1}u\,d\lambda.\end{aligned} \tag{2.18}$$

As $t \to 0+$ the last integral converges to $\int_\Gamma \lambda^{-1}(\lambda + A)^{-1}u\,d\lambda = 0$. Therefore, owing to the uniqueness of the limit, we obtain $A^{-1}u = A^{-1}\xi$, i.e. $\xi = u$. □

We have thus proved that, if $\theta \in (2-\alpha-\beta, 1)$, the mapping $u \to e^{-tA}u$, $t \in [0, +\infty)$, maps $\widetilde{X}_A^\theta$ into $C([0, +\infty); X)$.

Let $u \in \mathcal{D}(A)$ and $\lambda > 0$. Then $t \to e^{-t\lambda}e^{-tA}$ belongs to $L^1(\mathbb{R}_+; X)$ since $1 - \beta < \alpha$. Moreover,

$$\begin{aligned}\int_0^{+\infty} \lambda e^{-t\lambda}e^{-tA}u\,dt &= \int_0^{+\infty} D_t\big(-e^{-t\lambda}\big)e^{-tA}u\,dt \\ &= -[e^{-t\lambda}e^{-tA}u]_0^{+\infty} - \int_0^{+\infty} e^{-t\lambda}e^{-tA}Au\,dt \\ &= u - \int_0^{+\infty} e^{-t\lambda}e^{-tA}Au\,dt,\end{aligned} \tag{2.19}$$

implying

$$\int_0^{+\infty} e^{-t\lambda}e^{-tA}(\lambda u + Au)\,dt = u, \quad \forall u \in \mathcal{D}(A).$$

But this implies the equality

$$(\lambda I + A)^{-1}u = \int_0^{+\infty} e^{-t\lambda}e^{-tA}u\,dt, \quad \forall u \in X. \tag{2.20}$$

Indeed, if $u \in X$, there exists $v \in \mathcal{D}(A)$ such that $u = (\lambda I + A)v$. Then

$$(\lambda I + A)^{-1}u = v = \int_0^{+\infty} e^{-t\lambda}e^{-tA}(\lambda I + A)v\,dt = \int_0^{+\infty} e^{-t\lambda}e^{-tA}u\,dt.$$

Consequently, for all $u \in \widetilde{X}_A^\theta$ and $t \in \mathbb{R}_+$, for $\theta > 2-\alpha-\beta$ we have

$$\begin{aligned} \|A(t+A)^{-1}u\|_X &= \Big\| \int_0^{+\infty} e^{-t\lambda} A e^{-\lambda A} u \, d\lambda \Big\|_X \\ &\le C \|u\|_{\widetilde{X}_A^\theta} \int_0^{+\infty} e^{-t\lambda} \lambda^{(\theta+\beta-2)/\alpha} \, d\lambda \\ &= C \|u\|_{\widetilde{X}_A^\theta} t^{-(\theta+\alpha+\beta-2)/\alpha} \int_0^{+\infty} e^{-\xi} \xi^{(\theta+\beta-2)/\alpha} \, d\xi. \end{aligned} \tag{2.21}$$

Summing up, we have proved that the continuous embeddings

$$\widetilde{X}_A^\theta \hookrightarrow X_A^{(\theta+\alpha+\beta-2)/\alpha} \hookrightarrow (X, \mathcal{D}(A))_{(\theta+\alpha+\beta-2)/\alpha,\infty}, \tag{2.22}$$

hold for any pair $(\alpha,\beta) \in (0,1] \times (0,1]$ satisfying $0 < \beta \le \alpha \le 1$, $\alpha+\beta > 1$ and $2-\alpha-\beta < \theta < 1$. (Note that $(\theta+\alpha+\beta-2) < \alpha$ implies $\theta < 2-\beta$.)

3. Spatial regularity of solutions to Cauchy problems

Consider the problem

$$\begin{gathered} y'(t) + Ay(t) = f(t), \quad t \in [0,\tau], \\ y(0) = y_0. \end{gathered} \tag{3.1}$$

We look for a strict solution to the Cauchy problem (1.4), (1.5), i.e. for a function $y \in C^1([0,\tau];X) \cap C([0,\tau];\mathcal{D}(A))$, related to spatial regular data. For this purpose we assume

$$f \in C([0,\tau];X) \cap B([0,\tau];\widetilde{X}_A^\theta), \quad y_0 \in \mathcal{D}(A), \quad Ay_0 \in \widetilde{X}_A^\theta, \tag{3.2}$$

$$0 < \beta \le \alpha \le 1, \quad \alpha+\beta > 3/2, \quad 2(2-\alpha-\beta) < \theta < 1. \tag{3.3}$$

We recall that, for any Banach space Y, $B([0,\tau];Y)$ denotes the Banach space of all bounded Y-valued functions f, when endowed with the norm $\|f\|_{B([0,\tau];Y)} = \sup_{t\in[0,\tau]} \|f(t)\|_Y$.

Necessarily the solution to (1.4), (1.5) (cf. [16]) is given by

$$y(t) = e^{-tA} y_0 + \int_0^t e^{-(t-s)A} f(s) \, ds, \quad t \in [0,\tau]. \tag{3.4}$$

Set now

$$y_1(t) = e^{-tA} y_0, \quad y_2(t) = \int_0^t e^{-(t-s)A} f(s) \, ds, \quad t \in [0,\tau]. \tag{3.5}$$

It is immediate to check that the properties of the semigroup $\{e^{-tA}\}_{t>0}$ established previously guarantee that y_1 is differentiable in $(0,\tau]$. Moreover, for $0 \le s < t \le \tau$, we have

$$\begin{aligned} \|y_1'(t) - y_1'(s)\|_X = \|Ay_1(t) - Ay_1(s)\|_X &= \Big\| \int_s^t A e^{-rA} A y_0 \, dr \Big\|_X \\ &\le C \|Ay_0\|_{\widetilde{X}_A^\theta} \int_s^t r^{(\theta+\beta-2)/\alpha} \, dr \\ &\le C' \|Ay_0\|_{\widetilde{X}_A^\theta} (t-s)^{[\theta-(2-\alpha-\beta)]/\alpha}. \end{aligned} \tag{3.6}$$

Therefore, we have proved that $y_1', Ay_1 \in C^{[\theta-(2-\alpha-\beta)]/\alpha}([0,\tau];X)$. Consider now the relations

$$\begin{aligned}
&\sup_{0\le s\le\tau}\sup_{t>0} s^{(2-\beta-\theta)/\alpha}\|Ae^{-sA}Ae^{-tA}y_0\|_X \\
&= \sup_{0\le s\le\tau}\sup_{t>0} s^{(2-\beta-\theta)/\alpha}\|Ae^{-(s+t)A}Ay_0\|_X \\
&= \sup_{0\le s\le\tau}\sup_{t>0} \Big(\frac{s}{s+t}\Big)^{(2-\beta-\theta)/\alpha}(s+t)^{(2-\beta-\theta)/\alpha}\|Ae^{-(s+t)A}Ay_0\|_X \\
&\le C\|Ay_0\|_{\widetilde{X}_A^\theta}.
\end{aligned} \tag{3.7}$$

Therefore, concerning the regularity of y_1, we obtain

$$y_1', Ay_1 \in C^{[\theta-(2-\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^\theta). \tag{3.8}$$

Let us now consider y_2 and let us notice that, for $0\le s<t\le\tau$, we have

$$\begin{aligned}
&Ay_2(t)-Ay_2(s)\\
&= \int_0^s \big[Ae^{-(t-\sigma)A}-Ae^{-(s-\sigma)A}\big]f(\sigma)\,d\sigma + \int_s^t Ae^{-(t-\sigma)A}f(\sigma)\,d\sigma \\
&=: F_1(s,t)+F_2(s,t).
\end{aligned} \tag{3.9}$$

As far as F_2 is concerned we obtain

$$\|Ae^{-(t-\sigma)A}f(\sigma)\|_X \le (t-\sigma)^{(\theta+\beta-2)/\alpha}\|f(\sigma)\|_{\widetilde{X}_\theta} \le (t-\sigma)^{(\theta+\beta-2)/\alpha}\|f\|_{B([0,\tau];\widetilde{X}_\theta)}.$$

Hence

$$\begin{aligned}
\|F_2(s,t)\|_X &\le \int_s^t (t-\sigma)^{(\theta+\beta-2)/\alpha}\|f\|_{B([0,\tau];\widetilde{X}_\theta)}\,d\sigma \\
&= \frac{(t-s)^{[\theta-(2-\alpha-\beta)]/\alpha}}{[\theta-(2-\alpha-\beta)]/\alpha}\|f\|_{B([0,\tau];\widetilde{X}_\theta)}.
\end{aligned} \tag{3.10}$$

Further, since

$$\begin{aligned}
\|A^2e^{-rA}f(\sigma)\|_X &= \|Ae^{-(r/2)A}\big[Ae^{-(r/2)A}f(\sigma)\big]\|_X \\
&\le Cr^{(\beta-2)/\alpha}r^{(\beta+\theta-2)/\alpha}\|f(\sigma)\|_{\widetilde{X}^\theta} \\
&\le Cr^{-2+[\theta-2(2-\alpha-\beta)]/\alpha}\|f\|_{B([0,\tau];\widetilde{X}_A^\theta)},
\end{aligned}$$

we have

$$\begin{aligned}
\|F_1(s,t)\|_X &= \Big\|\int_0^s d\sigma\int_{s-\sigma}^{t-\sigma} A^2e^{-rA}f(\sigma)\,dr\Big\|_X \\
&\le C\int_0^s d\sigma\int_{s-\sigma}^{t-\sigma} r^{-2+[\theta-2(2-\alpha-\beta)]/\alpha}\,dr\|f\|_{B([0,\tau];\widetilde{X}_A^\theta)} \\
&\le C\|f\|_{B([0,\tau];\widetilde{X}_A^\theta)}(t-s)^{[\theta-2(2-\alpha-\beta)]/\alpha}
\end{aligned} \tag{3.11}$$

(recall that $\alpha+\beta>3/2$). In other words, we have proved that

$$y_2\in C^{[\theta-2(2-\alpha-\beta)]/\alpha}([0,\tau];X).$$

Concerning space regularity, first we consider the identity

$$Ae^{-\xi A}Ay_2(t) = \int_0^t A^2e^{-(t-s+\xi)A}f(s)\,ds. \tag{3.12}$$

Recalling that $f \in B([0,\tau];\widetilde{X}_A^\theta)$ we have

$$\begin{aligned}\|A^2 e^{-(t-s+\xi)A} f(s)\|_X &= \|A e^{-[(t-s+\xi)/2]A} A e^{-[(t-s+\xi)/2]A} f(s)\|_X \\ &\le C(t-s+\xi)^{(\beta-2)/\alpha}(t-s+\xi)^{(\beta+\theta-2)/\alpha}\|f(s)\|_{\widetilde{X}_A^\theta} \\ &\le C\|f\|_{B([0,\tau];\widetilde{X}_A^\theta)}(t-s+\xi)^{[\theta-2(2-\beta)]/\alpha}.\end{aligned}$$

Hence noting that

$$[\theta-2(2-\beta)]/\alpha = (\theta+2\beta-4)/\alpha < (2\beta-3)/\alpha \le -1,$$

we have

$$\begin{aligned}\|Ae^{-\xi A}Ay_2(t)\|_X &\le C\|f\|_{B([0,\tau];\widetilde{X}_A^\theta)}\int_{-\infty}^t (t-s+\xi)^{[\theta-2(2-\beta)]/\alpha}\,ds \\ &= C\|f\|_{B([0,\tau];\widetilde{X}_A^\theta)}\frac{\alpha\xi^{(\alpha+2\beta+\theta-4)/\alpha}}{4-\alpha-2\beta-\theta}.\end{aligned}$$

Therefore,

$$\sup_{0\le t\le\tau}\sup_{\xi>0}\xi^{(4-\alpha-2\beta-\theta)/\alpha}\|Ae^{-\xi A}Ay_2(t)\|_X < +\infty. \tag{3.13}$$

Since $(4-\alpha-2\beta-\theta)/\alpha = [2-\beta-(\alpha+\beta+\theta-2)]/\alpha$, (3.9), (3.10), (3.11), (3.13) imply

$$\begin{gathered}Ay_2 \in C^{[\theta-2(2-\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}), \\ \|Ay_2\|_{B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)})} \le C\|f\|_{B([0,\tau];\widetilde{X}_A^\theta)}.\end{gathered}$$

It follows from (3.2) and (3) that $y_2' = f - Ay_2 \in C([0,\tau];X)$. Summing up, we have proved the following theorem.

Theorem 3.1. *Let the pairs (f,y_0) and (α,β) satisfy* (3.2) *and* (3.3)*, respectively. Then Problem* (3.1) *admits a unique strict solution y with the following regularity properties:*

$$y' \in C([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}), \tag{3.14}$$

$$Ay \in C^{[\theta-2(2-\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}). \tag{3.15}$$

Taking into account the inclusions proved in Section 2, we can also establish the following result concerning spaces X_A^θ.

Theorem 3.2. *Let $2\alpha+\beta>2$, $3-2\alpha-\beta<\theta<1$, $y_0\in D(A)$, $Ay_0\in(X,D(A))_{\theta,\infty}$, $f\in C([0,\tau];X)\cap B([0,\tau];(X,D(A))_{\theta,\infty})$. Then Problem* (3.1) *admits a unique strict solution y such that*

$$\begin{gathered}y' \in C([0,\tau];X)\cap B([0,\tau];X_A^{[\theta-(3-2\alpha-\beta)]/\alpha}), \\ Ay \in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];X_A^{[\theta-(3-2\alpha-\beta)]/\alpha}).\end{gathered}$$

Proof. We use the notation in the proof of Theorem 3.1. One has $Ay_0\in\widetilde{X}_A^\theta$ by virtue of our assumption and (2.14). Hence, owing to the proof of Theorem 3.1 $y_1(t)=e^{-tA}y_0$ satisfies

$$\begin{aligned}y_1', Ay_1 &\in C^{[\theta-(2-\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^\theta) \\ &\subset C^{[\theta-(2-\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];X_A^{[\theta-(2-\alpha-\beta)]/\alpha}).\end{aligned} \tag{3.16}$$

From (2.15) and (3.10) one deduces the estimate

$$\begin{aligned}\|F_2(s,t)\|_X &\le C\|f\|_{B([0,\tau];\tilde{X}_A^\theta)}(t-s)^{[\theta-(2-\alpha-\beta)]/\alpha}\\ &\le C\|f\|_{B([0,\tau];(X,D(A))_{\theta,\infty})}(t-s)^{[\theta-(2-\alpha-\beta)]/\alpha}.\end{aligned} \tag{3.17}$$

Likewise, from the inequalities

$$\begin{aligned}\|A^2e^{-rA}f(\sigma)\|_X &\le Cr^{(\beta-3+\theta)/\alpha}\|f(\sigma)\|_{(X,D(A))_{\theta,\infty}}\\ &\le Cr^{(\beta-3+\theta)/\alpha}\|f\|_{B([0,\tau];(X,D(A))_{\theta,\infty})},\end{aligned} \tag{3.18}$$

one obtains

$$\begin{aligned}\|F_1(s,t)\|_X &= \Big\|\int_0^s d\sigma\int_{s-\sigma}^{t-\sigma}A^2e^{-rA}f(\sigma)dr\Big\|_X\\ &\le C\int_0^s d\sigma\int_{s-\sigma}^{t-\sigma}r^{(\beta-3+\theta)/\alpha}\|f\|_{B([0,\tau];(X,D(A))_{\theta,\infty})}dr\\ &\le C\|f\|_{B([0,\tau];(X,D(A))_{\theta,\infty})}(t-s)^{(2\alpha+\beta-3+\theta)/\alpha}.\end{aligned} \tag{3.19}$$

It follows from (3.17) and (3.19) that

$$Ay_2\in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau];X). \tag{3.20}$$

Using

$$\begin{aligned}\|A^2e^{-(t-s+\xi)A}f(s)\|_X &\le C(t-s+\xi)^{(\beta-3+\theta)/\alpha}\|f(s)\|_{(X,D(A))_{\theta,\infty}}\\ &\le C(t-s+\xi)^{(\beta-3+\theta)/\alpha}\|f\|_{B([0,\tau];(X,D(A))_{\theta,\infty})}\end{aligned}$$

one obtains

$$\begin{aligned}\|Ae^{-\xi A}Ay_2(t)\|_X &= \Big\|\int_0^t A^2e^{-(t-s+\xi)A}f(s)ds\Big\|_X\\ &\le C\|f\|_{B([0,\tau];(X,D(A))_{\theta,\infty})}\int_0^t(t-s+\xi)^{(\beta-3+\theta)/\alpha}\,ds\\ &\le C\|f\|_{B([0,\tau];(X,D(A))_{\theta,\infty})}\xi^{(\alpha+\beta-3+\theta)/\alpha}.\end{aligned} \tag{3.21}$$

Hence one obtains

$$Ay_2\in B([0,\tau];\widetilde{X}_A^{\alpha+\theta-1})\subset B([0,\tau];X_A^{[\theta-(3-2\alpha-\beta)]/\alpha}). \tag{3.22}$$

From (3.16), (3.20) and (3.22), it follows that

$$Ay\in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];X_A^{[\theta-(3-2\alpha-\beta)]/\alpha}). \tag{3.23}$$

The hypothesis on f and (3.22) imply

$$y_2'=f-Ay_2\in B([0,\tau];X_A^{[\theta-(3-2\alpha-\beta)]/\alpha}). \tag{3.24}$$

By (3.16) and (3.24), one concludes that

$$y'\in B([0,\tau];X_A^{[\theta-(3-2\alpha-\beta)]/\alpha}).$$

□

In view of embedding (1.6), Theorem 3.2 leads to the following corollary, where Y_A^γ stands for anyone of the spaces X_A^γ or $(X,D(A))_{\gamma,\infty}$.

Corollary 3.3. *Let $2\alpha+\beta>2$, $3-2\alpha-\beta<\theta<1$, $y_0\in D(A)$, $Ay_0\in Y_A^\theta$, $f\in C([0,\tau];X)\cap B([0,\tau];Y_A^\theta)$. Then Problem (3.1) admits a unique strict solution y such that*

$$y'\in C([0,\tau];X)\cap B([0,\tau];Y_A^{[\theta-(3-2\alpha-\beta)]/\alpha}),$$
$$Ay\in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];Y_A^{[\theta-(3-2\alpha-\beta)]/\alpha}).$$

4. A first identification problem

Consider the identification problem (1.1) in Section 1. We want to determine a pair $(y,f)\in[C^1([0,\tau];X)\cap C([0,\tau];\mathcal{D}(A))]\times C([0,\tau];\mathbb{C})$ satisfying (1.1) under the following assumptions:

$$\begin{gathered} y_0\in\mathcal{D}(A),\quad Ay_0,z\in\widetilde{X}_A^\theta,\\ g\in C^1([0,\tau];\mathbb{C}),\quad h\in C([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^\theta);\end{gathered}\tag{4.1}$$

$$0<\beta\le\alpha\le1,\quad \alpha+\beta>3/2,\quad 1>\theta>2(2-\alpha-\beta);\tag{4.2}$$

$$\Phi\in X^*,\quad \Phi[y_0]=g(0),\quad \Phi[z]\neq0.\tag{4.3}$$

If (y,f) is the solution sought for, we immediately deduce that (y,f) solves the equation

$$g'(t)+\Phi[Ay(t)]=f(t)\Phi[z]+\Phi[h(t)],\quad t\in[0,\tau].\tag{4.4}$$

Therefore, taking advantage of Theorem 3.1, we obtain the integral equation for f

$$\begin{aligned} f(t)&=\frac{g'(t)-\Phi[h(t)]+\Phi[Ay(t)]}{\Phi[z]}\\ &=\frac{g'(t)-\Phi[h(t)]+\Phi[Ae^{-tA}y_0]}{\Phi[z]}+\frac{1}{\Phi[z]}\int_0^t\Phi[Ae^{-(t-s)A}z]f(s)\,ds\\ &\quad+\frac{1}{\Phi[z]}\int_0^t\Phi[Ae^{-(t-s)A}h(s)]\,ds=:b(t)+Sf(t),\quad t\in[0,\tau].\end{aligned}\tag{4.5}$$

Note that $t\to Ae^{-tA}y_0$ is continuous in $[0,\tau]$ by Lemma 2.1. Since $z\in\widetilde{X}_A^\theta$, we obtain

$$\|Ae^{-(t-s)A}z\|_X\le C(t-s)^{-(2-\beta-\theta)/\alpha}\le C(t-s)^{-(1-\theta_0)}.\tag{4.6}$$

where $\theta_0=[\theta-(2-\alpha-\beta)]/\alpha$. Whence we deduce the inequality

$$|Sf(t)|\le\frac{C}{|\Phi[z]|}\|\Phi\|_{X^*}\|z\|_{\widetilde{X}_A^\theta}\int_0^t(t-s)^{-1+\theta_0}|f(s)|\,ds,\quad t\in(0,\tau].$$

Repeating the arguments and techniques in [2] we can deduce the following estimates involving the iterates S^n of operator S:

$$|S^nf(t)|\le\big[C(\Phi[z])^{-1}\|\Phi\|_{X^*}\|z\|_{\widetilde{X}_A^\theta}\big]^n\frac{\Gamma(\theta_0)^nt^{n\theta_0}}{\Gamma(n\theta_0)n\theta_0}\|f\|_{C([0,\tau];\mathbb{C})},\quad t\in(0,\tau].\tag{4.7}$$

Since $[\Gamma(n\theta_0)]^{1/n}\to+\infty$ as $n\to+\infty$, we conclude that the operator S has spectral radius equal to 0. Therefore equation (4.5) admits a unique solution $f\in C([0,\tau];\mathbb{C})$.

In view of Theorem 3.1 we conclude that the solution y corresponding to such an f has the regularity

$$\begin{gathered} y\in C^1([0,\tau];X)\cap C([0,\tau];\mathcal{D}(A)),\\ y'\in C([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}),\end{gathered}\tag{4.8}$$

$$Ay \in C^{[\theta-2(2-\alpha-\beta)]/\alpha}([0,\tau];X) \cap B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}). \tag{4.9}$$

Summing up, we have proved the following theorem.

Theorem 4.1. *Under assumptions* (4.1) *and* (4.2)*, the identification problem* (1.1) *in Section 1 admits a unique strict solution* (y, f) *satisfying* (4.8)*,* (4.9)*.*

We change now a bit our assumptions on the data: (4.1) is replaced with the following, where we change the condition on the pair (y_0, z):

$$\begin{gathered} y_0 \in \mathcal{D}(A), \quad Ay_0, z \in X_A^{\theta}, \quad g \in C^1([0,\tau];\mathbb{C}), \\ h \in C([0,\tau];X) \cap B([0,\tau];X_A^{\theta}), \end{gathered} \tag{4.10}$$

If (y, f) is the solution sought for, we deduce, as above, that f solves the integral equation (4.5). Reasoning as above and taking advantage of Corollary 3.3, we obtain the following result.

Theorem 4.2. *Let* Y_A^{γ} *be anyone of the spaces* $(X, D(A))_{\gamma,\infty}$ *or* X_A^{γ}*. Let* $2\alpha+\beta > 2$*,* $\theta > 3-2\alpha-\beta$ *and let*

$$\begin{gathered} y_0 \in D(A), \quad Ay_0, z \in Y_A^{\theta,\infty}, \quad g \in C^1([0,\tau];\mathbb{C}), \\ h \in C([0,\tau];Y_A^{\theta,\infty}), \quad \Phi[z] \neq 0. \end{gathered} \tag{4.11}$$

Then the identification problem

$$\begin{gathered} y'(t) + Ay(t) = f(t)z + h(t), \quad t \in [0,\tau], \\ y(0) = y_0, \\ \Phi[y(t)] = g(t), \quad t \in [0,\tau]. \end{gathered} \tag{4.12}$$

admits a unique strict solution $(y, f) \in [C^1([0,\tau];X) \cap C([0,\tau];D(A))] \times C([0,\tau];\mathbb{C})$ *such that*

$$\begin{gathered} y' \in C([0,\tau];X) \cap B([0,\tau];Y_A^{(2\alpha+\beta-3+\theta)/\alpha,\infty}), \\ Ay \in C^{(2\alpha+\beta-3+\theta)/\alpha}([0,\tau];X) \cap B([0,\tau];Y_A^{(2\alpha+\beta-3+\theta)/\alpha,\infty}). \end{gathered}$$

Proof. When $Y_A^{\gamma} = (X, D(A))_{\gamma,\infty}$, it suffices to observe that from (2.14) we deduce estimate (4.6) and that the same argument in the proof of Theorem 4.1 runs well, since $3-2\alpha-\beta = (2-\alpha-\beta)+1-\alpha \geq 2-\alpha-\beta$. When $Y_A^{\gamma} = X_A^{\gamma}$, the assertion follows from [16, Corollary 3.3 and Proposition 3.4]. □

5. A latter identification problem

In this section we consider the problem consisting in recovering two unknown scalar functions $f_1, f_2 \in C([0,\tau];\mathbb{C})$ and a vector function $y \in C^1([0,\tau];X) \cap C([0,\tau];\mathcal{D}(A))$ such that

$$\begin{gathered} y'(t) + Ay(t) = f_1(t)z_1 + f_2(t)z_2 + h(t), \quad t \in [0,\tau], \\ y(0) = y_0, \\ \Phi_j[y(t)] = g_j(t), \quad t \in [0,\tau],\ j = 1,2, \end{gathered} \tag{5.1}$$

where $\Phi_j \in X^*$, $g_j \in C^1([0,\tau];\mathbb{C})$, $z_j \in X$, $j = 1,2$, $h \in C([0,\tau];X) \cap B([0,\tau];X_A^{\theta})$, and $y_0 \in \mathcal{D}(A)$ are given. Let

$$\mathcal{A} = \begin{bmatrix} \Phi_1[z_1] & \Phi_1[z_2] \\ \Phi_2[z_1] & \Phi_2[z_2] \end{bmatrix}, \quad \det \mathcal{A} \neq 0.$$

Then we obtain the following fixed-point integral system for (f_1, f_2),

$$\begin{aligned}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} &= \mathcal{A}^{-1}\begin{bmatrix} g_1'(t) \\ g_2'(t) \end{bmatrix} + \mathcal{A}^{-1}\begin{bmatrix} \Phi_1[e^{-tA}Ay_0] \\ \Phi_2[e^{-tA}Ay_0] \end{bmatrix} \\ &\quad - \mathcal{A}^{-1}\begin{bmatrix} \Phi_1[h(t)] \\ \Phi_2[h(t)] \end{bmatrix} + \mathcal{A}^{-1}\int_0^t \begin{bmatrix} \Phi_1[Ae^{-(t-s)A}h(s)] \\ \Phi_2[Ae^{-(t-s)A}h(s)] \end{bmatrix} ds \\ &\quad + \mathcal{A}^{-1}\int_0^t \begin{bmatrix} \Phi_1[Ae^{-(t-s)A}z_1] & \Phi_1[Ae^{-(t-s)A}z_2] \\ \Phi_2[Ae^{-(t-s)A}z_1] & \Phi_2[Ae^{-(t-s)A}z_2] \end{bmatrix} \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds \\ &=: \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} + S\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \quad t \in [0,\tau]. \end{aligned} \tag{5.2}$$

We introduce in $C([0,\tau];\mathbb{C}^2)$ the sup-norm

$$\|(f_1,f_2)\|_{C([0,\tau];\mathbb{C}^2)} = \max_{t\in[0,\tau]}|f_1(t)| + \max_{t\in[0,\tau]}|f_2(t)|.$$

For any pair $(z_1,z_2) \in (\widetilde{X}_A^\theta)^2$ and $(f_1,f_2) \in C([0,\tau];\mathbb{C}^2)$ we obtain the bounds

$$\begin{aligned} &\|S\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t)\| \le \|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathbb{C}^2)}\int_0^t \sum_{j,k=1}^2 |\Phi_j[Ae^{-(t-s)A}z_k]||f_k(s)|\,ds \\ &\le \|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathbb{C}^2)}\sum_{j=1}^2\|\Phi_j\|_{X^*}\int_0^t\sum_{k=1}^2\|Ae^{-(t-s)A}z_k\|_X|f_k(s)|\,ds \\ &\le C\|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathbb{C}^2)}\sum_{j=1}^2\|\Phi_j\|_{X^*}\int_0^t\sum_{k=1}^2\|z_k\|_{\widetilde{X}_A^\theta}(t-s)^{(\beta+\theta-2)/\alpha}|f_k(s)|\,ds \\ &\le C\|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathbb{C}^2)}\sum_{j=1}^2\|\Phi_j\|_{X^*}\max_{1\le k\le 2}\|z_k\|_{\widetilde{X}_A^\theta}\int_0^t(t-s)^{(\beta+\theta-2)/\alpha}\sum_{k=1}^2|f_k(s)|\,ds. \end{aligned}$$

Proceeding by induction, we can prove the bounds for the iterates S^n of operator S (cf. Section 4):

$$\|S^n\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t)\|_2 \le C_1^n\frac{\Gamma(\theta_0)^n t^{n\theta_0}}{\Gamma(n\theta_0)n\theta_0}\|\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\|_{C([0,\tau];\mathbb{C}^2)},$$

where we have set

$$C_1 = C\|\mathcal{A}^{-1}\|_{\mathcal{L}(\mathbb{C}^2)}\sum_{j=1}^2\|\Phi_j\|_{X^*}\max_{1\le k\le 2}\|z_k\|_{\widetilde{X}_A^\theta}.$$

Since $[\Gamma(n\theta_0)]^{1/n} \to +\infty$ as $n \to +\infty$, we can conclude that operator S has spectral radius equal to 0, so that problem (5.1) admits a unique solution $(f_1,f_2) \in C([0,\tau];\mathbb{C}^2)$. Using Theorem 3.1 we easily deduce the following result.

Theorem 5.1. *Let $\alpha+\beta > 3/2$ and $\theta \in (2(2-\alpha-\beta),1)$. Let $y_0 \in \mathcal{D}(A)$, $Ay_0 \in \widetilde{X}_A^\theta$, $z_j \in \widetilde{X}_A^\theta$, $\Phi_j \in X^*$, $g_j \in C^1([0,\tau];\mathbb{C})$, $j = 1,2$ and $h \in C([0,\tau];X) \cap B([0,\tau];\widetilde{X}_A^\theta)$ such that*

$$\Phi_1[z_1]\Phi_2[z_2] - \Phi_2[z_1]\Phi_1[z_2] \ne 0, \quad \Phi_j[y_0] = g_j(0),\ j = 1,2. \tag{5.3}$$

Then problem (5.1) admits a unique strict solution $(y,f_1,f_2) \in [C^1([0,\tau];X) \cap C([0,\tau];\mathcal{D}(A))] \times C([0,\tau];\mathbb{C}) \times C([0,\tau];\mathbb{C})$ such that

$$y' \in B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}),\ Ay \in C^{[\theta-2(2-\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}).$$

We conclude this section by two easy extensions of Problem (5.1) to the case of n unknown functions f.

Corollary 5.2. *Let $\alpha+\beta>3/2$ and $\theta\in(2(2-\alpha-\beta),1)$. Let $y_0\in\mathcal{D}(A)$, $Ay_0\in\widetilde{X}_A^\theta$, $z_j\in\widetilde{X}_A^\theta$, $g_j\in C^1([0,\tau];\mathbb{R})$, $h\in C([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^\theta)$, $\Phi_j\in X^*$, $\Phi_j[y_0]=g_j(0)$, $j=1,\dots,n$ be such that*

$$\det\begin{bmatrix}\Phi_1[z_1] & \dots & \Phi_1[z_n]\\ \dots & \dots & \dots\\ \Phi_n[z_1] & \dots & \Phi_n[z_n]\end{bmatrix}\neq 0.$$

Then the identification problem

$$\begin{gathered}y'(t)+Ay(t)=\sum_{j=1}^n f_j(t)z_j+h(t),\quad t\in[0,\tau],\\ y(0)=y_0,\\ \Phi_j[y(t)]=g_j(t),\quad t\in[0,\tau],\ j=1,\dots,n,\end{gathered}\tag{5.4}$$

admits a unique strict solution $(y,f_1,\dots,f_n)\in[C^1([0,\tau];X)\cap C([0,\tau];\mathcal{D}(A))]\times C([0,\tau];\mathbb{R})^n$ such that

$$\begin{gathered}y'\in B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}),\\ Ay\in C^{[\theta-2(2-\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];\widetilde{X}_A^{\theta-(2-\alpha-\beta)}).\end{gathered}$$

Corollary 5.3. *Let Y_A^γ be anyone of the spaces $(X,D(A))_{\gamma,\infty}$ or X_A^γ. Let $2\alpha+\beta>2$ and $\theta\in(3-2\alpha-\beta,1)$. Let $y_0\in\mathcal{D}(A)$, $Ay_0\in Y_A^{\theta,\infty}$, $z_j\in Y_A^{\theta,\infty}$, $g_j\in C^1([0,\tau];\mathbb{R})$, $h\in C([0,\tau];X)\cap B([0,\tau];Y_A^{\theta,\infty})$, $\Phi_j\in X^*$, $\Phi_j[y_0]=g_j(0)$, $j=1,\dots,n$ be such that*

$$\det\begin{bmatrix}\Phi_1[z_1] & \dots & \Phi_1[z_n]\\ \dots & \dots & \dots\\ \Phi_n[z_1] & \dots & \Phi_n[z_n]\end{bmatrix}\neq 0.$$

Then the identification problem (5.4) *admits a unique strict solution $(y,f_1,\dots,f_n)$ in $[C^1([0,\tau];X)\cap C([0,\tau];\mathcal{D}(A))]\times C([0,\tau];\mathbb{R})^n$ such that*

$$\begin{gathered}y'\in B([0,\tau];Y_A^{[\theta-(3-2\alpha-\beta)]/\alpha,\infty}),\\ Ay\in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau];X)\cap B([0,\tau];Y_A^{[\theta-(3-2\alpha-\beta)]/\alpha,\infty}).\end{gathered}$$

6. Inverse problems for systems of differential boundary value problems

Let A, B, C, D be linear closed operators acting in the Banach space X satisfying the following relations:

$$\mathcal{D}(A)\subset\mathcal{D}(C),\quad \mathcal{D}(D)\subset\mathcal{D}(B),\tag{6.1}$$

$$\|(\lambda+A)^{-1}\|_{\mathcal{L}(X)}\le c|\lambda|^{-\beta_1},\quad \|(\lambda+D)^{-1}\|_{\mathcal{L}(X)}\le c|\lambda|^{-\beta_2},\quad \lambda\in\Sigma_\alpha,\tag{6.2}$$

$$\|C(\lambda+A)^{-1}\|_{\mathcal{L}(X)}\le c|\lambda|^{-\gamma_1},\quad \|B(\lambda+D)^{-1}\|_{\mathcal{L}(X)}\le c|\lambda|^{-\gamma_2},\quad \lambda\in\Sigma_\alpha,\tag{6.3}$$

with

$$\gamma_1+\gamma_2\in\mathbb{R}_+.\tag{6.4}$$

In the Banach space $X \times X$ we consider the problem consisting in determining a quadruplet (y_1, y_2, f_1, f_2), with

$$(y_1, y_2) \in [C^1([0,\tau]; X) \cap C([0,\tau]; \mathcal{D}(A))] \times [C^1([0,\tau]; X) \cap C([0,\tau]; \mathcal{D}(D))], \quad (6.5)$$

$$(f_1, f_2) \in C([0,\tau]; \mathbb{C}) \times C([0,\tau]; \mathbb{C}), \quad (6.6)$$

solving the identification problem

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \\ &= f_1(t)\begin{bmatrix} z_{1,1} \\ z_{2,1} \end{bmatrix} + f_2(t)\begin{bmatrix} z_{1,2} \\ z_{2,2} \end{bmatrix} + \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}, \quad t \in [0,\tau], \end{aligned} \quad (6.7)$$

$$y_j(0) = y_{0,j}, \quad j = 1, 2, \quad (6.8)$$

$$\Psi_j[y_j(t)] = g_j(t), \quad t \in [0,\tau],\ j = 1, 2, \quad (6.9)$$

with

$$\Psi_j \in X^*, \quad \Psi_j[y_{0,j}] = g_j(0), \quad j = 1, 2. \quad (6.10)$$

Let us now introduce the linear operator $\mathcal{A}$ defined by

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(D), \quad \mathcal{A}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Ay_1 + By_2 \\ Cy_1 + Dy_2 \end{bmatrix}.$$

It is shown in [14] that, for large positive R,

$$\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \le c|\lambda|^{-\beta}, \quad \lambda \in \Sigma_\alpha \cap B(0, R)^c, \quad (6.11)$$

where

$$\beta = \min\{\beta_1, \beta_2, \beta_1 + \gamma_2, \beta_2 + \gamma_1\}. \quad (6.12)$$

Using the change of variables $(y_1(t), y_2(t)) \mapsto (e^{-kt}y_1(t), e^{-kt}y_2(t))$ with a sufficiently large positive constant k, we can assume that bound (6.11) holds for all $\lambda \in \Sigma_\alpha$.

Set now

$$E = X \times X, \quad \xi = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad z_1 = \begin{bmatrix} z_{1,1} \\ z_{2,1} \end{bmatrix}, \quad z_2 = \begin{bmatrix} z_{1,2} \\ z_{2,2} \end{bmatrix}.$$

Then the direct problem (6.7), (6.8) takes the simpler form

$$\xi'(t) + \mathcal{A}\xi(t) = f_1(t)z_1 + f_2(t)z_2, \quad t \in [0,\tau], \quad (6.13)$$

$$\xi(0) = \begin{bmatrix} y_{0,1} \\ y_{0,2} \end{bmatrix} =: \xi_0. \quad (6.14)$$

Define the norm in $X \times X$ by $\|(y_1, y_2)\|_{X\times X} = (\|y_1\|_X^2 + \|y_2\|_X^2)^{1/2}$ and introduce the functionals $\Phi_1, \Phi_2 \in E^* \sim X^* \times X^*$ (cf. [17, p. 164]) defined by

$$\Phi_j[\xi] = \Phi_j\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Psi_j[y_j], \quad j = 1, 2.$$

Applying Φ_j, $j = 1, 2$, to both sides in (6.7), we easily obtain the following system, for all $t \in [0,\tau]$,

$$\begin{gathered} g_1'(t) + \Psi_1[Ay_1(t) + By_2(t)] = f_1(t)\Psi_1[z_{1,1}] + f_2(t)\Psi_1[z_{1,2}] + \Psi_1[h(t)], \\ g_2'(t) + \Psi_2[Cy_1(t) + Dy_2(t)] = f_1(t)\Psi_2[z_{2,1}] + f_2(t)\Psi_2[z_{2,2}] + \Psi_2[h(t)], \end{gathered}$$

Now assume that

$$\Phi_1[z_1]\Phi_2[z_2] - \Phi_1[z_2]\Phi_2[z_1] = \Psi_1[z_{1,1}]\Psi_2[z_{2,2}] - \Psi_1[z_{1,2}]\Psi_2[z_{2,1}] \ne 0. \quad (6.15)$$

Then it is easy to realize that Theorem 5.1 and Corollary 5.2 yield the following results.

Theorem 6.1. *Let operators A, B, C, D satisfy conditions* (6.1)–(6.4) *and let β be defined by* (6.12). *Let $\alpha+\beta > 3/2$, $\theta \in (2(2-\alpha-\beta), 1)$, $g_1, g_2 \in C^1([0,\tau];\mathbb{C})$ and $h \in C([0,\tau];X^2) \cap B([0,\tau];\widetilde{X}^\theta_{\mathcal{A}})$. Moreover, let $[y_{0,1}, y_{0,2}] \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}[y_{0,1}, y_{0,2}]^T \in \widetilde{X}^\theta_{\mathcal{A}}$, $[z_{1,1}, z_{2,1}]^T, [z_{1,2}, z_{2,2}]^T \in \widetilde{X}^\theta_{\mathcal{A}}$ and let* (6.10) *and* (6.15) *be satisfied. Then problem* (6.7)-(6.9) *admits a unique strict solution (y_1, y_2, f_1, f_2) in the space defined by* (6.5), (6.6) *such that*

$$
\begin{gathered}
[y_1', y_2']^T \in B([0,\tau]; \widetilde{X}^{\theta-(2-\alpha-\beta)}_{\mathcal{A}}),\\
[Ay_1 + By_2, Cy_1 + Dy_2]^T\\
\in C^{[\theta-2(2-\alpha-\beta)]/\alpha}([0,\tau]; X \times X) \cap B([0,\tau]; \widetilde{X}^{\theta-(2-\alpha-\beta)}_{\mathcal{A}}).
\end{gathered}
$$

We can conclude this section by stating the following corollaries that take into account Corollary 5.3.

Corollary 6.2. *Let operators A, B, C, D satisfy conditions* (6.1)–(6.4) *and let β be defined by* (6.12). *Let $2\alpha+\beta > 2$, $\theta \in (3-2\alpha-\beta, 1)$, $g_1, g_2 \in C^1([0,\tau];\mathbb{R})$ and $h \in C([0,\tau];X^2) \cap B([0,\tau];X^\theta_{\mathcal{A}})$. Moreover, let $[y_{0,1}, y_{0,2}] \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}[y_{0,1}, y_{0,2}]^T \in X^\theta_{\mathcal{A}}$, $[z_{1,1}, z_{2,1}]^T, [z_{1,2}, z_{2,2}]^T \in X^\theta_{\mathcal{A}}$, and let* (6.10) *and* (6.15) *be satisfied. Then problem* (6.7)–(6.9) *admits a unique strict solution (y_1, y_2, f_1, f_2) in the space defined by* (6.5), (6.6) *such that*

$$
\begin{gathered}
[y_1', y_2']^T \in B([0,\tau]; X^{[\theta-(3-2\alpha-\beta)]/\alpha}_{\mathcal{A}}),\\
[Ay_1 + By_2, Cy_1 + Dy_2]^T\\
\in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau]; X \times X) \cap B([0,\tau]; X^{[\theta-(3-2\alpha-\beta)]/\alpha}_{\mathcal{A}}).
\end{gathered}
$$

Corollary 6.3. *Let operators A, B, C, D satisfy conditions* (6.1)–(6.4) *and let β be defined by* (6.12). *Let $2\alpha + \beta > 2$, $\theta \in (3 - 2\alpha - \beta, 1)$ and $g_1, g_2 \in C^1([0,\tau];\mathbb{R})$. Moreover, let $[y_{0,1}, y_{0,2}] \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}[y_{0,1}, y_{0,2}]^T \in (X \times X, \mathcal{D}(\mathcal{A}))_{\theta,\infty}$, $[z_{1,1}, z_{2,1}]^T$, $[z_{1,2}, z_{2,2}]^T \in (X \times X, \mathcal{D}(\mathcal{A}))_{\theta,\infty}$, and let* (6.10) *and* (6.15) *be satisfied. Then problem* (6.7)-(6.9) *admits a unique strict solution (y_1, y_2, f_1, f_2) in the space defined by* (6.5), (6.6) *such that*

$$
\begin{gathered}
[y_1', y_2']^T \in B([0,\tau]; (X \times X, \mathcal{D}(\mathcal{A}))_{[\theta-(3-2\alpha-\beta)]/\alpha,\infty}),\\
[Ay_1 + By_2, Cy_1 + Dy_2]^T\\
\in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau]; X \times X) \cap B([0,\tau]; (X \times X, \mathcal{D}(\mathcal{A}))_{[\theta-(3-2\alpha-\beta)]/\alpha,\infty}).
\end{gathered}
$$

Remark 6.4. The conclusions of Theorem 6.1 and Corollaries 6.2 and 6.3 may be true even in cases when the domain of the operator matrix $\mathcal{A}$ is not $\mathcal{D}(A) \times \mathcal{D}(D)$ (cf. Problem 8.1 in Section 8).

Remark 6.5. In the optimal situation, when $\alpha = \beta = 1$ and, e.g., the operators A and D generate two analytic semigroups on X and B and C are bounded operators, the previous conditions reduce to the following for some $\theta \in (0, 1)$:

$$
\begin{gathered}
(Ay_{0,1} + By_{0,2}, Cy_{0,1} + Dy_{0,2}) \in (X, \mathcal{D}(A))_{\theta,\infty}) \times (X, \mathcal{D}(D))_{\theta,\infty})\\
z_{1,1}, z_{1,2} \in (X, \mathcal{D}(A))_{\theta,\infty}), \quad z_{2,1}, z_{2,2} \in (X, \mathcal{D}(D)_{\theta,\infty}),
\end{gathered}
$$

Then (y_1, y_2, f_1, f_2) also satisfies

$$y_1', Ay_1 + By_2 \in B([0,\tau]; (X, \mathcal{D}(A))_{\theta,\infty}),$$
$$y_2', Cy_1 + Dy_2 \in B([0,\tau]; (X, \mathcal{D}(A))_{\theta,\infty}),$$
$$Ay_1 + By_2 \in C^\theta([0,\tau]; X), \quad Cy_1 + Dy_2 \in C^\theta([0,\tau]; X).$$

7. Weakly coupled identification problems

In this section we deal with the following weakly coupled identification problem

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} y_1(t) \\ \dots \\ y_n(t) \end{bmatrix} + \begin{bmatrix} A_{1,1}+B_{1,1} & B_{1,2} & \dots & B_{1,n} \\ B_{2,1} & A_{2,2}+B_{2,2} & \dots & B_{2,n} \\ \dots & \dots & \dots & \dots \\ B_{n,1} & B_{n,2} & \dots & A_{n,n}+B_{n,n} \end{bmatrix} \begin{bmatrix} y_1(t) \\ \dots \\ y_n(t) \end{bmatrix} \\ &= \begin{bmatrix} h_1(t) \\ \dots \\ h_n(t) \end{bmatrix} + \sum_{j=1}^n f_j(t) \begin{bmatrix} z_{1,j} \\ \dots \\ z_{n,j} \end{bmatrix}, \quad t \in [0,\tau], \end{aligned} \tag{7.1}$$

$$y_j(0) = y_{0,j}, \quad j = 1, \dots, n, \tag{7.2}$$

$$\Psi_j[y_j(t)] = g_j(t), \quad t \in [0,\tau],\ j = 1, \dots, n, \tag{7.3}$$

with

$$\Psi_j \in X^*, \quad \Psi_j[y_{0,j}] = g_j(0), \quad j = 1, \dots, n, \tag{7.4}$$

where $A_{i,i}$, $B_{i,j}$ are closed linear operators acting in the Banach space X. Now we introduce the operator matrices A and B defined by

$$A = \begin{bmatrix} A_{1,1} & O & \dots & O \\ O & A_{2,2} & \dots & O \\ \dots & \dots & \dots & \dots \\ O & O & \dots & A_{n,n} \end{bmatrix}, \quad B = \begin{bmatrix} B_{1,1} & B_{1,2} & \dots & B_{1,n} \\ B_{2,1} & B_{2,2} & \dots & B_{2,n} \\ \dots & \dots & \dots & \dots \\ B_{n,1} & B_{n,2} & \dots & B_{n,n} \end{bmatrix}$$

Assume now that $\rho(A_{j,j}) \supset \Sigma_\alpha$, $j = 1, \dots, n$, and

$$\|(\lambda I + A_{j,j})^{-1}\|_{\mathcal{L}(X)} \le C(1+|\lambda|)^{-\beta}, \quad \lambda \in \Sigma_\alpha, \tag{7.5}$$

Then $\lambda I - A$ is invertible for all $\lambda \in \Sigma_\alpha$ and

$$(\lambda I + A)^{-1} = \begin{bmatrix} (\lambda I + A_{1,1})^{-1} & O & \dots & O \\ O & (\lambda I + A_{2,2})^{-1} & \dots & O \\ \dots & \dots & \dots & \dots \\ O & O & \dots & (\lambda I + A_{n,n})^{-1} \end{bmatrix}.$$

Further, let the linear closed operators $B_{i,j} : \mathcal{D}(B_{i,j}) \subset X \to X$, $\mathcal{D}(B_{i,j}) \supset \mathcal{D}(A_{j,j})$, $i, j = 1, \dots, n$, satisfy, for some $\sigma > 0$, the estimates

$$\|B_{i,j}(\lambda I + A_{j,j})^{-1}\|_{\mathcal{L}(X)} \le C(1+|\lambda|)^{-\sigma}, \quad \lambda \in \Sigma_\alpha. \tag{7.6}$$

Observe now that

$$\mathcal{D}(A+B) = \mathcal{D}(A) = \prod_{j=1}^n \mathcal{D}(A_{j,j}), \quad \lambda I + A + B = \big[I + B(\lambda I + A)^{-1}\big](\lambda I + A).$$

Since

$$\begin{bmatrix} B_{1,1} & \dots & B_{1,n} \\ \dots & \dots & \dots \\ B_{n,1} & \dots & B_{n,n} \end{bmatrix} \begin{bmatrix} (\lambda I + A_{1,1})^{-1} & \dots & O \\ \dots & \dots & \dots \\ O & \dots & (\lambda I + A_{n,n})^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} B_{1,1}(\lambda I + A_{1,1})^{-1} & \dots & B_{1,n}(\lambda I + A_{n,n})^{-1} \\ \dots & \dots & \dots \\ B_{n,1}(\lambda I + A_{1,1})^{-1} & \dots & B_{n,n}(\lambda I + A_{n,n})^{-1} \end{bmatrix},$$

assumption (7.6)) implies that $I + B(\lambda I + A)^{-1}$ has a bounded inverse for each $\lambda \in \Sigma_\alpha$, with a large enough modulus, satisfying

$$\|\big[I + B(\lambda I + A)^{-1}\big]^{-1}\|_{\mathcal{L}(X^n)} \le 2, \quad \lambda \in \Sigma_\alpha \cap B(0, C)^c,$$

where X^n is the product space $X \times \dots \times X$ (n times).

On the other hand, the change of the vector unknown defined by $y(t) = e^{kt}w(t)$, where $y(t) = [y_1(t), \dots, y_n(t)]^T$ and $w(t) = [w_1(t), \dots, w_n(t)]^T$, transforms our equation into

$$w'(t) = -(kI + A + B)w(t) + e^{-kt}\sum_{j=1}^{n} f_j(t)z_j + e^{-kt}h(t), \quad t \in [0, \tau],$$

where $h(t) = [h_1(t), \dots, h_n(t)]^T$. Observe now that

$$(\lambda I+kI+A+B)^{-1} = ((\lambda+k)I+A+B)^{-1} = ((\lambda+k)I+A)^{-1}\big[I+B((\lambda+k)I+A)^{-1}\big].$$

Therefore, we conclude that $\lambda I + kI + A + B$ is invertible for large enough k and $(\lambda I + kI + A + B)^{-1} \in \mathcal{L}(X^n)$ for $\lambda \in \Sigma_\alpha$. Then set

$$\mathcal{A} = A + B, \quad \mathcal{D}(\mathcal{A}) = \prod_{j=1}^{n} \mathcal{D}(A_{j,j}), \quad y = (y_1, \dots, y_n)^T,$$

$$z_j = (z_{1,j}, \dots, z_{n,j})^T, \quad \Phi_j \in (X^n)^* = (X^*)^n, \quad \Phi_j[y] = \Psi_j[y_j], \quad j = 1, \dots, n.$$

Consider the equality

$$\begin{bmatrix} \Phi_1[z_1] & \dots & \Phi_1[z_n] \\ \dots & \dots & \dots \\ \Phi_n[z_1] & \dots & \Phi_n[z_n] \end{bmatrix} = \begin{bmatrix} \Psi_1[z_{1,1}] & \dots & \Psi_1[z_{1,n}] \\ \dots & \dots & \dots \\ \Psi_n[z_{n,1}] & \dots & \Psi_n[z_{n,n}] \end{bmatrix}.$$

Thus we need to assume that

$$\det \begin{bmatrix} \Psi_1[z_{1,1}] & \dots & \Psi_1[z_{1,n}] \\ \dots & \dots & \dots \\ \Psi_n[z_{n,1}] & \dots & \Psi_n[z_{n,n}] \end{bmatrix} \neq 0. \tag{7.7}$$

Then we characterize the space $X^\theta_{\mathcal{A}}$ in the following Lemma.

Lemma 7.1. *The following relations hold for all $\theta \in (0, \beta)$:*

$$X^{\theta+1-\beta}_{\mathcal{A}} \hookrightarrow X^\theta_A, \quad X^{\theta+1-\beta}_A \hookrightarrow X^\theta_{\mathcal{A}}, \quad X^\theta_A = \prod_{j=1}^{n} X^\theta_{A_{j,j}}. \tag{7.8}$$

We postpone the proof of this lemma to the end of this section and state our conclusive theorem.

Theorem 7.2. *Let $\alpha, \beta \in (0, 1]$, $\alpha + 2\beta + \alpha\beta > 3$ and $3 - \alpha - \beta - \alpha\beta < \theta < \beta$. Let $z_j = [z_{1,j}, \dots, z_{n,j}] \in X^{\theta+1-\beta}_A (\subset X^\theta_{\mathcal{A}})$, $j = 1, \dots, n$, $y_0 = (y_{0,1}, \dots, y_{0,n}) \in \mathcal{D}(A)$, $(A + B)y_0 \in X^{\theta+1-\beta}_A$, $h \in C([0, \tau]; X^n) \cap B([0, \tau]; X^{\theta+1-\beta}_A)$ satisfy condition* (7.7)*. Then the identification problem* (9.40)-(9.42) *admits a unique strict solution $(y, f_1, \dots, f_n) \in C([0, \tau]; X) \cap C([0, \tau]; \mathbb{C})^n$ such that*

$$y' \in B([0, \tau]; X_A^{[\theta-(3-2\alpha-\beta)-\alpha(1-\beta)]/\alpha}),$$

$$(A+B)y \in C^{[\theta-(3-2\alpha-\beta)]/\alpha}([0,\tau];X^n) \cap B([0,\tau];X_A^{[\theta-(3-2\alpha-\beta)-\alpha(1-\beta)]/\alpha}).$$

The proof of the above theorem follows easily from our assumptions and Corollary 5.3. We conclude this section with the proof of Lemma 7.1.

Proof of Lemma 7.1. To show the first embedding in (7.8) we recall that $\tau+A+B$ admits a continuous inverse for $\tau \geq t_0$, t_0 being positive and large enough. Then we make use of the following identity, with $t > 0$:

$$\begin{aligned}&(t_0+A+B)(t+t_0+A+B)^{-1}-(t_0+A+B)(t+A)^{-1}\\&=-(t_0+A+B)(t+t_0+A+B)^{-1}(t_0+B)(t+A)^{-1}\end{aligned} \tag{7.9}$$

Whence we deduce

$$\begin{aligned}&\sup_{t>0}(1+t)^{\theta}\|(t_0+A+B)(t+t_0+A+B)^{-1}u\|\\&\leq \|(t_0+A+B)A^{-1}\|_{\mathcal{L}(X)}\sup_{t>0}(1+t)^{\theta}\|A(t+A)^{-1}u\|\\&\quad+\sup_{t>0}C(1+t)^{1-\beta}\|(t_0+B)A^{-1}\|_{\mathcal{L}(X)}(1+t)^{\theta}\|A(t+A)^{-1}u\|\\&\leq C'\sup_{t>0}(1+t)^{\theta+1-\beta}\|A(t+A)^{-1}u\|.\end{aligned}$$

These inequalities imply the embedding

$$X_A^{\theta+1-\beta} \hookrightarrow X_{t_0+A+B}^{\theta}.$$

Interchanging the roles of t_0+A+B and A, we obtain the embedding

$$X_{t_0+A+B}^{\theta+1-\beta} \hookrightarrow X_A^{\theta}, \quad \text{if } \theta \in (0,\beta).$$

We have thus shown the first two relations in (7.7).

Now we show that $X_A^{\theta} = X_{t_0+A}^{\theta}$ for all $\theta \in (0,1)$ and $t_0 \in \mathbb{R}_+$. For this purpose first we consider the following identities:

$$\begin{aligned}&(t_0+A)(t+t_0+A)^{-1}-A(t+A)^{-1}\\&=\big[(t_0+A)(t+t_0+A)^{-1}(t+A)A^{-1}-I\big]A(t+A)^{-1}\\&=\big[(t_0+A)(t+t_0+A)^{-1}(t+t_0+A-t_0)A^{-1}-I\big]A(t+A)^{-1}\\&=\big[t_0A^{-1}+I-I-t_0(t_0A^{-1}+I)(t+t_0+A)^{-1}\big]\big[A(t+A)^{-1}\big]\end{aligned}$$

Observe that

$$\|(t+t_0+A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{(1+t+t_0)^{\beta}} \leq M, \quad t \in [0,+\infty).$$

Therefore we easily get the estimate

$$\sup_{t>0}(1+t)^{\theta}\|(t_0+A)(t+t_0+A)^{-1}u\| \leq C(t_0,A)\sup_{t>0}(1+t)^{\theta}\|A(t+A)^{-1}u\|,$$

where

$$C(t_0,A) \leq 1+t_0\|A^{-1}\|_{\mathcal{L}(X)}+t_0M\big[t_0\|A^{-1}\|_{\mathcal{L}(X)}+1\big].$$

This inequality implies, for all $t_0 > 0$, the embedding

$$X_A^{\theta} \hookrightarrow X_{t_0+A}^{\theta}.$$

Interchanging the roles of A and t_0+A, we obtain the identities

$$\begin{aligned}&A(t+A)^{-1}-(t_0+A)(t+t_0+A)^{-1}\\&=\big[A(t+A)^{-1}(t+A+t_0)(t_0+A)^{-1}-I\big](t_0+A)(t+t_0+A)^{-1}\end{aligned}$$

$$= \big[A(t_0+A)^{-1} + t_0 A(t_0+A)^{-1}(t+A)^{-1} - I\big](t_0+A)(t+t_0+A)^{-1}$$

Recalling that

$$\|(t+A)^{-1}\|_{\mathcal{L}(X)} \le c(1+t)^{-\beta} \le c,$$
$$\|A(t_0+A)^{-1}\|_{\mathcal{L}(X)} \le \|I - t_0(t_0+A)^{-1}\|_{\mathcal{L}(X)} \le 1 + ct_0^{1-\beta},$$

we deduce the estimate

$$\|A(t+A)^{-1}u\| \le \big[(1+ct_0^{1-\beta})(1+ct_0)+1\big]\|(t_0+A)\big(t+t_0+A\big)^{-1}u\|$$

Whence we deduce the following inequality holding for all $\theta \in (0,1)$:

$$\begin{aligned}&\sup_{t>0}(1+t)^{\theta}\|A(t+A)^{-1}u\| \\ &\le \big[(1+ct_0^{1-\beta})(1+ct_0)+1\big]\sup_{t>0}(1+t)^{\theta}\|(t_0+A)(t+t_0+A)^{-1}u\|.\end{aligned}$$

Whence we deduce the following inequality holding for all $\theta \in (0,1)$:

$$\sup_{t>0}(1+t)^{\theta}\|A(t+A)^{-1}u\| \le C' \sup_{t>0}(1+t)^{\theta}\|(t_0+A)(t+t_0+A)^{-1}u\|.$$

We have thus proved the reverse embedding, holding for all $t_0 > 0$ and all $\theta \in (0,1)$:

$$X^{\theta}_{t_0+A} \hookrightarrow X^{\theta}_{A}.$$

Finally, we have shown the first two relations in (7.8).

The third equality follows from the fact that A is a diagonal operator-matrix operator, so that, for all $t > 0$, we have

$$(tI+A)^{-1} = \begin{bmatrix} (tI+A_1)^{-1} & O & \dots & O \\ O & (tI+A_2)^{-1} & \dots & O \\ \dots & \dots & \dots & \dots \\ O & O & \dots & (tI+A_n)^{-1} \end{bmatrix}.$$

If we define the norm in X^n by

$$\|(x_1,\dots,x_n)\|_{X^n} = \sum_{j=1}^{n} \|x_j\|_X,$$

then

$$\sup_{t>0}(1+t)^{\theta}\|A(t+A)^{-1}x\|_{X^n} = \sup_{t>0}(1+t)^{\theta}\sum_{j=1}^{n}\|A_j(t+A_j)^{-1}x_j\|_X \le \sum_{j=1}^{n}\|x_j\|_{X^{\theta}_{A_j}}.$$

Therefore,

$$\prod_{j=1}^{n} X^{\theta}_{A_j} \hookrightarrow (X^n)^{\theta}_{A}.$$

Conversely, if $\sup_{t>0}(1+t)^{\theta}\|A(t+A)^{-1}x\|_{X^n} < +\infty$, then

$$\sup_{t>0}(1+t)^{\theta}\sum_{j=1}^{n}\|A_j(t+A_j)^{-1}x_j\|_X \le \sup_{t>0}(1+t)^{\theta}\|A(t+A)^{-1}x\|_{X^n} < +\infty,$$

for all $j = 1,\dots,n$, so that the embedding

$$(X^n)^{\theta}_{A} \hookrightarrow \prod_{j=1}^{n} X^{\theta}_{A_j}.$$

follows immediately.

Now we show that $X_A^\theta = X_{t_0+A}^\theta$. For this purpose first we consider the following identity obtained from (7.9) setting $B = O$:

$$(t_0 + A)(t + t_0 + A)^{-1} - (t_0 + A)(t + A)^{-1} = -t_0(t_0 + A)(t + t_0 + A)^{-1}(t + A)^{-1}.$$

Whence we deduce

$$\begin{aligned}
&\sup_{t>0}(1+t)^\theta \|(t_0 + A)(t + t_0 + A)^{-1}u\| \\
&\leq \|(t_0 + A)A^{-1}\|_{\mathcal{L}(X)} \sup_{t>0}(1+t)^\theta \|A(t + A)^{-1}u\| \\
&\quad + \sup_{t>0} M(1+t)^\theta (1 + t + t_0)^{-\beta} t_0 \|(t_0 + A)A^{-1}\|_{\mathcal{L}(X)} \|A(t + A)^{-1}u\| \\
&\leq C \sup_{t>0}(1+t)^\theta \|A(t + A)^{-1}u\|.
\end{aligned}$$

So, we have proved the embedding $X_A^\theta \hookrightarrow X_{t_0+A}^\theta$. Interchanging the roles of $t_0 + A$ and A, we obtain the set equality $X_A^\theta = X_{t_0+A}^\theta$ with equivalence of the corresponding norms.

Finally, the third embedding in (7.8) is obvious. □

Remark 7.3. Corollary 5.3 applies if the regularity assumptions on the data concern the spaces $(X, \mathcal{D}(A))_{\theta,\infty} = (X, \mathcal{D}(\lambda_0 + A + B))_{\theta,\infty}$. Notice that, if operator B, with $\mathcal{D}(A) \subset \mathcal{D}(B)$ satisfies the following estimate, similar to the ones satisfied by A (cf.(1.2), (1.3)):

$$\|(\lambda + \lambda_0 + A + B)^{-1}\|_{\mathcal{L}(X)} \leq c'(1 + |\lambda|)^{-\beta} \tag{7.10}$$

for all λ in the sector

$$\Sigma_\alpha := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -c'(1 + |\operatorname{Im}\lambda|)^\alpha\}, \quad 0 < \beta \leq \alpha \leq 1, \tag{7.11}$$

then $(X, \mathcal{D}(A))_{\theta,\infty} = (X, \mathcal{D}(\lambda_0 + A + B))_{\theta,\infty}$ with the equivalence of their norms.

8. Identification problems for singular non-classical first-order in time systems of PDE's corresponding to $\beta = 1$

In this section some applications related to the regular and singular parabolic equations will be given.

Problem 8.1. We will consider a problem related to a reaction diffusion model describing a man-environment epidemic system investigated in [6]. Such a model consists in a parabolic equation coupled with an ordinary differential equation via a boundary feedback operator (cf. also [9]). To obtain stability results in [6] the authors linearize the model and arrive at the following evolution system, where $u(t, x)$ and $v(t, x)$ stand, respectively, for the concentration of the infection agent and the density of the infective population at time t and point x:

$$\begin{aligned}
D_t u(t,x) &= \Delta u(t,x) - a(x)u(t,x) + f_1(t)z_{1,1}(x) + f_2(t)z_{1,2}(x),\\
&\qquad (t,x) \in (0,\tau)\times\Omega,\\
D_t v(t,x) &= c(x)u(t,x) - d(x)v(t,x) + f_1(t)z_{2,1}(x) + f_2(t)z_{2,2}(x),\\
&\qquad (t,x) \in (0,\tau)\times\Omega,\\
u(0,x) &= u_0(x), \quad v(0,x) = v_0(x), \quad x\in\Omega,\\
D_\nu u(t,x) + \beta(x)u(t,x) &= \int_\Omega k(x,y)v(t,y)\,dy, \quad (t,x)\in(0,\tau)\times\partial\Omega,\\
\int_\Omega u(t,x)\,d\mu_1(x) &= g_1(t), \quad t\in[0,\tau],\\
\int_\Omega v(t,x)\,d\mu_2(x) &= g_2(t), \quad t\in[0,\tau],
\end{aligned} \tag{8.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, Δ is the Laplacian, a, c, $d \in C(\overline{\Omega})$, $\beta\in C(\partial\Omega)$, $k\in W^{1,\infty}(\partial\Omega; L^\infty(\Omega))$ are non-negative functions and D_ν denotes the outward normal derivative on $\partial\Omega$. Finally, μ_1 and μ_2 are two positive Borel measure on $\overline{\Omega}$.

We define $E = C(\overline{\Omega})$, $X = E\times E$ and denote by M_h the multiplication operator induced by the function h. Moreover, we introduce the operator-matrix

$$\mathcal{A} = \begin{bmatrix} \Delta - M_a & O\\ M_c & -M_d\end{bmatrix}, \tag{8.2}$$

$$\begin{aligned}
\mathcal{D}(\mathcal{A}) = \Big\{(u,v)\in X : u\in H^2(\Omega), \quad \Delta u\in E,\\
D_\nu u(\cdot) + \beta(\cdot)u(\cdot) = \int_\Omega k(\cdot,y)v(y)\,dy \text{ on } \partial\Omega\Big\}.
\end{aligned} \tag{8.3}$$

It can be proved (cf. [9, p. 26]) that A generates an analytic semigroup on X with $\alpha=\beta=1$ and $\theta\in(0,1)$. Therefore we can apply Theorem 6.1 and its Corollaries 6.2, 6.3.

Let us assume that $(u_0,v_0)\in D(\mathcal{A})$, $((\Delta - a(\cdot))u_0, c(\cdot)u_0 - d(\cdot)v_0) \in (C(\overline{\Omega})\times C(\overline{\Omega}), D(\mathcal{A}))_{\theta,\infty}$,

$$\int_\Omega u_0(x)\mu_1(dx) = g_1(0), \quad \int_\Omega v_0(x)\,d\mu_2(x) = g_2(0),$$

$g_1, g_2 \in C^1([0,\tau];\mathbb{C})$, $z_{ik}\in C(\overline{\Omega})$, $i,k=1,2$, $(z_{11},z_{21}), (z_{12},z_{22}) \in (C(\overline{\Omega})\times C(\overline{\Omega}), \mathcal{D}(\mathcal{A}))_{\theta,\infty}$,

$$\int_\Omega z_{1,1}\,d\mu_1(x)\int_\Omega z_{2,2}\,d\mu_2(x) - \int_\Omega z_{1,2}\,d\mu_1(x)\int_\Omega z_{2,1}\,d\mu_2(x) \neq 0.$$

Then the identification problem (8.1), admits a unique global strict solution

$$((u,v), f_1, f_2)\in C([0,\tau]; D(\mathcal{A}))\times C([0,\tau];\mathbb{C})\times C([0,\tau];\mathbb{C})$$

such that

$$(D_t u, D_t v),\ \mathcal{A}(u,v)^T \in B([0,\tau]; (C(\overline{\Omega})\times C(\overline{\Omega}), \mathcal{D}(\mathcal{A}))_{\theta,\infty}).$$

We can characterize the interpolation space $(C(\overline{\Omega})\times C(\overline{\Omega}), D(\mathcal{A}))_{\theta,\infty}$ taking advantage of (cf. [21, Theorem 1.14.3, p. 93]) and the representation of $\mathcal{A}-\lambda I$ as a product of suitable operator matrices (cf. [9], p. 126).

Problem 8.2. Let us consider the weakly coupled identification vector problem occurring in the theory of semiconductors. Here we will deal with the problem consisting of recovering the three scalar functions $f_j : [0,\tau] \to \mathbb{R}$, $1 \le j \le 3$, in the singular problem

$$\begin{gathered} D_t u_1 = a\Delta u_1 - d\Delta u_3 + f_1(t)\zeta_1, \quad \text{in } (0,\tau)\times\Omega, \\ D_t u_2 = b\Delta u_2 + e\Delta u_3 + f_2(t)\zeta_2, \quad \text{in } (0,\tau)\times\Omega, \\ 0 = u_1 - u_2 - c\Delta u_3 + f_3(t)\zeta_3, \quad \text{in } (0,\tau)\times\Omega, \\ u_1(0,\cdot) = u_{0,1}, \quad u_2(0,\cdot) = u_{0,2}, \quad \text{in } \Omega, \\ u_1 = u_2 = u_3 = 0, \quad \text{in } (0,\tau)\times\partial\Omega, \end{gathered} \tag{8.4}$$

under the following three additional conditions

$$\langle u_i(t,\cdot),\varphi_i\rangle := \int_\Omega u_i(t,x)\varphi_i(x)\,dx = g_i(t), \quad t\in(0,\tau),\ i=1,2,3, \tag{8.5}$$

where $\zeta_i \in L^p(\Omega)$, $i=1,2,3$, $p\in(1,+\infty]$, $a,b\in\mathbb{R}_+$, $c,e\in\mathbb{R}\backslash\{0\}$, $d\in\mathbb{R}$ and $\varphi_i \in L^{p'}(\Omega)$, $1/p+1/p'=1$, $i=1,2,3$.

We notice that Theorem 7.2 cannot be directly applied to this identification problem, since such a problem is singular due to the lack of the term $D_t u_3$. However, since $\Delta : W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega) \to L^p(\Omega)$ is a linear isomorphism, we can solve the elliptic equation for u_3:

$$u_3 = c^{-1}\Delta^{-1}[u_1 - u_2 + f_3(t)\zeta_3], \quad \text{in } (0,\tau)\times\Omega. \tag{8.6}$$

Assume now

$$\chi_3^{-1} := (\Delta^{-1}\zeta_3, \varphi_3)_{L^2(\Omega)} \neq 0. \tag{8.7}$$

Consequently, from (8.6) and the additional equation $(u_3(t,\cdot),\varphi_3)_{L^2(\Omega)} = g_3(t)$ we deduce the following formula for f_3:

$$f_3(t) = c\chi_3 g_3(t) - \chi_3\langle\Delta^{-1}(u_1-u_2)(t,\cdot),\varphi_3\rangle. \tag{8.8}$$

Therefore, our inverse problem is equivalent to the following problem:

$$\begin{gathered} D_t u_1 = a\Delta u_1 + \Big[-dc^{-1}u_1 + dc^{-1}\chi_3\langle\Delta^{-1}u_1,\varphi_3\rangle\zeta_3 + dc^{-1}u_2 \\ - dc^{-1}\chi_3\langle\Delta^{-1}u_2,\varphi_3\rangle\zeta_3\Big] - d\chi_3 g_3(t)\zeta_3 + f_1(t)\zeta_1, \quad \text{in } (0,\tau)\times\Omega, \\ D_t u_2 = b\Delta u_2 + \Big[ec^{-1}u_1 - ec^{-1}\chi_3\langle\Delta^{-1}u_1,\varphi_3\rangle\zeta_3 - ec^{-1}u_2 \\ + ec^{-1}\chi_3\langle\Delta^{-1}u_2,\varphi_3\rangle\zeta_3\Big] + e\chi_3 g_3(t)\zeta_3 + f_2(t)\zeta_2, \quad \text{in } (0,\tau)\times\Omega, \\ u_1(0,\cdot) = u_{0,1}, \quad u_2(0,\cdot) = u_{0,2}, \quad \text{in } \Omega, \\ u_1 = u_2 = 0, \quad \text{in } (0,\tau)\times\partial\Omega, \\ \langle u_i(t,\cdot),\varphi_i\rangle = g_i(t), \quad t\in(0,\tau),\ i=1,2. \end{gathered} \tag{8.9}$$

Define $\{e^{t\Delta}\}_{t>0}$ as the analytic semigroup generated by Δ with the domain $\Delta : W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega) \to L^p(\Omega)$ and observe that the semigroups $\{T_1(t)\}_{t>0}$ and $\{T_2(t)\}_{t>0}$ generated by $a\Delta$ and $b\Delta$ are defined, respectively, by

$$T_1(t) = e^{at\Delta}, \quad T_2(t) = e^{bt\Delta}. \tag{8.10}$$

In this case we have

$$X_A^\theta = (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty} \times (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty}.$$

Such spaces were characterized by Grisvard (a proof can be found, for the reader's convenience, in [21, p. 321]). Then we define

$$B_{1,1}u_1 = -dc^{-1}u_1 + dc^{-1}\chi_3\langle \Delta^{-1}u_1, \varphi_3\rangle\zeta_3, \tag{8.11}$$

$$B_{1,2}u_2 = dc^{-1}u_2 - dc^{-1}\chi_3\langle \Delta^{-1}u_2, \varphi_2\rangle\zeta_3, \tag{8.12}$$

$$B_{2,1}u_1 = ec^{-1}u_1 - ec^{-1}\chi_3\langle \Delta^{-1}u_1, \varphi_3\rangle\zeta_3, \tag{8.13}$$

$$B_{2,2}u_2 = -ec^{-1}u_2 + ec^{-1}\chi_3\langle \Delta^{-1}u_2, \varphi_3\rangle\zeta_3, \tag{8.14}$$

$$h_1(t) = -d\chi_3 g_3(t)\zeta_3, \quad h_2(t) = e\chi_3 g_3(t)\zeta_3, \tag{8.15}$$

$$z_{1,1} = \zeta_1, \quad z_{2,2} = \zeta_2, \quad z_{1,2} = z_{2,1} = 0. \tag{8.16}$$

Assume further that

$$\Big[a\Delta u_{0,1} - dc^{-1}u_{0,1} + dc^{-1}\chi_3\langle \Delta^{-1}u_{0,1}, \varphi\rangle\zeta_3 + dc^{-1}u_{0,2} - dc^{-1}\chi_3\langle \Delta^{-1}u_{0,2}, \varphi_3\rangle\zeta_3\Big] \in (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty}, \tag{8.17}$$

$$\Big[ec^{-1}u_{0,1} - ec^{-1}\chi_3\langle \Delta^{-1}u_{0,1}, \varphi_3\rangle\zeta_3 + b\Delta u_{0,2} - ec^{-1}u_{0,2} + ec^{-1}\chi_3\langle \Delta^{-1}u_{0,2}, \varphi_3\rangle\zeta_3\Big] \in (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty}, \tag{8.18}$$

$$u_{0,1}, u_{0,2} \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega), \quad \zeta_3 \in (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty}, \tag{8.19}$$

$$\langle\zeta_1, \varphi_1\rangle\langle\zeta_2, \varphi_2\rangle\langle\Delta^{-1}\zeta_3, \varphi_3\rangle \neq 0. \tag{8.20}$$

Then we can apply Theorem 7.2 with $(\alpha, \beta) = (1, 1)$, to problem (8.9) to ensure that there exists a quadruplet $(u_1, u_2, f_1, f_2) \in \mathcal{X} = \big\{C^1([0,\tau]; L^p(\Omega)^2) \cap C([0,\tau]; \big[W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\big]^2)\big\} \times C([0,\tau]; \mathbb{C}^2)$ solving (8.9). Finally, we observe that the pair (u_3, f_3) is defined by formulae (8.6) and (8.7). Therefore it belongs to $C([0,\tau]; W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)) \times C([0,\tau]; \mathbb{C})$.

The same technique applies when our additional information is

$$(u_i(t,\cdot), \varphi_i)_{L^2(\Omega)} = g_i(t),\ i = 1,2, \quad (u_1(t,\cdot), \varphi_3)_{L^2(\Omega)} = g_3(t), \quad t \in (0,\tau).$$

In this case the solvability condition changes to

$$\int_\Omega \varphi_2(x)\zeta_2(x)\,dx \begin{bmatrix} \int_\Omega \varphi_1(x)\zeta_1(x)\,dx & \int_\Omega \varphi_1(x)\zeta_3(x)\,dx \\ \int_\Omega \varphi_3(x)\zeta_1(x)\,dx & \int_\Omega \varphi_3(x)\zeta_3(x)\,dx \end{bmatrix} \neq 0. \tag{8.21}$$

In fact, from (8.6), we easily derive the new identification problem

$$\begin{gathered} D_t u_1 = \big(a\Delta - dc^{-1}\big)u_1 + bc^{-1}u_2 + f_1(t)\zeta_1 - dc^{-1}f_3(t)\zeta_3, \quad \text{in } (0,\tau)\times\Omega, \\ D_t u_2 = ec^{-1}u_1 + \big(b\Delta - ec^{-1}\big)u_2 + f_2(t)\zeta_2 + ec^{-1}f_3(t)\zeta_3, \quad \text{in } (0,\tau)\times\Omega, \\ u_1(0,\cdot) = u_{0,1}, \quad u_2(0,\cdot) = u_{0,2}, \quad \text{in } \Omega, \\ u_1 = u_2, \quad \text{in } (0,\tau)\times\partial\Omega, \\ (u_i(t,\cdot), \varphi_i)_{L^2(\Omega)} = g_i(t), \quad t \in (0,\tau),\ i = 1,2, \\ (u_1(t,\cdot), \varphi_3)_{L^2(\Omega)} = g_3(t), \quad t \in (0,\tau). \end{gathered} \tag{8.22}$$

Now Corollary 5.3 applies if the following solvability condition is satisfied

$$\begin{vmatrix} \int_\Omega \varphi_1(x)\zeta_1(x)\,dx & 0 & -dc^{-1}\int_\Omega \varphi_1(x)\zeta_3(x)\,dx \\ 0 & \int_\Omega \varphi_2(x)\zeta_2(x)\,dx & ec^{-1}\int_\Omega \varphi_2(x)\zeta_3(x)\,dx \\ \int_\Omega \varphi_3(x)\zeta_1(x)\,dx & 0 & -dc^{-1}\int_\Omega \varphi_3(x)\zeta_3(x)\,dx \end{vmatrix} \neq 0.$$

But this condition is nothing but (8.21). However, notice that the consistency conditions

$$(u_{0,i}(t,\cdot),\varphi_i)_{L^2(\Omega)} = g_i(0),\ i=1,2, \quad (u_{0,1}(t,\cdot),\varphi_3)_{L^2(\Omega)} = g_3(0),$$

must hold.

Assume now that the boundary condition involving u_3 is changed to the Neumann one, i.e. $D_\nu u_3 = 0$ on $(0,\tau)\times\partial\Omega$, where ν and D_ν denote, respectively, the outward unit vector normal to $\partial\Omega$ and the normal derivative on $\partial\Omega$. Then the elliptic problem

$$\begin{gathered} 0 = u_1 - u_2 - c\Delta u_3 + f_3(t)\zeta_3, \quad \text{in } (0,\tau)\times\Omega, \\ D_\nu u_3 = 0, \quad \text{in } (0,\tau)\times\partial\Omega, \end{gathered} \tag{8.23}$$

admits a unique solution in $W^{2,p}(\Omega)$, if and only if the following condition is satisfied

$$f_3(t)\langle\zeta_3,1\rangle = -\langle(u_1-u_2)(t,\cdot),1\rangle, \quad t\in[0,\tau],$$

where $\langle h,1\rangle = \int_\Omega h(x)\,dx$, $h\in L^1(\Omega)$. Assuming that

$$\chi_3^{-1} := \langle\zeta_3,1\rangle \neq 0,$$

we obtain

$$f_3(t) = -\chi_3\langle(u_1-u_2)(t,\cdot),1\rangle, \quad t\in[0,\tau]. \tag{8.24}$$

Note that in this case we can get rid off of the third additional condition in (8.5). Consequently, an equivalent problem for (u_1,u_2,f_1,f_2) turns out to be the following:

$$\begin{gathered} D_t u_1 = a\Delta u_1 - dc^{-1}\Big[u_1 - u_2 - \chi_3\zeta_3\langle(u_1-u_2),1\rangle\Big] + f_1(t)\zeta_1, \quad \text{in } (0,\tau)\times\Omega, \\ D_t u_2 = b\Delta u_2 + ec^{-1}\big[u_1 - u_2 - \chi_3\zeta_3\langle(u_1-u_2),1\rangle\big] + f_2(t)\zeta_2, \quad \text{in } (0,\tau)\times\Omega, \\ u_1(0,\cdot) = u_{0,1}, \quad u_2(0,\cdot) = u_{0,2}, \quad \text{in } \Omega, \\ u_1 = u_2 = 0, \quad \text{in } (0,\tau)\times\partial\Omega, \\ \langle u_i(t,\cdot),\varphi_i\rangle = g_i(t), \quad t\in(0,\tau),\ i=1,2. \end{gathered} \tag{8.25}$$

Then we define

$$\begin{gathered} B_1(u_1,u_2) = -dc^{-1}\big[u_1 - u_2 - \chi_3\zeta_3\langle(u_1-u_2),1\rangle\big], \\ B_2(u_1,u_2) = ec^{-1}\big[u_1 - u_2 - \chi_3\zeta_3\langle(u_1-u_2),1\rangle\big], \\ h_1(t) = h_2(t) = 0, \\ z_{1,1} = \zeta_1, \quad z_{2,2} = \zeta_2, \quad z_{1,2} = z_{2,1} = 0. \end{gathered}$$

Assume further

$$\begin{gathered} a\Delta u_{0,1} - dc^{-1}\big[u_{0,1} - u_{0,2} - \chi_3\zeta_3\langle(u_{0,1}-u_{0,2}),1\rangle\big] \\ \in (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty}, \\ b\Delta u_{0,2} - dc^{-1}\big[u_{0,1} - u_{0,2} - \chi_3\zeta_3\langle(u_{0,1}-u_{0,2}),1\rangle\big] \\ \in (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty}, \\ u_{0,1},u_{0,2}\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega), \quad \zeta_3\in (L^p(\Omega); W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))_{\theta,\infty}, \\ \langle\zeta_1,\varphi_1\rangle\langle\zeta_2,\varphi_2\rangle\langle\zeta_3,1\rangle \neq 0. \end{gathered} \tag{8.26}$$

Then we can apply Theorem 7.2 with $(\alpha,\beta)=(1,1)$, to problem (8.25) to ensure that there exists a quadruplet

$$
\begin{aligned}
&(u_1,u_2,f_1,f_2)\in\mathcal{X}\\
&=\big\{C^1([0,\tau];L^p(\Omega)^2)\cap C^1([0,\tau];\big[W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\big]^2)\big\}\times C([0,\tau];\mathbb{C}^2)
\end{aligned}
$$

solving (8.9). Finally, we observe that the pair (u_3,f_3) is defined by formulae (8.6) and (8.24). Therefore it belongs to $C([0,\tau];W^{2,p}(\Omega))\times C([0,\tau];\mathbb{R})$.

Now we change the boundary conditions in the previous direct problem to the following ones of mixed Dirichlet-Neumann type

$$u_1=u_2=u_3=0,\quad \text{in } (0,\tau)\times\Gamma_D, \tag{8.27}$$

$$D_\nu u_1=D_\nu u_2=D_\nu u_3=0,\quad \text{in } (0,\tau)\times\Gamma_N. \tag{8.28}$$

Here Γ_D is an non-empty open subset and $\Gamma_N=\partial\Omega\setminus\Gamma_D$. Moreover, Ω must satisfy the exterior sphere condition

$$m_n(B(x_0,R)\cap\Omega^c)\ge cR^n,\quad \forall x_0\in\partial\Omega, \tag{8.29}$$

m_n denoting the n-dimensional Lebesgue measure. In particular the latter property holds if $\partial\Omega$ is Lipschitz (cf. [5, 24]). Explicitly, we consider the identification problem (8.11) consisting in recovering the three scalar functions $f_i\in C([0,\tau];\mathbb{C})$ such that

$$
\begin{gathered}
D_t u_1=a\Delta u_1-d\Delta u_3+f_1(t)\zeta_1,\quad \text{in } (0,\tau)\times\Omega,\\
D_t u_2=b\Delta u_2+e\Delta u_3+f_2(t)\zeta_2,\quad \text{in } (0,\tau)\times\Omega,\\
0=u_1-u_2-c\Delta u_3+f_3(t)\zeta_3,\quad \text{in } (0,\tau)\times\Omega,\\
u_1(0,\cdot)=u_{0,1},\quad u_2(0,\cdot)=u_{0,2},\quad \text{in } \Omega,\\
u_1=u_2=u_3=0,\quad \text{in } (0,\tau)\times\Gamma_D,\\
D_\nu u_1=D_\nu u_2=D_\nu u_3=0,\quad \text{in } (0,\tau)\times\Gamma_N,\\
{}_{H^1_D(\Omega)}\langle u_i(t,\cdot),\varphi_i\rangle_{H^1_D(\Omega)^*}=g_i(t),\quad t\in(0,\tau),\ i=1,2,3,
\end{gathered}
\tag{8.30}
$$

for given $\varphi_i\in H^1_D(\Omega)$ and $g_i\in C^1([0,\tau];\mathbb{C})$, $i=1,2,3$.

Identifying $L^2(\Omega)$ with its antidual space, we introduce the Hilbert space

$$H^1_D(\Omega)=\{u\in H^1(\Omega): u=0 \text{ on } \Gamma_D\} \tag{8.31}$$

and we denote its antidual space by $H_D(\Omega)^*$. Then we define the linear operator $\Lambda\in\mathcal{L}(H^1_D(\Omega);H^1_D(\Omega)^*)$ by the bilinear form

$$(\Lambda u,v)_{L^2(\Omega)}=\int_\Omega \nabla u\cdot\overline{\nabla v}\,dx,\quad u\in H^1_D(\Omega),\ v\in H^1_D(\Omega)^*. \tag{8.32}$$

We note that Λ is the realization of $-\Delta$ in $H^1_D(\Omega)^*$ under the homogeneous Dirichlet condition on Γ_D and the homogeneous Neumann condition on Γ_N and that $-\Lambda$ generates an analytic semigroup on $H^1_D(\Omega)^*$ (cf. [5], [15, p.114] and [24]). Moreover, $-\Lambda$ is an isomorphism from $H^1_D(\Omega)$ to $H^1_D(\Omega)^*$.

Let us observe that, for any $\theta\in[1/2,1]$, we have

$$
\begin{aligned}
D(\Lambda^\theta)&=[L^2(\Omega),H^1_D(\Omega)]_{2\theta-1}=\big[[H^1_D(\Omega)^*,H^1_D(\Omega)]_{1/2},H^1_D(\Omega)\big]_{2\theta-1}\\
&=[H^1_D(\Omega)^*,H^1_D(\Omega)]_{1/2-(2\theta-1)/2}\\
&=[H^1_D(\Omega)^*,H^1_D(\Omega)]_\theta\hookrightarrow(H^1_D(\Omega)^*,H^1_D(\Omega))_{\theta,\infty}.
\end{aligned}
$$

Notice that $\Lambda u \in L^2(\Omega)$ needs not to imply $u \in H^2(\Omega)$ due to the boundary conditions of mixed type, while $D(\Lambda^{1/2}) = L^2(\Omega)$. Choose now $X = H^1_D(\Omega)^*$. Then from the equation

$$c\Lambda u_3 = -u_1 + u_2 - f_3(t)\zeta_3$$

we have

$$u_3 = -c^{-1}\Lambda^{-1}u_1 + c^{-1}\Lambda^{-1}u_2 - c^{-1}\Lambda^{-1}f_3(t)\zeta_3$$

and

$$\begin{aligned} g_3(t) &= {}_{H^1_D(\Omega)}\langle u_3(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*} \\ &= -c^{-1}_{H^1_D(\Omega)}\langle \Lambda^{-1}u_1(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*} + c^{-1}_{H^1_D(\Omega)}\langle \Lambda^{-1}u_2(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*} \\ &\quad - c^{-1}f_3(t)_{H^1_D(\Omega)}\langle \Lambda^{-1}\zeta_3, \varphi_3\rangle_{H^1_D(\Omega)^*}. \end{aligned}$$

If

$$\eta^{-1} =: {}_{H^1_D(\Omega)}\langle \Lambda^{-1}\zeta_3, \varphi_3\rangle_{H^1_D(\Omega)^*} \neq 0,$$

the latter equation uniquely determines f_3 as

$$f_3(t) = -c\eta g_3(t) - \eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_1(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*} + \eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_2(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*}.$$

Whence we easily deduce the formula

$$\begin{aligned} \Lambda u_3 &= -c^{-1}u_1 + c^{-1}u_2 + c^{-1}\eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_1(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*}\zeta_3 \\ &\quad - c^{-1}\eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_2(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*}\zeta_3 + \eta g_3(t)\zeta_3. \end{aligned}$$

So, problem (8.30) reduces to the following

$$\begin{gathered} D_t u_1 = -a\Lambda u_1 - c^{-1}du_1 + c^{-1}du_2 + c^{-1}d\eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_1(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*}\zeta_3 \\ - c^{-1}d\eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_2(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*}\zeta_3 + d\eta g_3(t)\zeta_3 + f_1(t)\zeta_1, \\ \text{in } (0,\tau)\times\Omega, \\ D_t u_2 = -b\Lambda u_2 + c^{-1}eu_1 - c^{-1}eu_2 - c^{-1}e\eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_1(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*}\zeta_3 \\ + c^{-1}e\eta_{H^1_D(\Omega)}\langle \Lambda^{-1}u_2(t,\cdot), \varphi_3\rangle_{H^1_D(\Omega)^*}\zeta_3 - e\eta g_3(t)\zeta_3 + f_2(t)\zeta_2, \\ \text{in } (0,\tau)\times\Omega, \\ u_1(0,\cdot) = u_{0,1}, \quad u_2(0,\cdot) = u_{0,2}, \quad \text{in } \Omega, \\ {}_{H^1_D(\Omega)}\langle u_i(t,\cdot), \varphi_i\rangle_{H^1_D(\Omega)^*} = g_i(t), \quad t \in (0,\tau),\ i = 1,2. \end{gathered} \tag{8.33}$$

Now assume that the data $(\varphi_1, \varphi_2, \varphi_3, \zeta_1, \zeta_2, \zeta_3, u_{0,1}, u_{0,2}, g_1, g_2, g_3)$ satisfy the following properties:

$$\begin{gathered} \varphi_i \in H^1_D(\Omega), \quad \zeta_i \in (H^1_D(\Omega)^*, H^1_D(\Omega))_{\theta,\infty}, \quad i = 1,2,3; \\ u_{0,1}, u_{0,2} \in H^1_D(\Omega), \quad \Delta u_{0,1}, \Delta u_{0,2} \in (H^1_D(\Omega)^*, H^1_D(\Omega))_{\theta,\infty}, \\ g_1, g_2, g_3 \in C([0,\tau];\mathbb{R}), \\ {}_{H^1_D(\Omega)}\langle u_{i,0}, \varphi_i\rangle_{H^1_D(\Omega)^*} = g_i(0), \quad i = 1,2,3, \\ {}_{H^1_D(\Omega)}\langle \Lambda^{-1}\zeta_3, \varphi_3\rangle_{H^1_D(\Omega)^*}\ {}_{H^1_D(\Omega)}\langle \zeta_1, \varphi_1\rangle_{H^1_D(\Omega)^*}\ {}_{H^1_D(\Omega)}\langle \zeta_2, \varphi_2\rangle_{H^1_D(\Omega)^*} \neq 0. \end{gathered}$$

Then, according to Theorem 7.2, with $\alpha = \beta = 1$, we can conclude that the identification problem (8.30) admits a unique solution

$$\begin{gathered} (u_1, u_2, u_3, f_1, f_2, f_3) \in C([0,\tau];[H^1_D(\Omega)^*]^3) \times C([0,\tau];\mathbb{C}), \\ D_t u_1, D_t u_2 \in B([0,\tau];(H^1_D(\Omega)^*, H^1_D(\Omega))_{\theta,\infty}), \end{gathered}$$

Remark 8.1. Note that sufficient conditions could be deduced simply by replacing the interpolation space $(H_D^1(\Omega)^*, H_D^1(\Omega))_{\theta,\infty}$ with $\mathcal{D}(\Lambda^\theta)$, since $\mathcal{D}(\Lambda^\theta)$ for $\theta \in [1/2, 1]$ coincides with the complex interpolation space $[H_D^1(\Omega)^*, H_D^1(\Omega)]_\theta$, which is included in $(H_D^1(\Omega)^*, H_D^1(\Omega))_{\theta,\infty}$, as we have already pointed out.

Problem 8.3. Let us consider the identification problem consisting of recovering the m scalar functions $f_j : [0, \tau] \to \mathbb{R}$ in the singular problem

$$\begin{gathered} D_t u = a_{1,1}\Delta u + a_{1,2}\Delta v + b_{1,1}(x)u + b_{1,2}(x)v + h_1(t,x) + \sum_{j=1}^m f_j(t) z_{1,j}, \\ \text{in } (0,\tau)\times\Omega, \\ D_t v = a_{2,1}\Delta u + a_{2,2}\Delta v + b_{2,1}(x)u + b_{2,2}(x)v + h_2(t,x) + \sum_{j=1}^m f_j(t) z_{2,j}, \\ \text{in } (0,\tau)\times\Omega, \\ u(0,\cdot) = u_{0,1}, \quad v(0,\cdot) = u_{0,2}, \quad \text{in } \Omega, \\ u = v = 0, \quad \text{in } (0,\tau)\times\partial\Omega, \end{gathered} \tag{8.34}$$

under the following m additional conditions

$$\Psi_j[u(t,\cdot)] = g_j(t), \quad t \in (0,\tau),\ j = 1,\dots,r, \tag{8.35}$$

$$\Psi_j[v(t,\cdot)] = g_j(t), \quad t \in (0,\tau),\ j = r+1,\dots,m, \tag{8.36}$$

where Ω is a (possibly unbounded) domain in $\mathbb{R}^n$ with a smooth boundary, $a_{i,j} \in \mathbb{R}$, $b_{i,j} \in C(\overline{\Omega};\mathbb{R})$, $i,j = 1,2$. Therefore, choosing $X_0 = L^p(\Omega)$, $p \in (1,+\infty)$, and $\mathcal{D}(\Delta) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, the well known resolvent estimates for our operator Δ hold in $L^p(\Omega)$, so that Δ generates an analytic semigroup of linear bounded operators.

To develop our strategy, we generalize to the case $p \in (1,+\infty)$ the results proved for $p = 2$ in [8]. For this purpose we introduce in the space $X = L^p(\Omega)\times L^p(\Omega)$ the linear unbounded operator $\mathcal{A}$ defined by

$$\mathcal{A} = \begin{bmatrix} a_{1,1}\Delta & a_{1,2}\Delta \\ a_{2,1}\Delta & a_{2,2}\Delta \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(\Delta)\times\mathcal{D}(\Delta). \tag{8.37}$$

Some simple algebraic computations yield the following formula for the resolvent

$$\begin{aligned} &(\mathcal{A}-\lambda I)^{-1} \\ &= \begin{bmatrix} a_{2,2}\Delta - \lambda I & -a_{1,2}\Delta \\ -a_{2,1}\Delta & a_{1,1}\Delta - \lambda I \end{bmatrix} \big[(a_{1,1}\Delta - \lambda I)(a_{2,2}\Delta - \lambda I) - a_{1,2}a_{2,1}\Delta^2\big]^{-1}. \end{aligned} \tag{8.38}$$

Observe that the determinant operator $D = (a_{1,1}\Delta - \lambda I)(a_{2,2}\Delta - \lambda I) - a_{1,2}a_{2,1}\Delta^2$ coincides with

$$D = \lambda^2 I - \lambda(a_{1,1} + a_{2,2})\Delta + (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})\Delta^2. \tag{8.39}$$

Suppose now that

$$a_{1,1} \ge 0, \quad a_{2,2} \ge 0, \quad a_{1,1} + a_{2,2} > 0, \quad a_{1,1}a_{2,2} - a_{1,2}a_{2,1} > 0, \tag{8.40}$$

$$(a_{1,1} - a_{2,2})^2 + 4a_{1,2}a_{2,1} \ge 0. \tag{8.41}$$

We note that the last inequality in (8.40) can be weakened to $\ge$ if Ω is bounded, while condition (8.41), not required in [8, Lemma 1, p.185], is necessary to ensure that the equation $\lambda^2 - \lambda(a_{1,1} + a_{2,2}) + (a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) = 0$ admits two real solutions $0 < \lambda_1 \le \lambda_2$, since (cf. (8.41))

$$(a_{1,1} + a_{2,2})^2 - 4(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) = (a_{1,1} - a_{2,2})^2 + 4a_{1,2}a_{2,1} \ge 0. \tag{8.42}$$

Now, in contrast with [8] we use the factorization $D = (\lambda - \lambda_1\Delta)(\lambda - \lambda_2\Delta)$. From the identity

$$D = \lambda_1\lambda_2(\lambda_1^{-1}\lambda - \Delta)(\lambda_2^{-1}\lambda - \Delta) \tag{8.43}$$

we deduce the resolvent estimate

$$\|(\mathcal{A} - \lambda I)^{-1}\|_{\mathcal{L}(X)} \le C(1 + |\lambda|)^{-1}, \quad \operatorname{Re}\lambda \ge 0. \tag{8.44}$$

Therefore, $\mathcal{A}$ generates an analytic semigroup on X. On the other hand the matrix operator

$$\mathcal{B} = \begin{bmatrix} b_{1,1}I & b_{1,2}I \\ b_{2,1}I & b_{2,2}I \end{bmatrix} \tag{8.45}$$

belongs to $\mathcal{L}(X)$, so that $\mathcal{A} + \mathcal{B}$, with $\mathcal{D}(\mathcal{A} + \mathcal{B}) = \mathcal{D}(\mathcal{A})$, generates an analytic semigroup on X, too. Now we make the following assumptions:

$$\det \begin{bmatrix} \Psi_1[z_{1,1}] & \dots & \Psi_1[z_{1,m}] \\ \dots & \dots & \dots \\ \Psi_r[z_{1,1}] & \dots & \Psi_r[z_{1,m}] \\ \Psi_{r+1}[z_{2,1}] & \dots & \Psi_{r+1}[z_{2,m}] \\ \dots & \dots & \dots \\ \Psi_m[z_{2,1}] & \dots & \Psi_1[z_{2,m}] \end{bmatrix} \ne 0,$$

$$u_0, v_0 \in \mathcal{D}(\Delta), \quad (\mathcal{A} + \mathcal{B})(u_0, v_0)^T \in (X, \mathcal{D}(\Delta))_{\theta,\infty} \times (X, \mathcal{D}(\Delta))_{\theta,\infty} := Z_\theta \times Z_\theta,$$
$$(z_{1,j}, z_{2,j})^T \in Z_\theta, \quad g_j \in C^1([0,\tau];\mathbb{R}),\ \varphi_j \in L^q(\Omega),\ j = 1,\dots,m,\ 1/p + 1/q = 1,$$
$$(h_1, h_2)^t \in C([0,\tau]; Z_\theta \times Z_\theta).$$

The characterization of the space Z_θ can be found in [21, Theorem 4.4.1, p.321].

Now we define the linear bounded functionals Ψ_j, $j = 1,\dots,m$, by

$$\Psi_j[u] = \int_\Omega \varphi_j(x)u(x)\,dx. \tag{8.46}$$

Then, by Corollary 5.2 we conclude that the identification problem (8.34)–(8.36) admits a strict solution $(u, v, f_1, \dots, f_m)$ with the following additional regularity:

$$D_t u, D_t v \in B([0,\tau]; Z_\theta),$$
$$a_{1,1}\Delta u + a_{1,2}\Delta v + b_{1,1}(\cdot)u + b_{1,2}(\cdot)v \in C([0,\tau]; L^p(\Omega)) \cap B([0,\tau]; Z_\theta),$$
$$a_{2,1}\Delta u + a_{2,2}\Delta v + b_{2,1}(\cdot)u + b_{2,2}(\cdot)v \in C([0,\tau]; L^p(\Omega)) \cap B([0,\tau]; Z_\theta).$$

Observe that this strategy works also if $L^p(\Omega)$ is replaced with $C(\overline{\Omega})$ and related functionals $\Psi_i \in C(\overline{\Omega})^*$.

9. Identification problems for PDE's corresponding to $\beta \in (0,1)$

Problem 9.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^∞-boundary $\partial\Omega$. We want to recover the scalar functions $f_j : [0,\tau] \to \mathbb{C}$, $j = 1,\dots,m$, in the initial boundary value problem

$$\frac{\partial u}{\partial t}(t,x) + A(x, D_x)u(t,x) = \sum_{j=1}^m f_j(t)z_j(x), \quad (t,x) \in (0,\tau) \times \Omega, \tag{9.1}$$

$$u(0,x)=u_0(x),\quad x\in\Omega, \tag{9.2}$$
$$a(x)D_\nu u(t,x)+\alpha(x)\cdot\nabla u(t,x)+b(t,x)u(t,x)=0,\quad (t,x)\in(0,\tau)\times\partial\Omega, \tag{9.3}$$

under the m additional pieces of information

$$\int_\Omega \eta_i(x)u(t,x)dx=g_i(t),\quad 0\le t\le\tau,\ i=1,\dots,m, \tag{9.4}$$

along with the consistency conditions

$$\int_\Omega \eta_i(x)u_0(x)dx=g_i(0),\quad i=1,\dots,m. \tag{9.5}$$

Here

$$-A(x,D_x)=\sum_{i,j=1}^n a_{i,j}(x)D_{x_i}D_{x_j}+\sum_{i=1}^n a_i(x)D_{x_i}+a_0(x) \tag{9.6}$$

is a second-order elliptic differential operator with real-valued C^∞-coefficients on $\overline{\Omega}$ such that

$$a_{j,i}(x)=a_{i,j}(x),\quad \sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j\ge c_0|\xi|^2,\quad (x,\xi)\in\overline{\Omega}\times\mathbb{R}^n, \tag{9.7}$$

c_0 being a positive constant. Concerning the linear boundary differential operator defined, for all $(t,x)\in(0,\tau)\times\partial\Omega$, by

$$\widehat{A}(x,D_x)u(t,x)=a(x)D_\nu u(t,x)+\alpha(x)\cdot\nabla u(t,x)+b(x)u(t,x), \tag{9.8}$$

we assume that a, b and α are real-valued C^∞-functions and a vector field on $\partial\Omega$, respectively, such that $Tu=\alpha\cdot\nabla u$ is a real C^∞-tangential operator on $\partial\Omega$, D_ν standing for the conormal derivative associated with the matrix $\big(a_{i,j}(x)\big)$; i.e.,

$$D_\nu=\Big(\sum_{i,j=1}^n a_{i,j}(x)n_i(x)n_j(x)\Big)^{-1}\sum_{i,j=1}^n a_{i,j}(x)D_{x_i}, \tag{9.9}$$

$n(x)=\big(n_1(x),\dots,n_n(x)\big)$ denoting the outward unit normal vector to $\partial\Omega$ at x.

Assume further (cf. [20, p. 515] that the vector field α does not vanish on $\Gamma_0=\{x\in\partial\Omega: a(x)=0\}$ and the function $t\to a(x(t,x_0))$ has zeros of even order not exceeding some value $2k_1$ along the integral curve $x'(t,x_0)=\alpha(x(t,x_0))$ satisfying the initial condition $x(0,x_0)=x_0$, with $x_0\in\Gamma_0$. In other words, the so-called $(H)_\delta$-condition holds with $\delta=\delta_1=(1+2k_1)^{-1}$. It is shown on p. 516 in [20] that the operator L defined by

$$\mathcal{D}(L)=\big\{u\in L^2(\Omega):A(\cdot,D_x)u\in L^2(\Omega),\ \widehat{A}(\cdot,D_x)u=0 \text{ on } \partial\Omega\big\}, \tag{9.10}$$
$$Lu=A(\cdot,D_x)u,\quad u\in\mathcal{D}(L). \tag{9.11}$$

satisfies in $L^2(\Omega)$ the resolvent estimate

$$\|(\lambda+L)^{-1}\|_{\mathcal{L}(L^2(\Omega))}\le C(1+|\lambda|)^{-(1+\delta)/2} \tag{9.12}$$

for all λ with a large enough modulus belonging to the sector

$$\Sigma_\varphi:=\{\lambda\in\mathbb{C}\setminus\{0\}:|\arg\lambda|\le\varphi\},\quad \varphi\in(\pi/2,\pi). \tag{9.13}$$

If we consider the subelliptic case $k=(1+\delta)/2>1/2$, we can immediately apply Corollary 5.3.

Problem 9.2. Here we deal with a problem - similar to Problem 9.1 - in the reference space of Hölder-continuous functions $X = \big(C^\alpha(\overline{\Omega}), \|\cdot\|_\alpha\big)$, when $\alpha \in (0,1)$ and the boundary $\partial\Omega$ of Ω is of class C^{4m} for some positive integer m. In this case the linear differential operator A is defined by

$$\mathcal{D}(A) = \big\{u \in C^{2m+\alpha}(\overline{\Omega}) : D^\gamma u = 0 \text{ on } \partial\Omega,\ |\gamma| \le m-1\big\}, \tag{9.14}$$

$$Au(x) = \sum_{|\gamma|\le 2m} a_\gamma(x) D^\gamma u(x), \quad x \in \Omega,\ u \in \mathcal{D}(A), \tag{9.15}$$

where β is a usual multi-index with $|\beta| = \sum_{j=1}^n \beta_j$ and $D^\beta = \prod_{j=1}^n (-iD_{x_j})^{\beta_j}$.

Assume that the coefficients $a_\gamma : \overline{\Omega} \to \mathbb{C}$ of A satisfy the following conditions

(i) $a_\gamma \in C^\alpha(\overline{\Omega};\mathbb{C})$ for all $|\gamma| \le 2m$;
(ii) $a_\gamma(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\gamma| = 2m$;
(iii) there exists a positive constant $M \ge 1$ such that

$$M^{-1}|\xi|^{2m} \le \sum_{|\gamma|=2m} a_\gamma(x)\xi^\gamma \le M|\xi|^{2m}, \quad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n. \tag{9.16}$$

Then there exist $\lambda, \varepsilon \in \mathbb{R}_+$ such that the spectrum of the operator $A+\lambda$ satisfies

$$\sigma(A+\lambda) \subset S_{(\pi/2)-\varepsilon} = \big\{z \in \mathbb{C}\setminus\{0\} : |\arg z| \le \frac{\pi}{2} - \varepsilon\big\} \cup \{0\}. \tag{9.17}$$

Moreover, for any $\mu \in (\pi/2, \pi)$ there exists a positive constant $C(\mu)$ such that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(C^\alpha(\overline{\Omega}))} \le C(\mu)|\lambda|^{(\alpha/2m)-1}, \quad \lambda \in S_\mu. \tag{9.18}$$

For details cf. Satz 1 and Satz 2 in [22], where we choose $l = \alpha$, $\beta = 1 - (\alpha/2m)$.

As an example, we can consider the problem consisting in recovering the vector-function $(u, f_1, \dots, f_p)$, where $f_j : [0,\tau] \to \mathbb{C}$, $j = 1,\dots,p$, satisfying

$$\frac{\partial u}{\partial t}(t,x) + (A+\lambda)u(t,x) = \sum_{j=1}^p f_j(t)z_j(x) + h(t,x), \quad (t,x) \in [0,\tau]\times\Omega, \tag{9.19}$$

$$u(0,x) = u_0(x), \quad x \in \Omega, \tag{9.20}$$

$$D^\gamma u(t,x) = 0, \quad (t,x) \in [0,\tau]\times\partial\Omega,\ |\gamma| \le m-1, \tag{9.21}$$

under the p additional conditions

$$u(t,\overline{x}_j) = g_j(t), \quad t \in [0,\tau],\ j = 1,\dots,p, \tag{9.22}$$

where $\overline{x}_j$, $j = 1,\dots,p$, are p fixed points in Ω.

We remark that A is not sectorial and $\mathcal{D}(A) \subset \big\{u \in C^\alpha(\Omega) : u = 0 \text{ on } \partial\Omega\big\}$. In view of Corollary 5.3 we can establish our identification result.

Theorem 9.1. *Let $\theta \in (\alpha/(2m), 1)$ and $\beta = 1 - \alpha/(2m) > 0$. Let $Y_A^{\gamma,\infty}$ be either of the spaces $(X, \mathcal{D}(A))_{\gamma,\infty}$ or X_A^γ. Let $u_0 \in \mathcal{D}(A)$, $Au_0 \in Y_A^{\gamma,\infty}$, $g_j \in C^1([0,\tau];\mathbb{C})$, $j = 1,\dots,p$, $h \in C([0,\tau];X) \cap B([0,\tau];Y_A^{\gamma,\infty})$, $u_0(\overline{x}_j) = g_j(0)$, $j = 1,\dots,p$, with*

$$\det\begin{bmatrix} z_1(\overline{x}_1) & \dots & z_p(\overline{x}_1) \\ \dots & \dots & \dots \\ z_1(\overline{x}_p) & \dots & z_p(\overline{x}_p)\end{bmatrix} \ne 0.$$

Then problem (9.19)-(9.22) *admits a unique strict solution*

$$(u, f_1, \dots, f_p) \in \big[C^1([0,\tau]; C^\alpha(\overline{\Omega}) \cap C([0,\tau];\mathcal{D}(A))\big] \times C([0,\tau];\mathbb{C})^p$$

such that $u(t,\cdot)\in C^{2m+\alpha}(\overline{\Omega})$ *for all* $t\in[0,\tau]$, $D_tu\in B([0,\tau];Y_A^{\theta-(\alpha/2m),\infty})$, $Au\in \big[C^{\theta-(\alpha/2m)}([0,\tau];C^\alpha(\overline{\Omega}))\cap B([0,\tau];Y_A^{\theta-(\alpha/2m),\infty})\big]$.

Problem 9.3. Under the same assumptions on m and Ω as in Problem 9.2 we introduce the linear operators A and D by the formulae.

$$\mathcal{D}(A)=\big\{u\in C^{2m+\alpha}(\overline{\Omega}): D^\beta u=0 \text{ on } \partial\Omega,\ |\beta|\le m-1\big\}, \tag{9.23}$$

$$Au(x)=\sum_{|\beta|\le 2m} a_\beta(x)D^\beta u(x),\quad x\in\Omega,\ u\in\mathcal{D}(A), \tag{9.24}$$

$$\mathcal{D}(D)=\big\{v\in C^{2p+\alpha}(\overline{\Omega}): D^\gamma v=0 \text{ on } \partial\Omega,\ |\gamma|\le p-1\big\}, \tag{9.25}$$

$$Dv(x)=\sum_{|\gamma|\le 2p} d_\gamma(x)D^\gamma v(x),\quad x\in\Omega,\ v\in\mathcal{D}(D), \tag{9.26}$$

where $a_\beta, d_\gamma\in C^\alpha(\overline{\Omega})$ and D^β, D^γ are defined as in Problem 9.2.

Let us introduce the operators B and C defined by

$$Bv(x)=\sum_{|\gamma|\le 2p-1} b_\gamma(x)D^\gamma v(x),\quad x\in\Omega,\ v\in C^{2p-1+\alpha}(\overline{\Omega}), \tag{9.27}$$

$$Cu(x)=\sum_{|\beta|\le 2m-1} c_\beta(x)D^\beta u(x),\quad x\in\Omega,\ u\in C^{2m-1+\alpha}(\overline{\Omega}). \tag{9.28}$$

In view of [22, Satz 1] the following estimate holds in the set $|\arg\lambda|\le(\pi/2)+\varepsilon$, $\operatorname{Re}\lambda\ge\lambda_0$:

$$\begin{aligned}&|\lambda|\|u\|_{C(\overline{\Omega})}+|\lambda|^{(2m-\alpha)/(2m)}\|u\|_{C^\alpha(\overline{\Omega})}+|\lambda|^{(1-\alpha)/(2m)}\|u\|_{C^{2m-1+\alpha}(\overline{\Omega})}+\|u\|_{C^{2m+\alpha}(\overline{\Omega})}\\&\le C_1\|(A+\lambda)u\|_{C^\alpha(\overline{\Omega})}.\end{aligned} \tag{9.29}$$

Whence we deduce the estimates

$$\|C(A+\lambda)^{-1}f\|_{C^\alpha(\overline{\Omega})}\le C_2|\lambda|^{(-1+\alpha)/(2m)}\|f\|_{C^\alpha(\overline{\Omega})}, \tag{9.30}$$

$$\|B(D+\lambda)^{-1}f\|_{C^\alpha(\overline{\Omega})}\le C_3|\lambda|^{(-1+\alpha)/(2p)}\|f\|_{C^\alpha(\overline{\Omega})}. \tag{9.31}$$

Consequently, conditions (6.1)-(6.3) hold with

$$\beta_1=1-\frac{\alpha}{2m},\quad \beta_2=1-\frac{\alpha}{2p},\quad \gamma_1=\frac{1-\alpha}{2m},\quad \gamma_2=\frac{1-\alpha}{2m}. \tag{9.32}$$

Therefore, we are allowed to apply Theorem 6.1 and Corollaries 6.2, 6.3 to the problem consisting in finding a quadruplet (u,v,f_1,f_2) solving

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x)+A(x,D_x)u(t,x)+B(x,D_x)v(t,x)\\&=f_1(t)z_{1,1}(x)+f_2(t)z_{1,2}(x)+h_1(t,x),\quad (t,x)\in(0,\tau)\times\Omega,\end{aligned} \tag{9.33}$$

$$\begin{aligned}&\frac{\partial v}{\partial t}(t,x)+C(x,D_x)u(t,x)+D(x,D_x)v(t,x)\\&=f_1(t)z_{2,1}(x)+f_2(t)z_{2,2}(x)+h_2(t,x),\quad (t,x)\in(0,\tau)\times\Omega,\end{aligned} \tag{9.34}$$

$$u(0,x)=u_0(x),\quad v(0,x)=v_0(x),\quad x\in\overline{\Omega}, \tag{9.35}$$

$$u(t,\overline{x})=g_1(t),\quad v(t,\widetilde{x})=g_2(t),\quad t\in[0,\tau], \tag{9.36}$$

where

$$\overline{x},\widetilde{x}\in\overline{\Omega},\quad u(0,\overline{x})=g_1(0),\quad v(0,\widetilde{x})=g_2(0). \tag{9.37}$$

Indeed, operator $\mathcal{A}$ defined in $X \times X = C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega})$ by

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(D), \quad \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Au + Bv \\ Cu + Dv \end{bmatrix}$$

satisfies, for all λ with large modulus belonging to the sector $|\arg \lambda| \le (\pi/2) + \varepsilon$, the resolvent estimate

$$\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega}))} \le c|\lambda|^{-\beta}, \tag{9.38}$$

where

$$\beta = \min\Big\{1 - \frac{\alpha}{2m}, 1 - \frac{\alpha}{2p}\Big\}. \tag{9.39}$$

We confine ourselves to translating Corollary 6.3 to this new situation.

Theorem 9.2. *Let β be defined by* (9.39) *and $\theta \in (1-\beta, 1)$, $g_1, g_2 \in C^1([0,\tau];\mathbb{C})$, $u_0 \in \mathcal{D}(A)$, $v_0 \in \mathcal{D}(D)$, $Au_0 + Bv_0 \in (C^\alpha(\overline{\Omega}); \mathcal{D}(A))_{\theta,\infty}$,*

$$Cu_0 + Dv_0 \in (C^\alpha(\overline{\Omega}); \mathcal{D}(D))_{\theta,\infty},$$
$$z_{1,1}, z_{1,2} \in (C^\alpha(\overline{\Omega}); \mathcal{D}(A))_{\theta,\infty}, \quad z_{2,1}, z_{2,2} \in (C^\alpha(\overline{\Omega}); \mathcal{D}(D))_{\theta,\infty},$$
$$h_1 \in C([0,\tau]; C^\alpha(\overline{\Omega})) \cap B([0,\tau]; (C^\alpha(\overline{\Omega}); \mathcal{D}(A))_{\theta,\infty}),$$
$$h_2 \in C([0,\tau]; C^\alpha(\overline{\Omega})) \cap B([0,\tau]; (C^\alpha(\overline{\Omega}); \mathcal{D}(D))_{\theta,\infty}),$$

satisfying the consistency conditions (9.37) *as well the solvability condition*

$$z_{1,1}(\overline{x}) z_{2,2}(\widetilde{x}) - z_{1,2}(\overline{x}) z_{2,1}(\widetilde{x}) \ne 0.$$

Then problem (9.33)-(9.36) *admits a unique strict solution (u, v, f_1, f_2) in the space $\big[C^1([0,\tau]; C^\alpha(\overline{\Omega}) \cap C([0,\tau]; \mathcal{D}(A))\big] \times \big[C^1([0,\tau]; C^\alpha(\overline{\Omega})) \cap C([0,\tau]; \mathcal{D}(D))\big]$ $\times C([0,\tau];\mathbb{C}) \times C([0,\tau];\mathbb{C})$ such that*

$$D_t u \in B([0,\tau]; (C^\alpha(\overline{\Omega}; \mathcal{D}(A))_{\theta+\beta-1,\infty}), \quad D_t v \in B([0,\tau]; (C^\alpha(\overline{\Omega}; \mathcal{D}(D))_{\theta+\beta-1,\infty}),$$
$$Au + Bv \in C^{\theta+\beta-1}([0,\tau]; C^\alpha(\overline{\Omega})), \quad Cu + Dv \in C^{\theta+\beta-1}([0,\tau]; C^\alpha(\overline{\Omega})).$$

We could also handle the system

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{bmatrix} y_1(t,x) \\ \dots \\ y_n(t.x) \end{bmatrix} + \begin{bmatrix} A_1 + B_{1,1} & B_{1,2} & \dots & B_{1,n} \\ B_{2,1} & A_2 + B_{2,2} & \dots & B_{2,n} \\ \dots & \dots & \dots & \dots \\ B_{n,1} & B_{n,2} & \dots & A_n + B_{n,n} \end{bmatrix} \begin{bmatrix} y_1(t.x) \\ \dots \\ y_n(t.x) \end{bmatrix} \\ &= \begin{bmatrix} h_1(t.x) \\ \dots \\ h_n(t.x) \end{bmatrix} + \sum_{j=1}^n f_j(t.x) \begin{bmatrix} z_{1,j} \\ \dots \\ z_{n,j} \end{bmatrix}, \quad t \in [0,\tau], \end{aligned} \tag{9.40}$$

$$y_j(0) = y_{0,j}, \quad j = 1, \dots, n, \tag{9.41}$$

$$\Psi_j[y_j(t)] = g_j(t), \quad t \in [0,\tau],\ j = 1, \dots, n, \tag{9.42}$$

in the space $\big[C^\alpha(\overline{\Omega})\big]^n$. Here the A_j's and the $B_{i,j}$'s are linear differential operators like in Problem 9.2 and Problem 9.1, respectively such that $\operatorname{ord} B_{i,j} < \operatorname{ord} A_j$, for all $i, j = 1, \dots, p$.

Since a bound of type (7.5) follows from [22, Satz 1] the previous argument applies immediately, e.g., when functionals Ψ_j are defined by $\Psi_j[y_j(\cdot)] = y_j(\overline{x}^{(j)})$, $j = 1, \dots, p$, $\overline{x}^{(1)}, \dots \overline{x}^{(p)}$ being p fixed points in $\overline{\Omega}$. The details are left to the reader.

Problem 9.4. Let us consider the degenerate parabolic system

$$\frac{\partial u}{\partial t}(t,x) = \Delta(a(x)u(t,x)) + b(x)v(t,x) + f_1(t)z_{1,1}(x) + f_2(t)z_{1,2}(x), \tag{9.43}$$

$$\frac{\partial v}{\partial t}(t,x) = c(x)u(t,x) + \Delta(d(x)v(t,x)) + f_1(t)z_{2,1}(x) + f_2(t)z_{2,2}(x), \tag{9.44}$$

$$(t,x) \in (0,\tau) \times \Omega,$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \Omega, \tag{9.45}$$

$$a(x)u(t,x) = 0 = d(x)v(t,x), \quad (t,x) \in (0,\tau) \times \partial\Omega, \tag{9.46}$$

$$\int_\Omega \eta_1(x)u(t,x)dx = g_1(t), \quad \int_\Omega \eta_2(x)v(t,x)dx = g_2(t), \ 0 \le t \le \tau, \tag{9.47}$$

along with the consistency conditions

$$\int_\Omega \eta_1(x)u_0(x)dx = g_1(0), \quad \int_\Omega \eta_2(x)v_0(x)dx = g_2(0), \tag{9.48}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 1$, with a C^2-boundary $\partial\Omega$, while a, b, c, d are functions in $C(\overline{\Omega};\mathbb{R})$ such that $a(x) > 0$ and $d(x) > 0$ a.e. in Ω. Moreover, $z_{i,j} \in L^2(\Omega)$, $i,j = 1,2$, $u_0, v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $g_i \in C^1([0,\tau];\mathbb{C})$, $i = 1,2$. Our task consists in recovering (u, v, f_1, f_2).

We recall [16, p. 83] that, if

$$a^{-1} \in L^r(\Omega) \quad \text{with} \begin{cases} r \ge 2 & \text{when } n = 1, \\ r > 2, & \text{when } n = 2, \\ r \ge n, & \text{when } n \ge 3, \end{cases}$$

then, for any function e enjoying the same properties as a, operator $K(e)$ defined by

$$\mathcal{D}(K(e)) := \{u \in L^2(\Omega) : eu \in H_0^1(\Omega) \cap H^2(\Omega)\}, \quad K(e)u := -\Delta(eu), \quad u \in \mathcal{D}(K)$$

satisfies the estimate

$$\|(\lambda I + K(e))^{-1} f\|_{L^2(\Omega)} \le c|\lambda|^{-(2r-n)/2r} \|f\|_{L^2(\Omega)},$$

for all λ in a sector containing the half-plane $\operatorname{Re} z \ge 0$. Therefore, $\alpha = 1$, $\beta = (2r-n)/2r$.

Let us assume $1/a \in L^{r_1}(\Omega)$, $1/d \in L^{r_2}(\Omega)$. Consequently, estimates (6.2)) hold with $\alpha = 1$, $\beta_1 = (2r_1 - n)/2r_1$, $\beta_2 = (2r_2 - n)/2r_2$ for operators $K(a)$ and $K(d)$. Since the multiplication operators generated by b and c are bounded in $L^2(\Omega)$, β in (6.12) is given by

$$\beta = \min\{\beta_1, \beta_2\} = \min\Big\{1 - \frac{n}{2r_1}, 1 - \frac{n}{2r_2}\Big\} \ge \frac{1}{2},$$

since $r_j \ge n$, $j = 1,2$. Let us assume

$$\int_\Omega \eta_1(x)z_{1,1}(x)dx \int_\Omega \eta_2(x)z_{2,2}(x)dx - \int_\Omega \eta_1(x)z_{1,2}(x)dx \int_\Omega \eta_2(x)z_{2,1}(x)dx \ne 0,$$

$$g_1, g_2 \in C^1([0,\tau];\mathbb{C}), \quad 1 - \beta < \theta < 1.$$

Let

$$\mathcal{A} = \begin{bmatrix} K(a) & -M_b \\ -M_c & K(d) \end{bmatrix}, \tag{9.49}$$

As $(L^2(\Omega)\times L^2(\Omega),\mathcal{D}(\mathcal{A}))_{\theta,\infty}=(L^2(\Omega),\mathcal{D}(K(a)))_{\theta,\infty}\times(L^2(\Omega),\mathcal{D}(K(d)))_{\theta,\infty}$ if

$$\begin{aligned}&(\Delta(a(\cdot)u_0)+b(\cdot)v_0,c(\cdot)u_0+\Delta(d(\cdot)v_0))\\&\in(L^2(\Omega),\mathcal{D}(K(a)))_{\theta,\infty}\times(L^2(\Omega),\mathcal{D}(K(d)))_{\theta,\infty},\end{aligned}$$

$z_{11},z_{12}\in(L^2(\Omega),D(K(a)))_{\theta,\infty}$, $z_{21},z_{22}\in(L^2(\Omega),\mathcal{D}(K(d)))_{\theta,\infty}$, from Corollary 6.3 we can conclude that problem (9.43)–(9.47), endowed with the consistency condition (9.48) admits a unique global strict solution $((u,v),f_1,f_2)\in C([0,\tau];\mathcal{D}(K(a))\times\mathcal{D}(K(d)))\times C([0,\tau];\mathbb{C})\times C([0,\tau];\mathbb{C})$ such that $(D_tu,D_tv)^T\in B([0,\tau];(L^2(\Omega)\times L^2(\Omega),\mathcal{D}(K(a))\times\mathcal{D}(K(d)))_{\theta-(1-\beta),\infty})$, $\mathcal{A}(u,v)^T\in C^{\theta-(1-\beta)}([0,\tau];L^2(\Omega)\times L^2(\Omega))\cap B([0,\tau];(L^2(\Omega)\times L^2(\Omega),\mathcal{D}(K(a))\times\mathcal{D}(K(d)))_{\theta-(1-\beta),\infty})$.

More generally, we could deal with an analogous doubly degenerate problem related to the system

$$\begin{gathered}\frac{\partial}{\partial t}(m(x)u(t,x))=\Delta(a(x)u(t,x))+b(x)v(t,x)+f_1(t)z_{1,1}(x)+f_2(t)z_{1,2}(x),\\\frac{\partial}{\partial t}(n(x)v(t,x))=c(x)u(t,x)+\Delta(d(x)v(t,x))+f_1(t)z_{2,1}(x)+f_2(t)z_{2,2}(x),\\(t,x)\in(0,\tau)\times\Omega\end{gathered}$$

m and n being positive and continuous functions on Ω, using the change of unknowns defined by $m(x)u=u_1$, $n(x)v=v_1$. Notice that then we must make continuity assumptions on the behaviour on the boundary of functions b/n, c/m, a/m, d/n.

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