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SOLVABILITY OF FRACTIONAL-ORDER MULTI-POINT BOUNDARY-VALUE PROBLEMS AT RESONANCE ON THE HALF-LINE

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Abstract. In this article, we study a fractional differential equation. By constructing two special Banach spaces and establishing an appropriate compactness criterion, we present some existence results about the boundary-value problem at resonance via Mawhin's continuation theorem of coincidence degree theory.

1. Introduction

In this article, we are concerned with the existence of solutions to the m-point boundary value problems involving Caputo fractional derivative

$$\begin{gathered} {}^C D_{0+}^{\alpha}\big(a(t)u'(t)\big) = f(t, u(t), {}^C D_{0+}^{\alpha} u(t), u'(t)), \quad t \in [0, +\infty), \\ u'(0) = 0, \quad \sum_{j=1}^{m-1} \sigma_j u(\xi_j) = \lim_{t\to+\infty} u(t), \end{gathered} \tag{1.1}$$

where ${}^C D_{0+}^{\alpha}$ is the Caputo fractional derivative, $0 < \alpha < 1$, $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies α-Carathéodory conditions, $a(t) \in C^1[0, +\infty)$, $a(t) > 0$, $\sigma_j \in \mathbb{R}$, $\sigma_j > 0$, $\xi_j > 0$, $j = 1, 2, \dots, m-1$, $m \in \mathbb{N}$, $m > 1$, and

$$\sum_{j=1}^{m-1} \sigma_j = 1, \tag{1.2}$$

which implies that (1.1) is at resonance. Problem (1.1) is at resonance in the sense that the kernel of the linear operator ${}^C D_{0+}^{\alpha}$ is not less than one-dimensional under the boundary value conditions.

Let $\nu > 0$ and $n = [\nu] + 1$, where $[\nu]$ denotes the largest integer less than ν. Then then the Riemann-Liouville fractional integral and derivative of order ν for a function $h : (0, \infty) \to \mathbb{R}$ is defined by (see [8, 16, 17])

$$D_{0+}^{-\nu} h(t) = I_{0+}^{\nu} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds, \tag{1.3}$$

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and

$$D^{\nu}_{0+}h(t) = \frac{1}{\Gamma(n-\nu)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\nu-1}h(s)ds, \tag{1.4}$$

respectively. Additionally, we have the Caputo fractional derivatives (see [8, 17]) of order ν

$${}^{C}D^{\nu}_{0+}h(t) = \frac{1}{\Gamma(n-\nu)}\int_0^t (t-s)^{n-\nu-1}h^{(n)}(s)ds. \tag{1.5}$$

Sequential fractional derivatives were defined by Podlubny [17] as

$$\mathcal{D}^{\nu}h(t) = D^{\nu_1}D^{\nu_2}\dots D^{\nu_p}h(t), \quad p \in \mathbb{N},\ p > 0, \tag{1.6}$$

where the symbol D^{ν_i} $(i = 1, 2, \dots p)$ is the Riemann-Liouville derivative, or the Caputo derivative. It is obvious that (1.6) is a generalized expression presented by Miller and Ross in [16].

Fractional calculus is a generalization of the ordinary differentiation and integration. It has played a significant role in science, engineering, economy, and other fields. For recent publication on on fractional calculus and fractional differential equations, we refer the reader to see[3, 4, 19, 7, 21, 2, 14, 18, 5, 10, 12, 13, 20].

In [2], the researchers studied the existence of solutions to boundary-value problems for fractional-order differential equation of the form

$$\begin{gathered}
{}^{C}D^{\alpha}_{0+}y(t) = f(t, y(t)), \quad t \in [0, +\infty),\\
y(0) = y_0, \quad y \text{ is bounded in } [0, +\infty),
\end{gathered}$$

where $1 < \alpha \le 2$ and $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. And the results are based on a fixed point theorem of Schauder combined with the diagonalization method. Then, Mouffak Benchohra and Naima Hamidi are concerned with the differential inclusions of the form above in [5].

Liu and Jia [14] studied the existence of multiple solutions of nonlocal boundary value problems of fractional order with integral boundary conditions on the half-line applying the fixed point theory and the upper and lower solutions method.

Su and Zhang [18] studied the following fractional differential equations on the half-line, using Schauder's fixed point theorem,

$$\begin{gathered}
D^{\alpha}_{0+}u(t) = f(t, u(t), D^{\alpha-1}_{0+}u(t)), \quad t \in (0, +\infty),\ 1 < \alpha \le 2,\\
u(0) = 0, \quad \lim_{t\to\infty} D^{\alpha-1}_{0+}u(t) = u_{\infty}.
\end{gathered}$$

In paper [10] and [20], the authors investigated the existence of global solutions for fractional differential equations on the half-axis. Liang and Shi [12] obtained some existence results of multiple positive solutions for m-point fractional boundary value problems with p-Laplacian operator on infinite interval by means of the properties of the Green function and some fixed-point theorems. And in [13], by a fixed point theorem due to Leggett-Williams, Liang and Zhang studied the existence of three positive solutions for the boundary value problem on the half-line.

However, the papers on the existence of solutions of fractional differential equations on the half-line are only handling with the problems under nonresonance conditions. And as far as we know, there is no paper dealing with the differential equations of sequential fractional order under resonance conditions on the half-line. Motivated by the papers [2, 14, 18, 5, 10, 12, 13, 20, 6, 11, 9], in this paper, we are concerned with the existence of the m-point boundary value problems (1.1).

Our methods are based on the Mawhin's continuation theorem of coincidence degree theory, unlike any other papers, the function f in the problem (1.1) satisfies the α-Carathéodory conditions, the definition of which will be given in Section 2. And the main difficulties are that we have to construct suitable Banach spaces for the problem and establish an appropriate compactness criterion.

The rest of the paper is organized as follows. Section 2, we give some results about fractional differential equations and an abstract existence theorem and present the special Banach spaces that will be used in the paper. Section 3, we obtain some existence results of the solutions for the problem (1.1) by applying the coincidence degree continuation theorem. Then an example is given in Section 4 to demonstrate the application of our results.

2. Preliminaries

First of all, we present some fundamental facts on the fractional calculus theory which we'll use in the next section. These can be found in [8, 16, 19].

Lemma 2.1 ([19]). *Let $\nu > 0$; then the differential equation ${}^C D^{\nu}_{0+} h(t) = 0$ has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\nu] + 1$.*

Lemma 2.2 ([19]). *Let $\nu > 0$; then $I^{\nu}_{0+} {}^C D^{\nu}_{0+} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, for some $c_i \in \mathbb{R}$, $i = 0, 1, \ 2, \dots, n-1$, where $n = [\nu] + 1$.*

Lemma 2.3 ([8, 16]). *If $\nu_1, \nu_2, \nu > 0$, $t \in [0, 1]$ and $h(t) \in L[0, 1]$, then we have*

$$I^{\nu_1}_{0+} I^{\nu_2}_{0+} h(t) = I^{\nu_1+\nu_2}_{0+} h(t), \quad {}^C D^{\nu}_{0+} I^{\nu}_{0+} h(t) = h(t). \tag{2.1}$$

Now let us recall some notation about the coincidence degree continuation theorem.

Let X, Z be real Banach spaces. Consider an operation equation $Lu = Nu$, where $L : \operatorname{dom} L \subset X \to Z$ is a linear operator, $N : X \to Z$ is a nonlinear operator. If $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z, then L is called a Fredholm mapping of index zero. And if L is a Fredholm mapping of index zero, there exist linear continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\ker L = \operatorname{Im} P$, $\operatorname{Im} L = \ker Q$ and $X = \ker L \oplus \ker P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Then it follows that $L_P = L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible. We denote the inverse of this map by K_P. If $\overline{\Omega}$ is an open bounded subset of X, the map N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I-Q)N : \overline{\Omega} \to X$ is compact. For $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J_{NL} : \operatorname{Im} Q \to \ker L$. Then we will give the the coincidence degree continuation theorem which is proved in [15].

Theorem 2.4. *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Suppose that the following conditions are satisfied:*

1. *$Lx \neq \lambda Nx$ for each $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;*
2. *$Nx \notin \operatorname{Im} L$ for each $x \in \ker L \cap \partial\Omega$;*
3. *$\deg(J_{NL} QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Z \to Z$ is a continuous projection as above with $\operatorname{Im} L = \ker Q$ and $J_{NL} : \operatorname{Im} Q \to \ker L$ is any isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Definition 2.5. We say that $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the α-Carathéodory conditions if

(1) for each $(x,y,z) \in \mathbb{R}^3$, the function $t \to f(t,x,y,z)$ is Lebesgue measurable;
(2) for almost every $t \in [0,+\infty)$, the function $t \to f(t,x,y,z)$ is continuous in $\mathbb{R}^3$;
(3) for each $r > 0$, there exists $\varphi_r(t) \in L^1[0,+\infty) \cap C[0,+\infty)$ subject to $\lim_{t\to+\infty} I^{\alpha}_{0+}\varphi_r(t) < +\infty$ such that for a.e. $t \in [0,+\infty)$ and all $(x,y,z) \in \mathbb{R}^3$ with $\|(x,y,z)\| \le r$,
$$|f(t,x,y,z)| \le \varphi_r(t)$$
where $\|\cdot\|$ is the norm in $\mathbb{R}^3$.

The assumptions on $a(t)$ are as follows:

(A1) $a(t) \in C^1[0,+\infty)$, $a(t) > 0$, for all $t \in [0,+\infty)$, and
$$L_a := \int_0^{+\infty} \frac{1}{a(t)} dt < +\infty;$$
(A2)
$$I_a := \lim_{t\to+\infty} I^{1-\alpha}_{0+} \frac{1}{a(t)} = 0.$$

If condition (A1) holds, then
$$M_a := \sup_{t\ge0} \frac{1}{a(t)} < +\infty.$$
Set
$$X = \big\{x \in C^1[0,+\infty) : \lim_{t\to+\infty} x(t),\ \lim_{t\to+\infty} {}^C D^{\alpha}_{0+}x(t) \text{ and } \lim_{t\to+\infty} x'(t) \text{ exist}\big\},$$
equipped with the norm
$$\|x\|_X = \sup_{t\ge0} |x(t)| + \sup_{t\ge0} |{}^C D^{\alpha}_{0+}x(t)| + \sup_{t\ge0} |x'(t)|\,.$$
Since $x(t) \in C^1[0,+\infty)$ implies that ${}^C D^{\alpha}_{0+}x(t) \in C[0,+\infty)$, the space $(X, \|\cdot\|_X)$ is well defined. It is easy to show that $(X, \|\cdot\|_X)$ is a Banach space.

Define
$$Z = \big\{z \in C[0,+\infty) \cap L^1[0,+\infty) : \lim_{t\to+\infty} I^{\alpha}_{0+}z(t) \text{ exists}\big\},$$
equipped with the norm
$$\|z\|_Z = \sup_{t\ge0} |z(t)| + \sup_{t\ge0} |I^{\alpha}_{0+}z(t)| + \int_0^{+\infty} |z(t)| dt\,.$$
The space $(Z, \|\cdot\|_Z)$ is well defined in virtue of the fact that $z(t) \in C[0,+\infty) \cap L^1[0,+\infty)$ leads to $\lim_{t\to+\infty} z(t) = 0$ and $I^{\alpha}_{0+}z(t) \in C[0,+\infty)$. Also, $(Z, \|\cdot\|_Z)$ is a Banach space.

Let
$$\operatorname{dom} L = \Big\{u : {}^C D^{\alpha}_{0+}\big(a(t)u'(t)\big) \in L^1[0,+\infty) \cap C[0,+\infty),\ \lim_{t\to+\infty} a(t)u'(t) \text{ exists},\\ u'(0) = 0,\ \sum_{j=1}^{m-1} \sigma_j u(\xi_j) = \lim_{t\to+\infty} u(t)\Big\} \cap X.$$

Define

$$L : \operatorname{dom} L \to Z, \quad u \mapsto {}^C D_{0+}^{\alpha}\big(a(t)u'(t)\big), \tag{2.2}$$

$$N : X \to Z, \quad u \mapsto f(t, u(t), {}^C D_{0+}^{\alpha} u(t), u'(t)). \tag{2.3}$$

Then the multi-point boundary-value problem (1.1) can be written as

$$Lu = Nu, \quad u \in \operatorname{dom} L.$$

Definition 2.6. A function $u \in X$ is called a solution of (1.1) if $u \in \operatorname{dom} L$ and u satisfies (1.1).

Next, similar to the compactness criterion in [1], we establish the following criterion.

Lemma 2.7. *The set $\mathcal{U}$ is relatively compact in X if and only if the following conditions are satisfied:*

(a) *$\mathcal{U}$ is uniformly bounded; that is, there exists a constant $R > 0$, such that for each $u \in \mathcal{U}$, $\|u\|_X \le R$;*

(b) *functions in $\mathcal{U}$ are equicontinuous on any compact subinterval of $[0, +\infty)$; that is, let J be a compact subinterval of $[0, +\infty)$, then, $\forall\, \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$,*

$$|u(t_1) - u(t_2)| < \varepsilon, \quad |u'(t_1) - u'(t_2)| < \varepsilon, \quad |{}^C D_{0+}^{\alpha} u(t_1) - {}^C D_{0+}^{\alpha} u(t_2)| < \varepsilon,$$

for all $u \in \mathcal{U}$;

(c) *functions from $\mathcal{U}$ are equiconvergent; that is, given $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$, such that for $s_1, s_2 > T$, for all $u \in \mathcal{U}$,*

$$|u(s_1) - u(s_2)| < \varepsilon, \quad |u'(s_1) - u'(s_2)| < \varepsilon, \quad |{}^C D_{0+}^{\alpha} u(s_1) - {}^C D_{0+}^{\alpha} u(s_2)| < \varepsilon.$$

Proof. We can prove the results by the fact that $\mathcal{U}$ is a relatively compact set in X if and only if $\mathcal{U}$ is totally bounded. The proof is analogous to the proof of the [21, Lemma 2.2]. Here we omit it. $\square$

3. Main Results

In this section, we establish the existence of solutions for (1.1) on the half-line. To prove our main results, we need the following lemmas.

Lemma 3.1. *Let $g \in Z$. Suppose that the condition* (A1) *holds. Then $u \in X$ is the solution of the fractional differential equation*

$$\begin{gathered} {}^C D_{0+}^{\alpha}\big(a(t)u'(t)\big) = g(t), \quad t \in [0, +\infty), \\ u'(0) = 0, \quad \sum_{j=1}^{m-1} \sigma_j u(\xi_j) = \lim_{t \to +\infty} u(t), \end{gathered} \tag{3.1}$$

if and only if u satisfies

$$u(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{a(s)} \int_0^s (s - \tau)^{\alpha - 1} g(\tau)\, d\tau\, ds, \quad c \in \mathbb{R}, \tag{3.2}$$

and

$$\int_0^{+\infty} \frac{1}{a(s)} \int_0^s (s - \tau)^{\alpha - 1} g(\tau)\, d\tau\, ds - \sum_{j=1}^{m-1} \sigma_j \int_0^{\xi_j} \frac{1}{a(s)} \int_0^s (s - \tau)^{\alpha - 1} g(\tau)\, d\tau\, ds = 0. \tag{3.3}$$

Proof. "Necessity". Assume that u is a solution of (3.1). By Lemma 2.2, we have

$$a(t)u'(t) = c_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad c_1 \in \mathbb{R}.$$

Since $u'(0) = 0$ and $a(t) > 0$, we have

$$u'(t) = \frac{1}{a(t)} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds.$$

Then we obtain

$$u(t) = c + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} g(\tau)\, d\tau\, ds, \quad c \in \mathbb{R}.$$

Since $\sum_{j=1}^{m-1} \sigma_j u(\xi_j) = \lim_{t\to+\infty} u(t)$, we have

$$\int_0^{+\infty} \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} g(\tau)\, d\tau\, ds = \sum_{j=1}^{m-1} \sigma_j \int_0^{\xi_j} \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} g(\tau)\, d\tau\, ds.$$

due to the fact that $\sum_{j=1}^{m-1} \sigma_j = 1$.

"Sufficiency". Conversely, suppose that (3.2) and (3.3) hold. In view of Lemma 2.3, we can easily certify that u is the solution of the equation (3.1). The proof is complete. □

Lemma 3.2. *Assume that the condition* (A1) *holds. Then L is a Fredholm mapping of index zero. Moreover,*

$$\ker L = \{u : u = c,\ c \in \mathbb{R}\} \subset X, \tag{3.4}$$

and

$$\operatorname{Im} L = \{g \in Z : g \text{ satisfies condition (3.3)}\} \subset Z. \tag{3.5}$$

Proof. It is obvious that Lemma 3.1 implies (3.4) and (3.5). Now, let us focus our minds to prove that L is a Fredholm mapping of index zero.

Define an auxiliary mapping $Q_1 : Z \to \mathbb{R}$:

$$Q_1 g = \int_0^{+\infty} \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} g(\tau)\, d\tau\, ds - \sum_{j=1}^{m-1} \sigma_j \int_0^{\xi_j} \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} g(\tau)\, d\tau\, ds,$$

where $g \in Z$. It is obvious that Q_1 is a continuous linear mapping.

Take an element $\mu(t) \in Z$ satisfying $\mu(t) > 0$ on $[0, +\infty)$, for example, $\mu(t) = e^{-at},\ a > 0$. In view of $\sum_{i=1}^{m-1} \sigma_j = 1$ and $\sigma_j > 0, j = 1, 2, \dots, m-1$, we have

$$\begin{aligned}
Q_1 \mu &= \int_0^{+\infty} \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} \mu(\tau)\, d\tau\, ds \\
&\quad - \sum_{j=1}^{m-1} \sigma_j \int_0^{\xi_j} \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} \mu(\tau)\, d\tau\, ds \\
&= \sum_{j=1}^{m-1} \sigma_j \int_{\xi_j}^{+\infty} \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} \mu(\tau)\, d\tau\, ds > 0.
\end{aligned}$$

Let the mapping $Q : Z \to Z$ be defined by

$$(Qg)(t) = \frac{Q_1 g}{Q_1 \mu} \mu(t), \tag{3.6}$$

where $g \in Z$. Evidently,

$$\operatorname{Im} Q = \{g : g = c\,\mu(t),\ c \in \mathbb{R}\},$$

and $Q : Z \to Z$ is a continuous linear projector. In fact, for an arbitrary $g \in Z$, we have

$$Q_1(Qg) = Q_1\big(\frac{Q_1 g}{Q_1 \mu}\mu(t)\big) = \frac{Q_1 g}{Q_1 \mu} Q_1(\mu) = Q_1 g,$$

$$Q^2 g = Q(Qg) = \frac{Q_1(Qg)}{Q_1 \mu}\mu(t) = \frac{Q_1 g}{Q_1 \mu}\mu(t) = Qg;$$

that is to say, $Q : Z \to Z$ is idempotent.

Observe that $g \in \operatorname{Im} L$ leads to $Q_1 g = 0$, then we can get that $Qg = \theta$, and $g \in \ker Q$, where we denote θ the zero element in Z. Conversely, if $g \in \ker Q$, we can have that $Q_1 g = 0$, that is to say, $g \in \operatorname{Im} L$. So, $\ker Q = \operatorname{Im} L$.

Let $g = g - Qg + Qg = (I - Q)g + Qg$, where $g \in Z$ is an arbitrary element. Since $Qg \in \operatorname{Im} Q$ and $(I - Q)g \in \ker Q$, we obtain that $Z = \operatorname{Im} Q + \ker Q$. Take $z_0 \in \operatorname{Im} Q \cap \ker Q$, then z_0 can be written as $z_0 = c\,\mu(t)$, $c \in \mathbb{R}$, for $z_0 \in \operatorname{Im} Q$. Since $z_0 \in \ker Q = \operatorname{Im} L$, by (3.5), we get that $Q_1(z_0) = Q_1(c\,\mu(t)) = cQ_1(\mu) = 0$, which implies that $c = 0$ and $z_0 = \theta$. Therefore $\operatorname{Im} Q \cap \ker Q = \{\theta\}$, thus $Z = \operatorname{Im} Q \oplus \ker Q = \operatorname{Im} Q \oplus \operatorname{Im} L$.

Now, $\dim \ker L = 1 = \dim \operatorname{Im} Q = \operatorname{codim} \ker Q = co\dim \operatorname{Im} L < +\infty$, and observing that $\operatorname{Im} L$ is closed in Z, so L is a Fredholm mapping of index zero. $\square$

Let $P : X \to X$ be defined by

$$(Pu)(t) = u(0), \quad u \in X. \tag{3.7}$$

It is clear that $P : X \to X$ is a linear continuous projector and

$$\operatorname{Im} P = \{u \big| u = c,\ c \in \mathbb{R}\} = \ker L.$$

Also, proceeding as the proof of Lemma 3.2, we can show that $X = \operatorname{Im} P \oplus \ker P = \ker L \oplus \ker P$.

Consider the mapping $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$,

$$(K_P g)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{a(s)} \int_0^s (s-\tau)^{\alpha-1} g(\tau)\, d\tau\, ds, \quad g \in \operatorname{Im} L.$$

Note that

$$(K_P L)u = K_P(Lu) = u, \quad \forall\, u \in \operatorname{dom} L \cap \ker P, \tag{3.8}$$

and

$$(LK_P)g = L(K_P g) = g, \quad \forall\, g \in \operatorname{Im} L.$$

Thus, $K_P = \big(L_P\big)^{-1}$, where $L_P = L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} P$.

Lemma 3.3. *Assume that* (A1) *and* (A2) *hold. Then, the operator* $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker L$ *is completely continuous. Further,*

$$\big\|K_P g\big\|_X \le (L_a + 2M_a)\big\|g\big\|_Z\,, \tag{3.9}$$

for each $g \in \operatorname{Im} L$.

Proof. We know that K_P is linear, we only need to prove that K_P is compact and (3.9) holds. For each $g \in \operatorname{Im} L$ and $t \in [0, +\infty)$, we have

$$\big|(K_P g)(t)\big| = \big| \int_0^t \frac{1}{a(s)} I_{0+}^{\alpha} g(s) ds \big| \le \int_0^t \frac{1}{a(s)} |I_{0+}^{\alpha} g(s)| ds$$

$$\le \sup_{t\ge 0}|I^\alpha_{0+}g(s)|\int_0^t \frac{1}{a(s)}ds \le L_a\cdot\|g\|_Z\,,$$

$$\big|(K_Pg)'(t)\big| = \Big|\frac{1}{\Gamma(\alpha)}\frac{1}{a(t)}\int_0^t (t-s)^{\alpha-1}g(s)ds\Big| = \frac{1}{a(t)}|I^\alpha_{0+}g(t)| \le M_a\cdot\|g\|_Z\,,$$

and

$$\begin{aligned}
\Big|{}^CD^\alpha_{0+}(K_Pg)(t)\Big| &= \Big|{}^CD^\alpha_{0+}\Big(\int_0^t \frac{1}{a(s)}I^\alpha_{0+}g(s)ds\Big)\Big| = \Big|I^{1-\alpha}_{0+}\Big(\frac{1}{a(t)}I^\alpha_{0+}g(t)\Big)\Big| \\
&= \Big|\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{-\alpha}\frac{1}{a(s)}\int_0^s (s-\tau)^{\alpha-1}g(\tau)d\tau\,ds\Big| \\
&= \Big|\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^t g(\tau)\int_\tau^t (s-\tau)^{\alpha-1}(t-s)^{-\alpha}\frac{1}{a(s)}ds\,d\tau\Big| \\
&= \Big|\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^t g(\tau)\int_0^1 s^{\alpha-1}(1-s)^{-\alpha}\frac{1}{a(\tau+s(t-\tau))}ds\,d\tau\Big| \\
&\le M_a\int_0^t |g(\tau)|d\tau \le M_a\cdot\|g\|_Z\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\big\|K_Pg\big\|_X &= \sup_{t\ge 0}|(K_Pg)(t)| + \sup_{t\ge 0}|{}^CD^\alpha_{0+}(K_Pg)(t)| + \sup_{t\ge 0}|(K_Pg)'(t)| \\
&\le (L_a+2M_a)\big\|g\big\|_Z\,.
\end{aligned}$$

Next, we show that K_P is compact. Let G be a bounded set in Z; i.e., there exists $r>0$ such that $\|g\|_Z\le r$, $\forall g\in G$. Then we need to validate that $K_P(G)$ is relatively compact via Lemma 2.7.

First, $K_P(G)$ is bounded in view of (3.9).

Second, $K_P(G)$ is equicontinuous on any compact subinterval J of $[0,+\infty)$. There exist two positive constants $T_1, T_2 (T_1<T_2)$ such that $J\subset[T_1,T_2]$. Since $1/a(t)$ is uniformly continuous on $[0,T_2]$, for all $\varepsilon>0$, there exists $\delta_1=\delta_1(\varepsilon)>0$ such that for $\tau_1,\tau_2\in[0,T_2]$, $|\tau_1-\tau_2|<\delta_1$, we have that $\big|\frac{1}{a(\tau_1)}-\frac{1}{a(\tau_2)}\big|<\frac{\varepsilon}{2r}$. Let

$$\delta=\min\Big\{\delta_1,\ \frac{\varepsilon}{2rM_a},\ \Big(\frac{\varepsilon\Gamma(\alpha+1)}{4rM_a}\Big)^{1/\alpha}\Big\}.$$

Then for every pair $t_1,t_2\in J$ and $|t_1-t_2|<\delta\,(t_1<t_2)$ we have

$$\big|(K_Pg)(t_1)-(K_Pg)(t_2)\big| \le \int_{t_1}^{t_2}\frac{1}{a(s)}|I^\alpha_{0+}g(s)|ds \le rM_a(t_2-t_1)\le\frac{\varepsilon}{2}<\varepsilon,$$

$$\begin{aligned}
&\big|{}^CD^\alpha_{0+}(K_Pg)(t_1) - {}^CD^\alpha_{0+}(K_Pg)(t_2)\big| \\
&\le \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\Big|\int_0^{t_1} g(\tau)\int_0^1 s^{\alpha-1}(1-s)^{-\alpha}\Big(\frac{1}{a(\tau+s(t_1-\tau))} \\
&\quad -\frac{1}{a(\tau+s(t_2-\tau))}\Big)ds\,d\tau\Big| \\
&\quad +\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\Big|\int_{t_1}^{t_2} g(\tau)\int_0^1 s^{\alpha-1}(1-s)^{-\alpha}\frac{1}{a(\tau+s(t_2-\tau))}ds\,d\tau\Big| \\
&< \frac{\varepsilon}{2r}\cdot\int_0^{t_1}|g(\tau)|d\tau + M_a\int_{t_1}^{t_2}|g(\tau)|d\tau
\end{aligned}$$

$$< \frac{\varepsilon}{2r} \cdot r + rM_a(t_2 - t_1) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and

$$\begin{aligned}
&\big|(K_P g)'(t_1) - (K_P g)'(t_2)\big| \\
&\le \Big|\Big(\frac{1}{a(t_1)} - \frac{1}{a(t_2)}\Big) I_{0+}^{\alpha} g(t_1)\Big| + \frac{1}{a(t_2)}\Big|I_{0+}^{\alpha} g(t_1) - I_{0+}^{\alpha} g(t_2)\Big| \\
&\le \frac{\varepsilon}{2} + \frac{M_a}{\Gamma(\alpha)}\Big(\int_0^{t_1} \big[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\big]|g(s)|ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}|g(s)|ds\Big) \\
&\le \frac{\varepsilon}{2} + \frac{rM_a}{\Gamma(\alpha+1)}\Big({t_1}^{\alpha} - {t_2}^{\alpha} + 2(t_2 - t_1)^{\alpha}\Big) \\
&\le \frac{\varepsilon}{2} + \frac{rM_a}{\Gamma(\alpha+1)} 2(t_2 - t_1)^{\alpha} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, $K_P(G)$ is equicontinuous on the compact subinterval J of $[0, +\infty)$.

Third, $K_P(G)$ is equiconvergent. Since (A1) and (A2) hold, $\lim_{t\to+\infty} 1/a(t) = 0$. For all $\varepsilon_1 > 0$, there exists a constant $T > 0$ such that for all $t, t_1, t_2 \ge T$ $(t_1 < t_2)$, we have

$$0 < \frac{1}{a(t)} < \frac{\varepsilon_1}{2r}, \quad 0 < \int_{t_1}^{t_2} \frac{1}{a(s)} ds < \frac{\varepsilon_1}{2r}, \quad \big|I_{0+}^{1-\alpha} \frac{1}{a(t)}\big| < \frac{\varepsilon_1}{2r}\,.$$

So, for all $t_1, t_2 \ge T$ $(t_1 < t_2)$, we have

$$\big|(K_P g)(t_1) - (K_P g)(t_2)\big| \le \int_{t_1}^{t_2} \frac{1}{a(s)} |I_{0+}^{\alpha} g(s)| ds \le r \int_{t_1}^{t_2} \frac{1}{a(s)} ds < r\frac{\varepsilon_1}{2r} < \varepsilon_1,$$

$$\begin{aligned}
\big|{}^C D_{0+}^{\alpha}(K_P g)(t_1) - {}^C D_{0+}^{\alpha}(K_P g)(t_2)\big| &\le r\Big|I_{0+}^{1-\alpha} \frac{1}{a(t_1)}\Big| + r\Big|I_{0+}^{1-\alpha} \frac{1}{a(t_2)}\Big| \\
&< \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1
\end{aligned}$$

and

$$\begin{aligned}
\big|(K_P g)'(t_1) - (K_P g)'(t_2)\big| &\le \frac{1}{a(t_1)} |I_{0+}^{\alpha} g(t_1)| + \frac{1}{a(t_2)} |I_{0+}^{\alpha} g(t_2)| \\
&< \frac{\varepsilon_1}{2r} r + \frac{\varepsilon_1}{2r} r = \varepsilon_1.
\end{aligned}$$

Hence, by Lemma 2.7, $K_P(G)$ is relatively compact, and the proof is complete. □

Lemma 3.4. *Let $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the α-Carathéodory conditions. Assume that the condition* (A1) *and* (A2) *hold. Then $K_{P,Q}N = K_P(I - Q)N : X \to X$ is completely continuous.*

Proof. In view of the continuity of K_p, $I - Q$ and the boundedness of N, combining with the Lemma 3.3, we can conclude that the claim of the lemma is true. □

The following assumptions that will be used later.

(H1) There exist four functions $\beta_1, \beta_2, \beta_3, \beta_4 \in Z$ such that $\beta_i(t) \ge 0, t \in [0, +\infty)$ $(i = 1, 2, 3, 4)$, and for $t \in [0, +\infty)$ and $(x, y, z) \in \mathbb{R}^3$, we have

$$|f(t, x, y, z)| \le \beta_1(t)|x| + \beta_2(t)|y| + \beta_3(t)|z| + \beta_4(t); \tag{3.10}$$

(H2)
$$0 < \eta_1(2L_a + 2M_a) < 1, \tag{3.11}$$
where η_1 is defined by $\eta_1 = \|\beta_1\|_Z + \|\beta_2\|_Z + \|\beta_3\|_Z$;

(H3) There exists a constant $\Lambda_1 > 0$ such that
$$Q_1(Nu) \neq 0, \tag{3.12}$$
for each $u \in \operatorname{dom} L \setminus \ker L$ satisfying $\big|u(t)\big| > \Lambda_1$;

(H4) There exists a constant $S > 0$ such that for any $c \in \mathbb{R}$, if $|c| > S$, then either
$$c\, Q_1\big(N(c)\big) < 0, \tag{3.13}$$
or
$$c\, Q_1\big(N(c)\big) > 0. \tag{3.14}$$

Lemma 3.5. *Set* $\Omega_1 = \Big\{u \in \operatorname{dom} L \setminus \ker L \Big| Lu = \lambda Nu,\ \lambda \in [0,1]\Big\}$*. Suppose that* (H1), (H2), (H3) *hold. Then,* Ω_1 *is bounded.*

Proof. Take $u \in \Omega_1$, then $u \in \operatorname{dom} L \setminus \ker L$ and $Lu = \lambda Nu$, so $\lambda \neq 0$ and $Nu \in \operatorname{Im} L = \ker Q \subset Z$. Hence, $Q(Nu) = \theta$; that is, $Q_1(Nu) = 0$. From (H3), we have that there exists $t_1 \in [0, +\infty)$ such that $|u(t_1)| \leq \Lambda_1$.

If $t_1 = 0$, then $|u(0)| \leq \Lambda_1$. If $t_1 > 0$, by the fact that
$$u'(t) = \frac{1}{a(t)} I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha}\big(a(t)u'(t)\big) = \frac{1}{a(t)} I_{0+}^{\alpha}(Lu)(t), \quad t \in [0, +\infty),$$
we obtain
$$\begin{aligned}|u(0)| &= \Big|u(t_1) - \int_0^{t_1} u'(s)ds\Big| = \Big|u(t_1) - \int_0^{t_1} \frac{1}{a(s)} I_{0+}^{\alpha}(Lu)(s)ds\Big|\\ &\leq |u(t_1)| + \int_0^{t_1} \frac{1}{a(s)} |I_{0+}^{\alpha}(Lu)(s)|ds\\ &\leq \Lambda_1 + L_a\|Lu\|_Z \leq \Lambda_1 + L_a\|Nu\|_Z\,.\end{aligned}$$
Again, for $u \in \Omega_1$, we obtain
$$\begin{aligned}\big\|Pu\big\|_X &= \sup_{t\geq 0}|(Pu)(t)| + \sup_{t\geq 0}|{}^{C}D_{0+}^{\alpha}(Pu)(t)| + \sup_{t\geq 0}|(Pu)'(t)|\\ &= |u(0)| \leq \Lambda_1 + L_a\|Nu\|_Z\,.\end{aligned} \tag{3.15}$$
In view of $(I-P)u \in \operatorname{dom} L \cap \ker P$, by (3.8) and Lemma 3.3, we have
$$\begin{aligned}\big\|(I-P)u\big\|_X &= \big\|K_pL(I-P)u\big\|_X \leq (L_a + 2M_a)\big\|L(I-P)u\big\|_Z\\ &= (L_a + 2M_a)\big\|Lu\big\|_Z \leq (L_a + 2M_a)\big\|Nu\big\|_Z\,.\end{aligned} \tag{3.16}$$
Combining (3.15) and (3.16), we obtain
$$\begin{aligned}\big\|u\big\|_X &= \big\|u - Pu + Pu\big\|_X\\ &\leq \big\|Pu\big\|_X + \big\|(I-P)u\big\|_X\\ &\leq \Lambda_1 + (2L_a + 2M_a)\big\|Nu\big\|_Z\,.\end{aligned} \tag{3.17}$$
From (H1), for each $u \in \Omega_1$, we have
$$\int_0^{+\infty} |(Nu)(s)|ds$$

$$\le \Big(\int_0^{+\infty} |\beta_1(s)|ds + \int_0^{+\infty} |\beta_2(s)|ds + \int_0^{+\infty} |\beta_3(s)|ds\Big) \|u\|_X + \int_0^{+\infty} |\beta_4(s)|ds\,,$$

$$\begin{aligned}
&\big|f(t, u(t), {}^C D_{0+}^{\alpha} u(t), u'(t))\big| \\
&\le \Big(\sup_{t\ge 0} |\beta_1(t)| + \sup_{t\ge 0} |\beta_2(t)| + \sup_{t\ge 0} |\beta_3(t)|\Big) \|u\|_X + \sup_{t\ge 0} |\beta_4(t)|\,,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|I_{0+}^{\alpha} f(t, u(t), {}^C D_{0+}^{\alpha} u(t), u'(t))\Big| \\
&\le \Big(\sup_{t\ge 0} |I_{0+}^{\alpha}\beta_1(s)| + \sup_{t\ge 0} |I_{0+}^{\alpha}\beta_2(s)| + \sup_{t\ge 0} |I_{0+}^{\alpha}\beta_3(s)|\Big)\|u\|_X + \sup_{t\ge 0} |I_{0+}^{\alpha}\beta_4(s)|\,.
\end{aligned}$$

Then, we can deduce that

$$\begin{aligned}
\|Nu\|_Z &\le \big(\|\beta_1\|_Z + \|\beta_2\|_Z + \|\beta_3\|_Z\big)\|u\|_X + \|\beta_4\|_Z \\
&= \eta_1\|u\|_X + \eta_2,
\end{aligned} \tag{3.18}$$

where we denote $\eta_2 = \|\beta_4\|_Z$. Thus, by (H2), (3.17) and (3.18) imply that

$$\|u\|_X \le \frac{\Lambda_1 + \eta_2(2L_a + 2M_a)}{1 - \eta_1(2L_a + 2M_a)}\,,$$

which clearly states that Ω_1 is bounded. □

Lemma 3.6. *Set $\Omega_2 = \Big\{u \in \ker L \Big| N u \in \operatorname{Im} L\Big\}$. Assume that (H4) holds, then Ω_2 is bounded.*

Proof. Let $u \in \Omega_2$, then $u \in \ker L$ and $u = c$, $c \in \mathbb{R}$. Since $Nu \in \operatorname{Im} L = \ker Q$, we have $Q(Nu) = \theta$; that is, $Q_1\big(N(c)\big) = 0$. Taking account of (H4), $|c| \le S$, which implies that Ω_2 is bounded. □

Lemma 3.7. *If* (3.13) *holds, set*

$$\Omega_3 = \big\{u \in \ker L \big| -\lambda\, u + (1-\lambda)\, J_{NL} Q N u = 0,\ \lambda \in [0, 1]\big\};$$

if (3.14) *holds, set*

$$\Omega_3 = \big\{u \in \ker L \big| \lambda u + (1-\lambda) J_{NL} Q N u = 0,\ \lambda \in [0, 1]\big\},$$

where $J_{NL} : \operatorname{Im} Q \to \ker L$ is a linear isomorphism defined as

$$J_{NL}(c\,\mu(t)) = c, \quad c \in \mathbb{R},\ t \in [0, +\infty). \tag{3.19}$$

Assume that (H4) *holds. Then Ω_3 is bounded.*

Proof. If (3.13) holds, for $u \in \Omega_3$, then we have $u = c$, $c \in \mathbb{R}$ and $\lambda u = (1 - \lambda) J_{NL} Q N u$. Thus,

$$\lambda c = (1-\lambda) \frac{Q_1\big(N(c)\big)}{Q_1(\mu(t))} \mu(t)\,.$$

Therefore, via (H4) and (3.13), we have $|c| \le S$, which shows that Ω_3 is bounded. If (3.14) holds, the proof is similar. □

Next, let us give the main results of the paper.

Theorem 3.8. *Let $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the α-Carathéodory conditions. Assume that the condition* (A1), (A2), (H1), (H2), (H3), (H4) *hold. Then problem* (1.1) *has at least one solution in* $\operatorname{dom} L$.

Proof. Let Ω be an bounded open set such that $\Omega \supset \cup_{i=1}^{3} \overline{\Omega}_i$ and we will prove that
$$\deg(J_{_{NL}} QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0.$$
The operator N is L-compact on $\overline{\Omega}$ due to the fact that $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I-Q)N : \overline{\Omega} \to X$ is compact by Lemma 3.4.

In view of Lemmas 3.5 and 3.6, we have that

(1) $L\,u \neq \lambda N u$ for each $(u, \lambda) \in [(\operatorname{dom} L \setminus \ker\ L) \cap \partial\Omega] \times (0, 1)$;
(2) $Nu \notin \operatorname{Im} L$ for each $u \in \ker L \cap \partial\Omega$.

Without loss of generality, we suppose that (3.14) holds. Define $H(u, \lambda) = \lambda I u + (1-\lambda) J_{_{NL}} QNu$, where I is the identity operator in X. According to the arguments in Lemma 3.7, we can get
$$H(u, \lambda) \neq 0, \quad \forall u \in \ker L \cap \partial\Omega,$$
and therefore, via the homotopy property of degree, we obtain that
$$\begin{aligned}
\deg\left(J_{_{NL}} QN|_{\ker L}, \Omega \cap \ker L, 0\right) &= \deg\left(H(\cdot, 0), \Omega \cap \ker L, 0\right) \\
&= \deg\left(H(\cdot, 1), \Omega \cap \ker L, 0\right) \\
&= \deg\left(I, \Omega \cap \ker L, 0\right) = 1,
\end{aligned}$$
which verifies the condition (3) of Theorem 2.4. Applying Theorem 2.4, we conclude that (1.1) has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. The proof is complete. □

For the next theorem, we use the assumption

(H3') There exists a constant $\Lambda_2 > 0$ such that
$$f(t, x, y, z) > 0, \quad t \in [0, +\infty), \tag{3.20}$$
or
$$f(t, x, y, z) < 0, \quad t \in [0, +\infty), \tag{3.21}$$
for each $|x| > \Lambda_2$.

Theorem 3.9. *Let $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ satisfy the α-Carathéodory conditions. Assume that the condition* (A1), (A2), (H1), (H2), (H4), (H3') *hold. Then* (1.1) *has at least one solution in* $\operatorname{dom} L$.

Proof. We assume that (3.20) holds. Then, for each $u \in \Omega_1$, we have $Q(Nu) = \theta$, that is, $Q_1(Nu) = 0$. So, there exists $t_2 \in [0, +\infty)$ such that $|u(t_2)| \leq \Lambda_2$. Analogous to the proof of Lemma 3.5, we have that
$$\|u\|_X \leq \frac{1 - \eta_1(2L_a + 2M_a)}{2\Lambda_1 + \eta_2(L_a + 2M_a)}\,,$$
Then, Ω_1 is bounded. Hence, similar to the proof of Theorem 3.8, we can conclude that the problem (1.1) has at least one solution in $\operatorname{dom} L$. □

4. Examples

To illustrate our main results, we present the following example.
$$\begin{gathered}
{}^{C}D_{0+}^{0.6}\big(a(t)u'(t)\big) = f(t, u(t), {}^{C}D_{0+}^{0.6}u(t), u'(t)), \quad t \in [0, +\infty), \\
u'(0) = 0, \quad \lim_{t \to +\infty} u(t) = \sum_{j=1}^{3} \sigma_j u(\xi_j),
\end{gathered} \tag{4.1}$$

where $a(t)=e^t$, and for $(x,y,z)\in\mathbb{R}^3$,

$$\begin{aligned} f(t,x,y,z)=(\sqrt[3]{|x|}-10)\Big(&\beta_1(t)\frac{|\sin x|}{12+x^2+y^2}+\beta_2(t)\frac{|y|}{11+y^2+x^2}\\ &+\beta_3(t)\frac{|z|e^{-|z|}}{(16+x^2)(1+|z|)}\Big); \end{aligned}$$

$$\beta_1(t)=\frac{e^{-t}(1+t)^{-\alpha}}{10(1+t^2)},\quad \beta_2(t)=\frac{e^{-2t}(2+t)^{-\alpha}}{20(1+t^3)},\quad \beta_3(t)=\frac{e^{-3t}(3+t)^{-\alpha}}{50(1+t^4)},$$

for $t\in[0,+\infty)$; and $\xi_1=0.1$, $\xi_2=0.2$, $\xi_3=0.5$, $\sigma_1=6$, $\sigma_2=0.5$, $\sigma_3=0.6$.

It is easy to verify that $a(t)$ satisfies conditions (A1) and (A2). Also, $\beta_i(t)\in Z$ $(i=1,2,3)$. Note that

$$f(t,x,y,z)\le\beta_1(t)|x|+\beta_2(t)|y|+\beta_3(t)|z|,$$

and that for $|x|>1000$,

$$f(t,x,y,z)>0,$$

Hence, (H1) and (H3') hold.

Meanwhile, by simple computation we see that $L_a=1$, $M_a=1$, $\eta_1<0.1315$, which leads to the condition (H2'). Also for $|c|>1000$, (H4) holds. Summing up the points indicated above, by Theorem 3.9, problem (4.1) has at least one solution.

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