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WEAK SOLUTIONS FOR A-DIRAC EQUATIONS WITH VARIABLE GROWTH IN CLIFFORD ANALYSIS

BINLIN ZHANG, YONGQIANG FU

Abstract. In this article we show the existence of weak solutions for obstacle problems for A-Dirac equations with variable growth in the setting of variable exponent spaces of Clifford-valued functions. We also obtain the existence of weak solutions to the scalar part of A-Dirac equations in space $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

1. Introduction

After Kováčik and Rákosník first discussed the $L^{p(x)}$ space and $W^{k,p(x)}$ space in [20], a lot of results have been obtained concerning these kinds of variable exponent spaces and their applications, for example, see [4, 5, 6, 7] and references therein. Recently the theory of nonlinear partial differential equations with nonstandard growth conditions has important applications in elasticity (see [27]), eletrorheological fluids (see [26]) and so on. For an overview of variable exponent spaces with various applications to differential equations we refer to [15] and the references quoted there.

Clifford algebras were introduced by Clifford as geometric algebras in 1878, which are a generalization of the complex numbers, the quaternions, and the exterior algebras, see [11]. As an active branch of mathematics over the past 40 years, Clifford analysis usually studies the solutions of the Dirac equations for functions defined on domains in Euclidean space and taking value in Clifford algebras, see [18]. Gürlebeck and Sprößig [12, 14] developed the theory of Clifford analysis to investigate elliptic boundary value problems of fluid dynamics, in particular the Navier-Stokes equations and related equations. Doran and Lasenby [2] gave in detail an overview of the intrinsic value and usefulness of Clifford algebras and Clifford analysis for mathematical physics.

Nolder [21, 22] introduced A-Dirac equations $DA(x, Du) = 0$ and investigated some properties of weak solutions to the scalar parts of above-mentioned equations, for example, the Caccioppoli estimate and the removability theorem. Fu and Zhang [8] first introduced the weighted variable exponent spaces in the context of Clifford algebras, and then discussed the properties of these spaces. As an application, they

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obtained the existence of weak solutions in space $W^{D,p(x)}(\Omega, \mathrm{C}\ell_n)$ to the scalar part of the nondegenerate A-Dirac equations $DA(x,Du)+B(x,u)=0$. Unfortunately, the proof of [8, Corollary 4.1] is invalid for the case in which $B(x,u)\equiv 0$. Motivated by such problems, the aim of this paper is to investigate the existence of solutions to the scalar part of A-Dirac equations. Note that when u is a real-valued function and $A:\Omega\times \mathrm{C}\ell_n^1(\Omega)\to \mathrm{C}\ell_n^1(\Omega)$, the scalar part of A-Dirac equations becomes $\operatorname{div} A(x,\nabla u)=0$; i.e., A-harmonic equations. These equations have been extensively studied with many applications, see [17].

In recent years, obstacle problems in the variable exponent setting have attracted a lot of interest, we refer to [5, 8, 9, 16, 25] and references therein. Inspired by their works, we are interested in the following obstacle problems:

$$\int_\Omega \left[\overline{A(x,Du)}D(v-u)\right]_0 \geq 0 \tag{1.1}$$

for v belonging to

$$K_\psi = \{v\in W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n) : v\geq \psi \text{ a.e. in } \Omega\} \tag{1.2}$$

where $\psi(x)=\Sigma\psi_I e_I\in \mathrm{C}\ell_n(\Omega)$, $\psi_I:\Omega\to[-\infty,\ +\infty]$, $v\geq\psi$, a.e. in Ω means that for any I, we have $v_I\geq\psi_I$ a.e. in Ω.

We will study the solution $u\in K_\psi$ for (1.1)-(1.2) as $A(x,\xi):\Omega\times\mathrm{C}\ell_n\to\mathrm{C}\ell_n$ satisfies the following growth conditions:

- (A1) $A(x,\xi)$ is measurable with respect x for $\xi\in\mathrm{C}\ell_n$ and continuous with respect to ξ for a.e. $x\in\Omega$,
- (A2) $|A(x,\xi)|\leq C_1|\xi|^{p(x)-1}+g(x)$ for a.e. $x\in\Omega$ and $\xi\in\mathrm{C}\ell_n$,
- (A3) $\left[\overline{A(x,\xi)}\xi\right]_0\geq C_2|\xi|^{p(x)}+h(x)$ for a.e. $x\in\Omega$ and $\xi\in\mathrm{C}\ell_n$,
- (A4) $\left[\overline{(A(x,\xi_1)-A(x,\xi_2))}(\xi_1-\xi_2)\right]_0>0$ for a.e. $x\in\Omega$ and $\xi_1\neq\xi_2\in\mathrm{C}\ell_n$,

where $g\in L^{p'(x)}(\Omega)$, $h\in L^1(\Omega)$, C_i $(i=1,2)$ are positive constants. Throughout the paper we always assume that Ω is a bounded domain in $\mathbb{R}^n (n\geq 2)$. And that (unless declare specially)

$$p\in P^{\log}(\Omega) \quad\text{and}\quad 1<p_-=:\inf_{x\in\bar\Omega}p(x)\leq p(x)\leq\sup_{x\in\bar\Omega}p(x):=p_+<\infty \tag{1.3}$$

This article is divided into four sections. In Section 2, we will recall some basic knowledge of Clifford algebras and variable exponent spaces of Clifford-valued functions, which will be needed later. In Section 3, we will prove the existence of solutions for the above-mentioned obstacle problems for A-Dirac equations with nonstandard growth. Furthermore, we also obtain the existence of solutions to the scalar part of A-Dirac equations in $W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)$.

2. Preliminaries

2.1. **Clifford algebra.** In this section we first recall some related notions and results from Clifford algebras. For a detailed account we refer to [11, 12, 13, 14, 15, 16, 17, 18].

Let $\mathrm{C}\ell_n$ for the real universal Clifford algebras over $\mathbb{R}^n$, then

$$\mathrm{C}\ell_n=\operatorname{span}\{e_0,e_1,e_2,\dots,e_n,e_1e_2,\dots,e_{n-1}e_n,\dots,e_1e_2\dots e_n\}$$

where $e_0=1$ (the identity element in $\mathbb{R}^n$), $\{e_1,e_2,\dots,e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_ie_j+e_je_i=-2\delta_{ij}$. Thus the dimension of $\mathrm{C}\ell_n$ is 2^n. For

$I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ with $1 \le i_1 < i_2 < \dots < i_n \le n$, put $e_I = e_{i_1} e_{i_2} \dots e_{i_r}$, while for $I = \emptyset$, $e_\emptyset = e_0$. For $0 \le r \le n$ fixed, the space Cl_n^r is defined by

$$\mathrm{C}\ell_n^r = \operatorname{span}\{e_I : |I| := \operatorname{card}(I) = r\}.$$

The Clifford algebras $\mathrm{C}\ell_n$ is a graded algebra as

$$\mathrm{C}\ell_n = \oplus_r \mathrm{C}\ell_n^r.$$

Any element $a \in \mathrm{C}\ell_n$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \dots + [a]_n$$

where $[\ ]_r : \mathrm{C}\ell_n \to \mathrm{C}\ell_n^r$ denotes the projection of $\mathrm{C}\ell_n$ onto $\mathrm{C}\ell_n^r$. It is customary to identify $\mathbb{R}$ with $\mathrm{C}\ell_n^0$ and identify $\mathbb{R}^n$ with $\mathrm{C}\ell_n^1$ respectively. For $u \in \mathrm{C}\ell_n$, we know that $[u]_0$ denotes the scalar part of u, that is the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{(e_{i_1} e_{i_2} \dots e_{i_r})} = (-1)^{\frac{r(r+1)}{2}} e_{i_1} e_{i_2} \dots e_{i_r}$$

For $A \in \mathrm{C}\ell_n$, $B \in \mathrm{C}\ell_n$, we have

$$\overline{AB} = \overline{B}\ \overline{A}, \quad \overline{\overline{A}} = A.$$

We denote

$$(A, B) = [\overline{A} B]_0.$$

Then an inner product is thus obtained, leading to the norm $|\cdot|$ on $\mathrm{C}\ell_n$ given by

$$|A|^2 = [\overline{A} A]_0.$$

From [13] we know that this norm is submultiplicative:

$$|AB| \le C_3 |A||B|. \tag{2.1}$$

where $C_3 \in [1, 2^{n/2}]$ is a constant.

A Clifford-valued function $u : \Omega \to \mathrm{C}\ell_n$ can be written as $u = \Sigma_I u_I e_I$, where the coefficients $u_I : \Omega \to \mathbb{R}$ are real valued functions.

The Dirac operator on Euclidean space used here is as follows:

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} e_j \partial_j.$$

If u is C^1 real-valued function defined on a domain Ω in $\mathbb{R}^n$, then $Du = \nabla u = (\partial_1 u, \partial_2 u, \dots, \partial_n u)$, where ∇ is the distributional gradient. Further $D^2 = -\Delta$, where Δ is the Laplace operator which operates only on coefficients. A function is left monogenic if it satisfies the equation $Du(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right monogenic function. An important example of a left monogenic function is the generalized Cauchy kernel

$$G(x) = \frac{1}{\omega_n} \frac{\overline{x}}{|x|^n},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator. Basic properties of left monogenic functions one can refer to [11, 12, 13, 14].

Next we recall some basic properties of variable exponent spaces. Let $P(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1, \infty)$. Given $p \in P(\Omega)$ we define the conjugate function $p'(x) \in P(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x) - 1}, \quad x \in \Omega.$$

We define the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u \in P(\Omega) : \int_\Omega |u|^{p(x)} dx < \infty\}.$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\{t > 0 : \int_\Omega \Big|\frac{u}{t}\Big|^{p(x)} dx \leq 1\}. \tag{2.2}$$

Definition 2.1 ([2]). A function $a : \Omega \to \mathbb{R}$ is globally log-Hölder continuous in Ω if there exist $L_i > 0$ $(i = 1, 2)$ and $a_\infty \in \mathbb{R}^n$ such that

$$|a(x) - a(y)| \leq \frac{L_1}{\log(e + 1/|x - y|)}, \quad |a(x) - a_\infty| \leq \frac{L_2}{\log(e + |x|)}$$

hold for all $x, y \in \Omega$. We define the following class of variable exponents

$$P^{\log}(\Omega) = \big\{p \in P(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous } \}.$$

Theorem 2.2 ([1]). *If $p(x) \in P(\Omega)$, then the inequality*

$$\int_\Omega |uv| dx \leq 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}$$

holds for every $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$.

Theorem 2.3 ([1]). *If $p(x) \in P(\Omega)$, then space $L^{p(x)}(\Omega)$ is complete and reflexive.*

Remark 2.4. We shall say that $f_n \in L^{p(x)}(\Omega)$ converge modularly to $f \in L^{p(x)}(\Omega)$ if $\lim_{n\to\infty} \int_\Omega |f_n - f|^{p(x)} dx = 0$. In [20] it is shown that the topology of $L^{p(x)}(\Omega)$ given by the norm (2.3) coincides with topology of modular convergence.

Variable exponent spaces of Clifford-valued functions. In this section, we first recall some notation of variable exponent spaces of Clifford-valued functions, for a detailed treat we refer to [8, 9].

We define the space

$$L^{p(x)}(\Omega, \mathrm{C}\ell_n) = \{u \in \mathrm{C}\ell_n : u = \sum_I u_I e_I, u_I \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} = \big\|\sum_I u_I e_I\big\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} = \sum_I \|u_I\|_{L^{p(x)}(\Omega)}$$

and the Sobolev space

$$W^{1,p(x)}(\Omega, \mathrm{C}\ell_n) = \{u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n) : \nabla u \in (L^{p(x)}(\Omega, \mathrm{C}\ell_n))^n\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)} = \|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} + \|\nabla u\|_{(L^{p(x)}(\Omega, \mathrm{C}\ell_n))^n} \tag{2.3}$$

By $C^\infty(\Omega, \mathrm{C}\ell_n)$ denote the space of Clifford-valued functions in Ω whose coefficients are infinitely differentiable in Ω and by $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ denote the subspace of

$C^\infty(\Omega, \mathrm{C}\ell_n)$ with compact support in Ω. Denote $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ by the closure of $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ in $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ with respect to the norm (2.3).

Remark 2.5 ([8])**.** A simple computation shows that

$$2^{-\frac{n(1+p_+)}{p_-}} \||u|\|_{L^{p(x)}(\Omega)} \le \|u\|_{L^{p(x)}(\Omega, Cl_n)} \le 2^n \||u|\|_{L^{p(x)}(\Omega)},$$

from which we can obtain that $\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}$ and $\||u|\|_{L^{p(x)}(\Omega)}$ are equivalent norms on $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$.

Theorem 2.6 ([8])**.** *If $p(x) \in P(\Omega)$, then the inequality*

$$\int_\Omega |uv| dx \le C(p, n)\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} \|v\|_{L^{p'(x)}(\Omega, \mathrm{C}\ell_n)}$$

holds for every $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ and $v \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$.

Theorem 2.7 ([9])**.** *If $p(x) \in P(\Omega)$, then $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is a reflexive Banach space.*

Theorem 2.8 ([9])**.** *If $u \in W_0^{1,p(x)}(\Omega, Cl_n)$, then*

$$\|u\|_{L^{p(x)}(\Omega, Cl_n)} \le C(n, \Omega)\|\partial u\|_{L^{p(x)}(\Omega, Cl_n)}.$$

Theorem 2.9 ([9])**.** *If $p(x)$ satisfies (1.1) and $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then the norms $\big|\big|u\big|\big|_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)}$ and $\big|\big|Du\big|\big|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}$ are equivalent on $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.*

Proof. By [9, Remark 2.3], we know that $\big|\big|\nabla u\big|\big|_{(L^{p(x)}(\Omega, Cl_n))^n}$ is equivalent to $\big|\big|Du\big|\big|_{L^{p(x)}(\Omega, Cl_n)}$ for $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. According to Theorem 2.8, the norms $\big|\big|u\big|\big|_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)}$ and $\big|\big|\nabla u\big|\big|_{(L^{p(x)}(\Omega, Cl_n))^n}$ are equivalent on $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Thus we obtain the desired conclusion. □

3. Weak solutions for obstacle problems for A-Dirac equations

In this section we will establish the existence of weak solutions for obstacle problems for A-Dirac equations with variable growth. As a corollary, the existence of weak solutions to the scalar part of A-Dirac equations is obtained. We first introduce a theorem of Kinderlehrer and Stampacchia.

Let X be a reflexive Banach space with dual X^* and let $\langle \cdot, \cdot \rangle$ denote a pairing between X and X^*. If $K \subset X$ is a closed convex set, then a mapping $A : K \to X^*$ is called monotone if

$$\langle Au - Av, u - v\rangle \ge 0$$

for all $u, v \in K$. Further A is called coercive on K if there exists $\varphi \in K$ such that

$$\frac{\langle Au_n - A\varphi, u_n - \varphi\rangle}{\|u_n - \varphi\|_X} \to \infty$$

whenever $\{u_n\} \subset K$ with $\|u_n - \varphi\|_X \to \infty$ as $n \to \infty$. Moreover A is called strongly-weakly continuous on K if $u_n \to \varphi$ in K, then $Au_n \to A\varphi$ weakly in X^*.

Proposition 3.1 ([19])**.** *Let K be a nonempty closed convex subset of X and let $A : K \to X^*$ be monotone, coercive and strongly-weakly continuous on K. Then there exists an element $u \in K$ such that*

$$\langle Au, v - u\rangle \ge 0$$

for all $v \in K$.

In the following discuss we set $X = W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n), K = K_\psi$ and let $\langle\cdot,\cdot\rangle$ be the usual pairing between X and X^*; i.e.,

$$\langle u, v\rangle = \int_\Omega \big[\overline{u}v\big]_0 dx,$$

where $u \in X, v \in X^*$. By the definition of K_ψ, it is immediate to obtain the following lemma.

Lemma 3.2. *K is a closed convex set in X.*

Next we define a mapping $T : K \to X^*$ by

$$\langle Tu, v\rangle = \int_\Omega \big[\overline{A(x, Du)}Dv\big]_0$$

for $v \in X$.

Lemma 3.3. *For any $u \in K$, we have $Tu \in X^*$.*

Proof. In view of (A2), (2.1) and the Hölder inequality, we obtain

$$\begin{aligned}
\Big|\int_\Omega \big[\overline{A(x, Du)}Dv\big]_0\Big| &\le \int_\Omega \big|\overline{A(x, Du)}Dv\big| \\
&\le C_3 \int_\Omega \big(C_1|Du|^{p(x)-1} + g(x)\big)|Dv| \\
&\le 2C_1C_3\big\|\,|Du|^{p(x)-1}\big\|_{L^{p'(x)}(\Omega)}\||Dv|\|_{L^{p(x)}(\Omega)} \\
&\quad + 2C_3\|g\|_{L^{p'(x)}(\Omega)}\||Dv|\|_{L^{p(x)}(\Omega)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|\,|Du|^{p(x)-1}\|_{L^{p'(x)}(\Omega)} &= \inf\Big\{t > 0 : \int_\Omega \frac{|Du|^{p(x)}}{t^{p'(x)}}dx \le 1\Big\} \\
&= \inf\Big\{t > 0 : \int_\Omega \Big(\frac{|Du|}{\lambda^{\frac{1}{p(x)-1}}}\Big)^{p(x)}dx \le 1\Big\} \\
&\le \max\Big\{\||Du|\|_{L^{p(x)}(\Omega)}^{p_+-1}, \||Du|\|_{L^{p(x)}(\Omega)}^{p_--1}\Big\}.
\end{aligned}$$

Then the assertion immediately follows from Remark 2.5 and Theorem 2.9. □

Lemma 3.4. *T is monotone and coercive on K.*

Proof. In view of (A4), it is immediate that T is monotone. Next we show that T is coercive. Given $\varphi \in K$. Then by (A2), (A3) and (2.1), we obtain

$$\begin{aligned}
&\langle Tu - T\varphi, u - \varphi\rangle \\
&\ge C_2\int_\Omega |Du|^{p(x)}dx + C_2\int_\Omega |D\varphi|^{p(x)}dx - 2\int_\Omega |h|dx - C_3\int_\Omega |Du||g|dx \\
&\quad - C_3\int_\Omega |D\varphi||g|dx - C_1C_3\int_\Omega |Du|^{p(x)-1}|D\varphi|dx - C_1C_3\int_\Omega |Du||D\varphi|^{p(x)-1}dx \\
&\ge C_2\int_\Omega |Du|^{p(x)}dx + C_2\int_\Omega |D\varphi|^{p(x)}dx - 2\int_\Omega |h|dx \\
&\quad - C_3\varepsilon\int_\Omega \frac{1}{p(x)}|Du|^{p(x)}dx - C_3\int_\Omega \frac{1}{p'(x)}\varepsilon^{\frac{1}{1-p'(x)}}|g|^{p'(x)}dx - C_3\int_\Omega |D\varphi||g|dx \\
&\quad - \varepsilon C_1C_3\int_\Omega \frac{1}{p'(x)}|Du|^{p(x)}dx - C_1C_3\int_\Omega \frac{1}{p(x)}\varepsilon^{\frac{1}{1-p'(x)}}|D\varphi|^{p(x)}dx
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon C_1 C_3 \int_\Omega \frac{1}{p(x)} |Du|^{p(x)} dx - C_1 C_3 \int_\Omega \frac{1}{p'(x)} \varepsilon^{\frac{1}{1-p'(x)}} |D\varphi|^{p(x)} dx \\
&\geq \Big(C_2 - C_3 \Big(\frac{1}{p_-} + C_1 \Big) \varepsilon \Big) \int_\Omega |Du|^{p(x)} dx - C_5.
\end{aligned}
$$

Taking $\varepsilon = \frac{C_2 p_-}{2C_3(1+C_1 p_-)}$, we have

$$
\begin{aligned}
\langle Tu - T\varphi, u - \varphi \rangle &\geq \frac{C_2}{2} \int_\Omega |Du|^{p(x)} dx - C_5 \\
&\geq \frac{C_2}{2} \int_\Omega 2^{-p_+} \Big(|Du - D\varphi|^{p(x)} - |D\varphi|^{p(x)} \Big) dx - C_5 \\
&\geq \frac{C_2}{2^{1+p_+}} \int_\Omega |Du - D\varphi|^{p(x)} dx - C_6.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{\int_\Omega \big|Du - D\varphi\big|^{p(x)} dx}{\||Du - D\varphi|\|_{L^{p(x)}(\Omega)}} \\
&= \int_\Omega \Big(\frac{\big|Du - D\varphi\big|}{2^{-1}\||Du - D\varphi|\|_{L^{p(x)}(\Omega)}} \Big)^{p(x)} \cdot \frac{\big(2^{-1}\||Du - D\varphi|\|_{L^{p(x)}(\Omega)}\big)^{p(x)}}{\||Du - D\varphi|\|_{L^{p(x)}(\Omega)}} dx,
\end{aligned}
$$

as $\||Du - D\varphi|\|_{L^{p(x)}(\Omega)} \geq 1$ we have

$$
\frac{\int_\Omega \big|Du - D\varphi\big|^{p(x)} dx}{\||Du - D\varphi|\|_{L^{p(x)}(\Omega)}} \geq 2^{-p_+} \||Du - D\varphi|\|_{L^{p(x)}(\Omega)}^{p_- - 1}.
$$

Therefore, in view of Remark 2.5 and Theorem 2.9, we obtain

$$
\frac{\langle Tu - Tv, u - v \rangle}{\|u - \varphi\|_X} \to \infty
$$

as $\|u - \varphi\|_X \to \infty$. That is to say, T is coercive on K. ☐

Lemma 3.5. *The operator T is strongly-weakly continuous on K.*

Proof. Let $\{u_k(x)\} \subset K$ be a sequence that converges to an element $u(x) \in K$ in X. Then $\{u_k\}$ is uniformly bounded in X. Moreover, by (A1) we can deduce that for each $v \in X$

$$
\big[\overline{A(x, Du_k)} Dv\big]_0 \to \big[\overline{A(x, Du)} Dv\big]_0 \quad \text{a.e. on } \Omega, \text{ as } k \to \infty.
$$

To see the equi-continuous integrability of the sequence $\big\{[\overline{A(x, Du_k)} Dv]_0\big\}$, we take a measurable subset $\Omega' \subset \Omega$, by (2.1) and (A2), for each $v \in X$, we have

$$
\begin{aligned}
&\Big| \int_{\Omega'} \big[\overline{A(x, Du_k)} Dv\big]_0 \Big| \\
&\leq 2C_1 \big(C_1 \big\| |Du_k|^{p(x)-1} \big\|_{L^{p'(x)}(\Omega')} + \|g\|_{L^{p'(x)}(\Omega')} \big) \||Dv|\|_{L^{p(x)}(\Omega')}.
\end{aligned} \tag{3.1}
$$

In view of Remark 2.4, Remark 2.5 and Theorem 2.9, we obtain that the first term of (3.1) is uniformly bounded in k. The second term of (3.1) is arbitrarily small if the measure of Ω' is chosen small enough. By the Vitali convergence theorem, we have

$$
\langle Tu_k, v \rangle = \int_\Omega \big[\overline{A(x, Du_k)} Dv\big]_0 \to \int_\Omega \big[\overline{A(x, Du)} Dv\big]_0 = \langle Tu, v \rangle
$$

as $k \to \infty$. That is to say, T is strongly-weakly continuous. ☐

Theorem 3.6. *Suppose $K \neq \emptyset$. Under conditions* (A1)-(A4)*, there exists a Clifford-valued solution $u \in K$ for the obstacle problems* (1.1)-(1.2)*. That is to say, there exists a Clifford-valued function $u \in K$ such that*

$$\int_\Omega \big[\overline{A(x, Du)} D(v - u)\big]_0 \geq 0$$

for any $v \in K$. Moreover, the solution to the scalar part of (1.1) *and* (1.2) *is unique up to a monogenic function, namely, if $u_1, u_2 \in K$ are solutions to the obstacle problem* (1.1)–(1.2)*, then $[Du_1]_0 = [Du_2]_0$ on Ω.*

Proof. Using Lemma 3.2, Lemma 3.5 and Proposition 3.1, it is immediate to obtain the existence of weak solutions for the obstacle problems (1.1)–(1.2). If there are two solutions $u_1, u_2 \in K$ to the obstacle problem (1.1)–(1.2), then

$$\int_\Omega \Big[\overline{A(x, Du_1)} D(u_2 - u_1)\Big]_0 dx \geq 0,$$
$$\int_\Omega \Big[\overline{A(x, Du_2)} D(u_1 - u_2)\Big]_0 dx \geq 0$$

So we have

$$\int_\Omega \Big[\overline{A(x, Du_1) - A(x, Du_2)} D(u_1 - u_2)\Big]_0 dx \leq 0$$

According to (A4), we can infer that

$$\int_\Omega \Big[\overline{A(x, Du_1) - A(x, Du_2)} D(u_1 - u_2)\Big]_0 dx = 0 \quad \text{on } \Omega.$$

That is to say, $[Du_1]_0 = [Du_2]_0$ on Ω. ☐

Corollary 3.7. *Under the conditions in Theorem 3.6, there exists one weak solution $u \in X$ to the scalar part of $DA(x, Du) = 0$. Namely, there exists at least one $u \in X$ satisfying*

$$\int_\Omega \big[\overline{A(x, Du)} D\varphi\big]_0 = 0 \tag{3.2}$$

for any $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

Proof. Let $\psi = \sum_I \psi_I e_I$, where $\psi_I = -\infty$ for any I. Let u be a solution for the obstacle problem of (1.1)–(1.2) in K. Since the Clifford-valued functions $u - \varphi \in K, u + \varphi \in K$ for any $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, we have

$$\int_\Omega \big[\overline{A(x, Du)} D\varphi\big]_0 \geq 0,$$
$$-\int_\Omega \big[\overline{A(x, Du)} D\varphi\big]_0 \geq 0.$$

Thus

$$\int_\Omega \big[\overline{A(x, Du)} D\varphi\big]_0 = 0.$$

The proof is complete. ☐

Example. If $A(x,\xi)=\xi$, then A-Dirac equations $DA(x,Du)=0$ becomes $-\triangle u=0$, that is, Clifford Laplacian equation. If $A(x,\xi)=|\xi|^{p-2}\xi$, then A-Dirac equations becomes $D(|Du|^{p-2}Du)=0$, that is, p-Dirac equation (see [23]). Moreover, if u is real-valued function, then the scalar part of A-Dirac equations is A-harmonic equations $-\operatorname{div}(A(x,\nabla u))=0$ (see [21, 22]). Therefore, by Corollary 3.7, we obtain the existence of a weak solution of the A-harmonic equations under the required conditions as in Theorem 3.6.

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Binlin Zhang
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
E-mail address: zhangbinlin2012@yahoo.cn

Yongqiang Fu
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
E-mail address: fuyqhagd@yahoo.cn