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ANTI-PERIODIC SOLUTIONS TO RAYLEIGH-TYPE EQUATIONS WITH TWO DEVIATING ARGUMENTS

MEIQIANG FENG, XUEMEI ZHANG

Abstract. In this article, the Rayleigh equation with two deviating arguments

$$x''(t) + f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t)$$

is studied. By using Leray-Schauder fixed point theorem, we obtain the existence of anti-periodic solutions to this equation. The results are illustrated with an example, which can not be handled using previous results.

1. Introduction

Consider the Rayleigh equation with two deviating arguments

$$x''(t) + f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t), \tag{1.1}$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $g_i \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$, $i = 1, 2$, $e, \tau_i \in C(\mathbb{R}, \mathbb{R})$, $i = 1, 2$, $g_i(t+T, x) = g_i(t, x)$, $g_i(t + \frac{T}{2}, -x) = -g_i(t, x)$, $\tau_i(t + T) = \tau_i(t)$, $\tau_i(t + \frac{T}{2}) = -\tau_i(t)$, $i = 1, 2$, and $e(t + T) = e(t)$, $e(t + \frac{T}{2}) = -e(t)$.

The dynamic behavior of Rayleigh equation have been widely investigated due to their applications in many fields such as physics, mechanics and the engineering technique fields. For example, an excess voltage of ferro-resonance known as some kind of nonlinear resonance having long duration arises from the magnetic saturation of inductance in an oscillating circuit of a power system, and a boosted excess voltage can give rise to some problems in relay protection. To probe this mechanism, a mathematical model was proposed in [12, 17, 26], which is a special case of the Rayleigh equation with two delays. This implies that (1.1) can represent analog voltage transmission. In a mechanical problem, f usually represents a damping or friction term, g_i represents the restoring force, e is an externally applied force and τ_i is the time lag of the restoring force (see [4]). Some other examples in practical problems concerning physics and engineering technique fields can be found in [15, 19, 28].

Arising from problems in applied sciences, it is well-known that anti-periodic problems of nonlinear differential equations have been extensively studied by many authors during the past twenty years, see [3, 7, 21, 22, 23, 29] and references therein. For example, anti-periodic trigonometric polynomials are important in the study of

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interpolation problems [8, 11], and anti-periodic wavelets are discussed in [6]. Recently, anti-periodic boundary conditions have been considered for the Schrödinger and Hill differential operator [9, 10]. Also anti-periodic boundary conditions appear in the study of difference equations [5, 27]. Moreover, anti-periodic boundary conditions appear in physics in a variety of situations [1, 2, 18]. There exist only few results for the existence of anti-periodic solutions for Rayleigh equation and Rayleigh type equations with and without deviating arguments in the literature. The main difficulty lies in the middle term $f(x'(t))$ of (1.1), the existence of which obstructs the usual method of finding a priori bounds for delay Duffing or Liénard equations from working. Thus, it is worthwhile to continue to investigate the anti-periodic solutions of Rayleigh equation in this case.

At the same time, the periodic solutions for Rayleigh equations with two deviating arguments have been studied by authors [20, 16, 25]. But all the results of [20, 16, 25] are periodic solutions, not anti-periodic solutions. Thus, it is worth discussing the existence of the anti-periodic solutions of Rayleigh equations with two deviating arguments in this case.

The main purpose of this paper is to establish sufficient conditions for the existence of anti-periodic solution of (1.1) by using the Leray-Schauder fixed theorem. We remark that our methods are different from those used in [20, 16, 25] to some degree. In particular, one example is also given to illustrate the effectiveness of our results.

For ease of exposition, we assume that $T > 0$, and define the following assumptions to be used in this article.

- (H1) $f \in C(\mathbb{R},\mathbb{R})$, $g_i \in C(\mathbb{R}^2,\mathbb{R})$, $\tau_i \in C(\mathbb{R},\mathbb{R})$, $i = 1, 2$, $e \in C(\mathbb{R},\mathbb{R})$, $g_i(t + T, x) = g_i(t, x)$, $\tau_i(t + T) = \tau_i(t)$, $g_i(t + \frac{T}{2}, -x) = -g_i(t, x)$, $\tau_i(t + \frac{T}{2}) = -\tau_i(t)$, $i = 1, 2$, and $e(t + T) = e(t)$, $e(t + \frac{T}{2}) = -e(t)$.
- (H2) $f(0) = 0$, and there exists $\gamma > 0$ such that $xf(x) \ge \gamma|x|^2$, for all $x \in \mathbb{R}$ (or $xf(x) \le -\gamma|x|^2$, for all $x \in \mathbb{R}$).
- (H3) g_i is differentiable with respect to t, and there exist $a_i > 0$, $b_i > 0$, $i = 1, 2$, such that
$$|g'_{it}(t, x)| \le a_i + b_i|x|, \quad \forall (t, x) \in \mathbb{R}^2,\ i = 1, 2.$$
- (H4) There exist $l_i > 0$ such that $|g_i(t, x_1) - g_i(t, x_2)| \le l_i|x_1 - x_2|$, $\forall t \in \mathbb{R}, x_1, x_2 \in \mathbb{R}$, $i = 1, 2$.
- (H5) There exist integers n_i such that $\delta_i := \max_{t\in[0,T]} |\tau_i(t) - n_iT| \le T$, $i = 1, 2$.

The main result in this article is the following theorem, which will be proved in Section 3.

Theorem 1.1. *If* (H1)–(H5) *hold, and* $(b_1 + b_2)\gamma^{-1}T^2 + 8\sqrt{2}(l_1\delta_1 + l_2\delta_2)\pi^2\gamma^{-1} < 8\pi^2$*, then* (1.1) *has at least one anti-periodic solution.*

2. Preliminaries

In this section, to establish the existence of anti-periodic solutions for (1.1), we provide some background definitions and some well-known results, which are crucial in our arguments.

Let X be a real Banach space, and $A : X \to X$ be a completely continuous operator.

Definition 2.1. Let $u : \mathbb{R} \to \mathbb{R}$ be continuous. $u(t)$ is said to be anti-periodic on $\mathbb{R}$ if

$$u(t+T) = u(t), \quad u(t+\frac{T}{2}) = -u(t), \quad \forall t \in \mathbb{R}.$$

Lemma 2.2 (Leray-Schauder Fixed point theorem [14, 30])**.** *Let X be a real Banach space, and $A : X \to X$ be a completely continuous operator. If*

$$\{x \in X : x = \lambda Ax,\ 0 < \lambda < 1\}$$

is bounded, then A has a fixed point $x^* \in \Omega$, where

$$\Omega = \{x \in X : \|x\| \le l\}, \quad l = \sup\{x \in X : x = \lambda Ax,\ 0 < \lambda < 1\}.$$

Lemma 2.3 (Wirtinger inequality [24])**.** *Suppose that $x(t) \in C^1(\mathbb{R},\mathbb{R}), x$ is T-periodic and $\int_0^T x(t)dt = 0$. Then $\int_0^T |x(t)|^2 dt \le \frac{T^2}{4\pi^2} \int_0^T |x'(t)|^2 dt$.*

Lemma 2.4 ([13])**.** *Let $0 \le \alpha \le T$ be constant, $s \in C(\mathbb{R},\mathbb{R})$ be periodic with period T, and $\max_{t\in[0,T]} |s(t)| \le \alpha$. Then for any $u \in C^1(\mathbb{R},\mathbb{R})$ which is periodic with period T, we have*

$$\int_0^T |u(t) - u(t - s(t))|^2 dt \le 2\alpha^2 \int_0^T |u'(t)|^2 dt.$$

3. Proof of Theorem 1.1

In this section, we will use Lemma 2.2 to prove Theorem 1.1. Let

$$X = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t),\ x(t+\frac{T}{2}) = -x(t)\},$$
$$Y = \{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+T) = x(t),\ x(t+\frac{T}{2}) = -x(t)\}.$$

Then X and Y are real Banach space endowed with the norms

$$\|x\|_\infty = \max_{t\in[0,T]} |x(t)| \quad \text{and} \quad \|x\| = \|x\|_\infty + \|x'\|_\infty,$$

respectively.

Choosing $m > 0$ with $m \ne (\frac{2k\pi}{T})^2$ $(k = 1, 2, \dots)$, then equation

$$x''(t) + mx(t) = 0$$

has only the trivial solution in Y. In fact, it is easy to see the general solution of $x''(t) + mx(t) = 0$ is

$$x(t) = c_1 \sin\sqrt{m}t + c_2 \cos\sqrt{m}t.$$

By the periodic properties we obtain that $x = 0$ is its unique solution in Y. Then for $h \in X$,

$$-x''(t) - mx(t) = h(t)$$

has unique solution $x \in Y$. Writing $x = Kh$, then $K : X \to Y$ is a completely continuous operator.

Define an operator $G : Y \to X$ by

$$(Gx)(t) = f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) - mx(t) - e(t), x \in Y.$$

Then $G : Y \to X$ is continuous and bounded. Let $A = KG : Y \to Y$. Then A is also a completely continuous operator. By Lemma 2.2, if

$$\{x \in Y : x = \lambda Ax,\ 0 < \lambda < 1\}$$

is bounded in Y, then A has a fixed point in Y. Thus (1.1) has anti-periodic solution.

Now suppose that $x \in Y$, $0 < \lambda < 1$ satisfying $x = \lambda Ax$. Then $x(t)$ is a solution of

$$x''(t) + \lambda f(x'(t)) + \lambda g_1(t, x(t - \tau_1(t))) + \lambda g_2(t, x(t - \tau_2(t))) + (1 - \lambda)mx(t) = \lambda e(t), \tag{3.1}$$

and $x(t)$ satisfies

$$\int_0^T x(t)dt = \int_0^{T/2} x(t)dt + \int_{\frac{T}{2}}^T x(t)dt = \int_0^{T/2} x(t)dt + \int_0^{T/2} x(t + \frac{T}{2})dt = 0.$$

Thus, there exists $\xi \in [0, T]$ such that $x(\xi) = 0$. So we have

$$|x(t)| = |x(\xi) + \int_\xi^t x'(s)ds| \le \sqrt{T}\|x'\|_{L^2}.$$

Then

$$\|x\|_\infty \le \sqrt{T}\|x'\|_{L^2},$$

where $\|\cdot\|_{L^2}$ is the norm of $L^2[0, T]$.

Multiplying (3.1) by $x'(t)$ and integrating from 0 to T, we have

$$\begin{aligned} \lambda \int_0^T f(x'(t))x'(t)dt &= -\lambda \int_0^T g_1(t, x(t - \tau_1(t)))x'(t)dt \\ &\quad - \lambda \int_0^T g_2(t, x(t - \tau_2(t)))x'(t)dt + \lambda \int_0^T e(t)x'(t)dt. \end{aligned} \tag{3.2}$$

By (H2), we know that

$$\int_0^T f(x'(t))x'(t)dt \ge \gamma \int_0^T |x'(t)|^2dt. \tag{3.3}$$

By Hölder's inequality, from (3.2) and (3.3), we have

$$\begin{aligned} &\gamma \int_0^T x'^2(t)dt \\ &\le |\int_0^T g_1(t, x(t - \tau_1(t)))x'(t)dt| + |\int_0^T g_2(t, x(t - \tau_2(t)))x'(t)dt| + \|e\|_{L^2}\|x'\|_{L^2} \\ &\le \int_0^T |g_1(t, x(t - \tau_1(t))) - g_1(t, x(t))||x'(t)|dt + |\int_0^T g_1(t, x(t))x'(t)dt| \\ &\quad + \int_0^T |g_2(t, x(t - \tau_2(t))) - g_2(t, x(t))||x'(t)|dt \\ &\quad + |\int_0^T g_2(t, x(t))x'(t)dt| + \|e\|_{L^2}\|x'\|_{L^2}. \end{aligned} \tag{3.4}$$

Since the functions $\int_0^{x(t)} g_i(t, v)dv$, $i = 1, 2$ are T-periodic , differentiable and

$$\frac{d}{dt}\int_0^{x(t)} g_i(t, v)dv = g_i(t, x(t))x'(t) + \int_0^{x(t)} g'_{it}(t, v)dv, \quad i = 1, 2,$$

we have

$$\int_0^T g_i(t, x(t))x'(t)dt = -\int_0^T dt \int_0^{x(t)} g'_{it}(t, v)dv, \quad i = 1, 2. \tag{3.5}$$

Combining (3.4) and (3.5) with (H3) and (H4) we obtain

$$\begin{aligned}
&\gamma \int_0^T x'^2(t)dt \\
&\le l_1 \int_0^T |x(t)-x(t-\tau_1(t))||x'(t)|dt + l_2 \int_0^T |x(t)-x(t-\tau_2(t))||x'(t)|dt \\
&\quad + \int_0^T dt \int_0^{|x(t)|} (a_1+b_1|v|)dv + \int_0^T dt \int_0^{|x(t)|} (a_2+b_2|v|)dv + \|e\|_{L^2}\|x'\|_{L^2} \\
&\le l_1\|x'\|_{L^2}\big(\int_0^T |x(t)-x(t-\tau_1(t)-n_1T)|^2dt\big)^{1/2} \\
&\quad + l_2\|x'\|_{L^2}\big(\int_0^T |x(t)-x(t-\tau_2(t)-n_2T)|^2dt\big)^{1/2} \\
&\quad + (a_1+a_2)\int_0^T |x(t)|dt + \frac{b_1+b_2}{2}\int_0^T |x(t)|^2dt + \|e\|_{L^2}\|x'\|_{L^2}.
\end{aligned} \tag{3.6}$$

By Lemma 2.2 we have

$$\int_0^T |x(t)|^2dt \le \frac{T^2}{4\pi^2}\|x'\|_{L^2}^2. \tag{3.7}$$

By (H5) and Lemma 2.3, we have

$$\Big(\int_0^T |x(t)-x(t-\tau_i(t)-n_iT)|^2dt\Big)^{1/2} \le \sqrt{2}\delta_i\Big(\int_0^T |x'(t)|^2dt\Big)^{1/2}, \quad i=1,2. \tag{3.8}$$

By Hölder's inequality and (3.7), we have

$$\int_0^T |x(t)|dt \le \sqrt{T}(\int_0^T |x(t)|^2dt)^{1/2} \le \sqrt{T}\frac{T}{2\pi}(\int_0^T |x'(t)|^2dt)^{1/2} = \frac{T^{3/2}}{2\pi}\|x'\|_{L^2}. \tag{3.9}$$

Thus, it follows from (3.6),(3.7) (3.8) and (3.9) that

$$\begin{aligned}
\gamma\|x'\|_{L^2}^2 &\le \sqrt{2}(l_1\delta_1+l_2\delta_2)\|x'\|_{L^2}^2 + \frac{(a_1+a_2)T^{3/2}}{2\pi}\|x'\|_{L^2} \\
&\quad + \frac{(b_1+b_2)T^2}{8\pi^2}\|x'\|_{L^2}^2 + \|e\|_{L^2}\|x'\|_{L^2}.
\end{aligned}$$

Combining this with $(b_1+b_2)\gamma^{-1}T^2 + 8\sqrt{2}(l_1\delta_1+l_2\delta_2)\pi^2\gamma^{-1} < 8\pi^2$, we know that there exists c_1 such that $\|x'\|_{L^2} \le c_1$. Then

$$\|x\|_\infty \le \sqrt{T}c_1 := M_1. \tag{3.10}$$

Multiplying (3.1) by $x''(t)$ and integrating from 0 to T, we have

$$\begin{aligned}
\|x''\|_{L^2}^2 &\le |-\lambda\int_0^T g_1(t,x(t-\tau_1(t)))x''(t)dt - \lambda\int_0^T g_2(t,x(t-\tau_2(t)))x''(t)dt \\
&\quad - (1-\lambda)m\int_0^T x(t)x''(t)dt + \lambda\int_0^T e(t)x''(t)dt| \\
&\le (g_{1M_1}+g_{2M_1})\sqrt{T}\|x''\|_{L^2} + mM_1\sqrt{T}\|x''\|_{L^2} + \|e\|_{L^2}\|x''\|_{L^2},
\end{aligned}$$

where

$$g_{1M_1} = \max_{t\in[0,T],\|x\|_\infty\le M_1} |g_1(t,x(t))|, \quad g_{2M_1} = \max_{t\in[0,T],\|x\|_\infty\le M_1} |g_2(t,x(t))|.$$

Thus

$$\|x''\|_{L^2} \le (g_{1M_1} + g_{2M_1})\sqrt{T} + mM_1\sqrt{T} + \|e\|_{L^2} := M_2.$$

Selecting $\eta \in [0,T]$ such that $x'(\eta) = 0$, we have

$$|x'(t)| \le \int_0^T |x''(t)|dt \le \sqrt{T}M_2. \tag{3.11}$$

Thus from (3.10) and (3.11), we know that $\|x\| \le M_1 + \sqrt{T}M_2 := M$. It is following that

$$\{x \in Y : x = \lambda Ax,\ 0 < \lambda < 1\}$$

is bounded. Therefore, by Lemma 2.2, we obtain that A has a fixed point $x^* \in \Omega$, where $\Omega = \{x \in Y : \|x\| \le M\}$. Therefore, (1.1) has an anti-periodic solution.

An example

In this section, we give one example to demonstrate the results obtained in previous sections. Consider the forced Rayleigh-type equation with period 2π,

$$x''(t) + f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t), \tag{3.12}$$

where

$$f(x) = \begin{cases} e^x - 1, & x \ge 0, \\ 1 - e^{-x}, & x \le 0, \end{cases} \tag{3.13}$$

and

$$\begin{gathered} g_1(t,x) = \frac{1}{9}\sin^2(t)x(t - \theta\cos t) + \cos t, \\ g_2(t,x) = \frac{1}{9}\cos^2(t)x(t - \theta\sin t) + \sin t, \\ e(t) = \sin t, \quad \tau_1(t) = \theta\cos t, \quad \tau_2(t) = \theta\sin t, \quad \theta \in (0,1). \end{gathered} \tag{3.14}$$

Then (3.12) has at least one anti-periodic solution with period 2π.

By (3.13) and (3.14), it is not difficult to see that condition (H1) holds, $T = 2\pi$, $|f(0)| = 0$,

$$|g'_{1t}(t,x)| = |\frac{1}{9}x\sin(2t) - \sin t| \le \frac{1}{9}|x| + 1, \quad \forall (t,x) \in \mathbb{R}^2,$$
$$|g'_{2t}(t,x)| = | - \frac{1}{9}x\sin(2t) + \cos t| \le \frac{1}{9}|x| + 1, \quad \forall (t,x) \in \mathbb{R}^2.$$

On the other hand, let $\gamma = 1$, $\delta_i = \theta$, $l_i = \frac{1}{9}$, $b_i = \frac{1}{9}$, $i = 1, 2$. If $\theta \in (0, \frac{\sqrt{2}}{2})$, then $xf(x) \ge |x|^2$, for all $x \in \mathbb{R}$,

$$(b_1 + b_2)\gamma^{-1}T^2 + 8\sqrt{2}(l_1\delta_1 + l_2\delta_2)\pi^2\gamma^{-1} < 8\pi^2.$$

Hence, (H1)–(H5) are satisfied. Thus, by Theorem 1.1, Equation (3.12) has at least one anti-periodic solution with period 2π.

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Meiqiang Feng
School of Applied Science, Beijing Information Science and Technology University, Beijing, 100192, China
E-mail address: meiqiangfeng@sina.com

Xuemei Zhang
Department of Mathematics and Physics, North China Electric Power University, Beijing, 102206, China
E-mail address: zxm74@sina.com