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EXISTENCE OF SOLUTIONS FOR SECOND-ORDER IMPULSIVE BOUNDARY-VALUE PROBLEMS

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Abstract. In this article we discuss the existence of solutions of second-order boundary-value problems subjected to impulsive effects. Our approach is based on fixed point theorems.

1. Introduction

Differential equations involving impulse effects arise naturally in the description of phenomena that are subjected to sudden changes in their states, such as population dynamics, biological systems, optimal control, chemotherapeutic treatment in medicine, mechanical systems with impact, financial systems. For typical examples see [9, 11]. For a general theory on impulsive differential equations the interested reader can consult the monographs [2, 7, 14], and the papers [1, 5, 6, 8, 10, 12, 13, 15] and the references therein. Our objective is to provide sufficient conditions on the data in order to ensure the existence of at least one solution of the problem

$$\begin{gathered}
(p(t)x'(t))' + q(t)x(t) = F(t, x(t), x'(t)), \quad t \neq t_k,\ t \in [0,1],\\
\Delta x(t_k) = U_k(x(t_k), x'(t_k)),\\
\Delta x'(t_k) = V_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m,\\
x(0) = x(1) = 0,
\end{gathered} \tag{1.1}$$

where $x \in \mathbb{R}$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a piecewise continuous function; U_k and V_k represent the jump discontinuities of x and x', respectively, at $t = t_k \in (0,1)$, called impulse moments, with $0 < t_1 < t_2 < \dots < t_m < 1$.

2. Preliminaries

In this section we introduce some definitions and notations that will be used in the remainder of the paper.

Let J denote the real interval $[0,1]$. Let $J' = J\backslash\{t_1, t_2, \dots, t_m\}$. $PC(J)$ denotes the space of all functions $x : J \to \mathbb{R}$ continuous on J', and for $i = 1, 2, \dots, m$, $x(t_i^+) = \lim_{\epsilon\to 0+} x(t_i + \epsilon)$ and $x(t_i^-) = \lim_{\epsilon\to 0} x(t_i - \epsilon)$ exist. We shall write $x(t_i^-) = x(t_i)$. This is a Banach space when equipped with the sup-norm; i.e.,

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$\|x\|_0 = \sup_{t\in J} |x(t)|$. Similarly, $PC^1(J)$ is the space of all functions $x \in PC(J)$, $x$is continuously differentiable on J', and for $i = 1, 2, \dots, m$, $x'(t_i^+)$ and $x'(t_i^-)$ exist and $x'(t_i) = x'(t_i^-)$. For $x \in PC^1(J)$ we define its norm by $\|x\|_1 = \|x\|_0 + \|x'\|_0$. Then $(PC^1(I), \|\cdot\|_1)$ is a Banach space.

The following linear problem plays an important role in our study.

$$\begin{gathered}
(p(t)x'(t))' + q(t)x(t) = f(t), \quad t \neq t_k, t \in [0,1], \\
\Delta x(t_k) = U_k(x(t_k), x'(t_k)), \\
\Delta x'(t_k) = V_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\
x(0) = x(1) = 0,
\end{gathered} \tag{2.1}$$

To study (2.1) we first consider the problem without impulses

$$\begin{gathered}
(p(t)x'(t))' + q(t)x(t) = f(t), \quad t \in [0,1] \\
x(0) = x(1) = 0.
\end{gathered} \tag{2.2}$$

We shall assume, throughout the paper, that the following condition holds.

- (H0) (i) $p \in C^1(J : \mathbb{R}), p(t) \geq p_0 > 0$, for all $t \in J$.
  - (ii) $q \in C(J : \mathbb{R}), q(t) \leq p_0\pi^2$, for all $t \in J$, and $q(t) < p_0\pi^2$ on a subset of J of positive measure.

Lemma 2.1. *If* (H0) *is satisfied, then for any nonzero $x \in C^2(J : \mathbb{R})$ with $x(0) = x(1) = 0$,*

$$\int_0^1 \{p(t)(x'(t))^2 - q(t)x^2(t)\}dt > 0.$$

Proof. The proof of this lemma is presented in [3]. We shall reproduce it here for the sake of completeness. Since $q(t) \leq p_0\pi^2$ on a subset of J of positive measure, we have

$$p(t)(x'(t))^2 - q(t)x^2(t) > p_0((x'(t))^2 - \pi^2 x^2(t)).$$

This inequality yields

$$\int_0^1 \{p(t)(x'(t))^2 - q(t)x^2(t)\}dt > p_0 \int_0^1 \{(x'(t))^2 - \pi^2 x^2(t)\}dt.$$

We show that

$$\mathcal{J}(x) = \int_0^1 \{(x'(t))^2 - \pi^2 x^2(t)\}dt \geq 0$$

for all functions $x \in C^2(J : \mathbb{R})$ with $x(0) = x(1) = 0$. The function u that minimizes $\mathcal{J}(x)$ satisfies the Euler-Lagrange equation (see [4])

$$u'' + \pi^2 u = 0,$$

and the boundary conditions $u(0) = u(1) = 0$. Then $u(t) = \sin \pi t$ or $u(t) = 0$, and $\mathcal{J}(u) = 0$. Since $\mathcal{J}(x) \geq \mathcal{J}(u)$ it follows that $\mathcal{J}(x) \geq 0$, and so

$$\int_0^1 \{p(t)(x'(t))^2 - q(t)x^2(t)\}dt > 0.$$

This completes the proof of the lemma. ∎

Lemma 2.2. *If* (H0) *is satisfied, then the linear problem*

$$\begin{gathered}(p(t)x'(t))' + q(t)x(t) = 0\\ x(0) = x(1) = 0.\end{gathered} \tag{2.3}$$

has only the trivial solution.

Proof. Assume on the contrary that (2.3) has a nontrivial solution x_0. Then (2.3) implies $[(p(t)x_0'(t))' + q(t)x_0(t)]x_0(t) = 0$ which yields

$$\begin{aligned}0 &= \int_0^1 [(p(t)x_0'(t))' + q(t)x_0(t)]x_0(t)\,dt\\ &= \int_0^1 [(p(t)x_0'(t))']x_0(t)\,dt + \int_0^1 q(t)x_0^2(t)\,dt\\ &= -\int_0^1 [p(t)x_0'^2(t) - q(t)x_0^2(t)]\,dt < 0.\end{aligned}$$

This is a contradiction. See Lemma 2.1. Therefore $x_0 \equiv 0$ is the only solution of (2.3). $\square$

It is well known that the unique solution of (2.2) is given by

$$x(t) = \int_0^1 G(t,s)f(s)ds,$$

where $G(\cdot,\cdot) : J \times J \to \mathbb{R}$ is the Green's function corresponding to (2.3).

Lemma 2.3. *The solution to* (2.1) *is*

$$\begin{aligned}x(t) = \int_0^1 G(t,s)f(s)ds - \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial s} p(t_k)U_k(x(t_k),x'(t_k))\\ + \sum_{k=1}^m G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)).\end{aligned} \tag{2.4}$$

Proof. We shall use of superposition principle and write $x(t) = y(t) + z(t) + w(t)$, where $y(t)$ solves the problem

$$\begin{gathered}(p(t)y'(t))' + q(t)y(t) = f(t), \quad t \in J,\\ \Delta y(t_k) = 0,\\ \Delta y'(t_k) = 0, \quad k = 1,2,\dots,m,\\ y(0) = y(1) = 0,\end{gathered} \tag{2.5}$$

while $z(t)$ solves the problem

$$\begin{gathered}(p(t)z'(t))' + q(t)z(t) = 0, \quad t \neq t_k, t \in J,\\ \Delta z(t_k) = U_k(x(t_k),x'(t_k)),\\ \Delta z'(t_k) = 0, \quad k = 1,2,\dots,m,\\ z(0) = z(1) = 0,\end{gathered} \tag{2.6}$$

and $w(t)$ solves the problem

$$\begin{gathered}(p(t)w'(t))' + q(t)w(t) = 0, \quad t \neq t_k, t \in J, \\ \Delta w(t_k) = 0, \\ \Delta w'(t_k) = V_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ w(0) = w(1) = 0.\end{gathered} \tag{2.7}$$

It is clear that

$$y(t) = \int_0^1 G(t,s)f(s)ds, \quad t \in I.$$

For $k = 1, 2, \dots, m$, set

$$\begin{gathered}z_k(t) = -\frac{\partial G(t,t_k)}{\partial s} p(t_k)U_k(x(t_k), x'(t_k)), \quad t \in J, \\ w_k(t) = G(t,t_k)p(t_k)V_k(x(t_k), x'(t_k)), \quad t \in J.\end{gathered}$$

Using the properties of Green's function and its derivatives we can prove that the functions z_k and $w_k, k = 1, 2, \dots, m$, are the solutions of problems (2.6) and (2.7), respectively. Consequently, $x = y + \sum_{k=1}^m z_k + \sum_{k=1}^m w_k$ is a solution of problem (2.1). □

3. Nonlinear Problem

In this section we present our main results on the existence of solutions for nonlinear boundary-value problems for the second-order impulsive control system. Consider the problem

$$\begin{gathered}(p(t)x'(t))' + q(t)x(t) = F(t, x(t), x'(t)), \quad t \neq t_k, t \in J, \\ \Delta x(t_k) = U_k(x(t_k), x'(t_k)), \\ \Delta x'(t_k) = V_k(x(t_k), x'(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) = x(1) = 0,\end{gathered} \tag{3.1}$$

where $x \in \mathbb{R}$ is the state variable; $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a piecewise continuous function; U_k and V_k are impulsive functions representing the jump discontinuities of x and x' at $t \in \{t_1, t_2, \dots, t_m\}$.

The nonlinear system

$$\begin{gathered}(p(t)\acute{x}(t))\acute{} + q(t)x(t) = F(t, x(t), x'(t)) \\ x(0) = x(1) = 0,\end{gathered} \tag{3.2}$$

is equivalent to the nonlinear integral equation

$$x(t) = \int_0^1 G(t,s)F(s, x(s), x'(s))ds, \quad \text{for all } t \in J$$

It follows from Lemma 2.3 that any solution of (3.1) satisfies

$$\begin{aligned}x(t) = \int_0^1 G(t,s)F(s, x(s), x'(s))ds &- \sum_{k=1}^m W(t,t_k)p(t_k)U_k(x(t_k), x'(t_k)) \\ &+ \sum_{k=1}^m G(t,t_k)p(t_k)V_k(x(t_k), x'(t_k)).\end{aligned} \tag{3.3}$$

where $W(t,t_k)=\frac{\partial G(t,t_k)}{\partial s}$. Let

$$K=\max\{|G(t,s)|:(t,s)\in J\times J\},\quad L=\max\{|W(t,s)|:(t,s)\in J\times J\},$$
$$M=\sup\{|\frac{\partial G(t,s)}{\partial t}|:(t,s)\in J\times J\},\quad N=\sup\{|\frac{\partial W(t,s)}{\partial t}|:(t,s)\in J\times J\},$$
$$P=\max\{K,L,M,N\}.$$

For the next theorem we use the following assumptions:

(H1) $F(\cdot,\cdot,\cdot)$ is continuous on J' and satisfies the Lipschitz condition
$$|F(t,x_1,y_1)-F(t,x_2,y_2)|\le\beta(|x_1-y_1|+|x_2-y_2|).$$

(H2) U_k and V_k are continuous and satisfy the Lipschitz conditions
$$|U_k(x_1,y_1)-U_k(x_2,y_2)|\le c_k(|x_1-y_1|+|x_2-y_2|),$$
$$|V_k(x_1,y_1)-V_k(x_2,y_2)|\le d_k(|x_1-y_1|+|x_2-y_2|),$$

(H3) $2P(\beta+R\sum_{k=1}^m c_k+R\sum_{k=1}^m d_k)<1$.

Theorem 3.1. *Under assumptions* (H0)–(H3)*, problem* (3.1) *has a unique solution.*

Proof. Define an operator $\omega:PC^1(J)\to PC^1(J)$ by
$$\omega(x)(t)=\int_0^1 G(t,s)F(s,x(s),x'(s))ds-\sum_{k=1}^m W(t,t_k)p(t_k)U_k(x(t_k),x'(t_k))$$
$$+\sum_{k=1}^m G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)).$$
It is clear that any solution of (3.1) is a fixed point of ω and conversely any fixed point of ω is a solution of (3.1).

We shall show that ω is a contraction. Let $x,y\in PC(J)$, then
$$\begin{aligned}
\|\omega(x)-\omega(y)\|_0 &\le \sup_{t\in J}\Big\{\int_0^1 |G(t,s)||F(s,x(s),x'(s))-F(s,y(s),y'(s))|ds\\
&\quad+\sum_{k=1}^m |W(t,t_k)|p(t_k)|U_k(x(t_k),x'(t_k))-U_k(y(t_k),y'(t_k))|\\
&\quad+\sum_{k=1}^m |G(t,t_k)|p(t_k)|V_k(x(t_k),x'(t_k))-V_k(y(t_k),y'(t_k))|\Big\}\\
&\le \sup_{t\in J}\Big\{\int_0^1 |G(t,s)|(\beta(\|x-y\|_0+\|x'-y'\|_0))ds\\
&\quad+R\sum_{k=1}^m |W(t,t_k)|c_k(\|x-y\|_0+\|x'-y'\|_0)\\
&\quad+R\sum_{k=1}^m |G(t,t_k)|d_k(\|x-y\|_0+\|x'-y'\|_0)\Big\}.
\end{aligned}$$
Now, by using (H1) and (H2), we have
$$\|\omega(x)-\omega(y)\|_0\le\beta K\|x-y\|_1+RL\sum_{k=1}^m c_k\|x-y\|_1+RK\sum_{k=1}^m d_k\|x-y\|_1. \tag{3.4}$$

We have

$$\begin{aligned}\frac{d}{dt}\omega(x)(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t} F(s,x(s),x'(s))ds - \sum_{k=1}^m \frac{\partial W(t,t_k)}{\partial t} U_k(x(t_k),x'(t_k)) \\ + \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial t}\ V_k(x(t_k),x'(t_k)).\end{aligned}$$

Let $x, y \in PC(J)$, then

$$\begin{aligned}\|\frac{d}{dt}\omega(x) - \frac{d}{dt}\omega(y)\|_0 \le \sup_{t\in J}\Big\{ \int_0^1 |\frac{\partial G(t,s)}{\partial t}\ ||F(s,x(s),x'(s)) - F(s,y(s),y'(s))|ds \\ + \sum_{k=1}^m |\frac{\partial W(t,t_k)}{\partial t}||U_k(x(t_k),x'(t_k)) - U_k(y(t_k),y'(t_k))| \\ + \sum_{k=1}^m |\frac{\partial G(t,t_k)}{\partial t}||V_k(x(t_k),x'(t_k)) - V_k(y(t_k),y'(t_k))|\Big\}.\end{aligned}$$

Conditions (H1) and (H2) imply

$$\|\frac{d}{dt}\omega(x) - \frac{d}{dt}\omega(y)\|_0 \le \beta M\|x-y\|_1 + RN\sum_{k=1}^m c_k\|x-y\|_1 + RM\sum_{k=1}^m d_k\|x-y\|_1. \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$\begin{aligned}\|\omega(x) - \omega(y)\|_1 &= \|\omega(x) - \omega(y)\|_0 + \|\frac{d}{dt}\omega(x) - \frac{d}{dt}\omega(y)\|_0 \\ &\le \Big(\beta K + RL\sum_{k=1}^m c_k + RK\sum_{k=1}^m d_k\Big)\|x-y\|_1 \\ &\quad + (\beta M + RN\sum_{k=1}^m c_k + RM\sum_{k=1}^m d_k)\|x-y\|_1 \\ &\le 2P\Big(\beta + R\sum_{k=1}^m c_k + R\sum_{k=1}^m d_k\Big)\|x-y\|_1\end{aligned}$$

Condition (H3) implies that ω is a contraction. By the Banach fixed point theorem ω has a unique fixed point x, which is the unique solution of (3.1). □

For the next Theorem, we use the following assumptions:

(H4) $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous on J' and there exists $h : J \times \mathbb{R}_+ \to \mathbb{R}_+$ a Caratheodory function, nondecreasing with respect to its second argument such that
$$|F(t,x,y)| \le h(t,|x|+|y|), \quad \text{a.e. } t \in [0,1].$$

(H5) U_k and V_k are continuous and there exist $a_k > 0$ and $b_k > 0$ such that
$$|U_k(x(t_k),y(t_k))| \le a_k, \quad |V_k(x(t_k),y(t_k))| \le b_k, \quad k = 1,2,\dots,m.$$

(H6) $\lim_{\varrho\to+\infty}\sup \frac{1}{\varrho}\big(\int_0^1 h(t,\varrho)dt + \sum_{k=1}^m R(a_k+b_k)\big) < 1/(2P)$.

Theorem 3.2. *Under assumptions* (H0), (H4)–(H6), *problem* (3.1) *has at least one solution.*

Proof. The proof is given in two steps.

Step 1. A priori bound on solutions. Let $x \in PC^1(J)$ be a solution of (3.1).

$$x(t) = \int_0^1 G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^m W(t,t_k)p(t_k)U_k(x(t_k),x'(t_k)) + \sum_{k=1}^m G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)),$$

and

$$x'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}F(s,x(s),x'(s))ds - \sum_{k=1}^m \frac{\partial W(t,t_k)}{\partial t}p(t_k)U_k(x(t_k),x'(t_k)) + \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial t}p(t_k)V_k(x(t_k),x'(t_k)).$$

It is easy to see that

$$|x(t)| \le K\int_0^1 |F(s,x(s),x'(s))|ds + RL\sum_{k=1}^m |U_k(x(t_k),x'(t_k))| + RK\sum_{k=1}^m |V_k(x(t_k),x'(t_k))|,$$

and

$$|x'(t)| \le M\int_0^1 |F(s,x(s),x'(s))|ds + RN\sum_{k=1}^m |U_k(x(t_k),x'(t_k))| + RM\sum_{k=1}^m |V_k(x(t_k),x'(t_k))|.$$

Conditions (H4), (H5) and (H6) lead to

$$\|x\|_0 + \|x'\|_0 \le (K+M)\int_0^1 h(s,\|x\|_0+\|x'\|_0)ds + \sum_{k=1}^m R((L+N)l_k + (K+M)p_k).$$

Since $\|x\|_1 = \|x\|_0 + \|x'\|_0$ and h is nondecreasing, then

$$\|x\|_1 \le 2P\int_0^1 h(s,\|x\|_1)ds + \sum_{k=1}^m R(2Pa_k + 2Pb_k),$$

or

$$\|x\|_1 \le 2P(\int_0^1 h(s,\|x\|_1)ds + \sum_{k=1}^m R(a_k+b_k)).$$

Let $\beta_0 = \|x\|_1$. Then the above inequality gives

$$\frac{1}{2P} \le \frac{1}{\beta_0}\Big(\int_0^1 h(s,\beta_0)ds + \sum_{k=1}^m R(a_k+b_k)\Big). \tag{3.6}$$

Condition (H6) implies that there exists $r > 0$ such that for all $\beta > r$, we have

$$\frac{1}{\beta}\Big(\int_0^1 h(s,\beta)ds + \sum_{k=1}^m R(a_k+b_k)\Big) < \frac{1}{2P}. \tag{3.7}$$

Comparing (3.6) and (3.7) we see that $\beta_0 \le r$. Hence we have $\|x\|_1 \le r$.

Step 2. Existence of solutions. Let $\Omega = \{x \in PC^1(J) : \|x\|_1 < r+1\}$. Then Ω is an open convex subset of $PC^1(J)$. Define an operator H by

$$\begin{aligned} H(\lambda, x)(t) &= \lambda \int_0^1 G(t,s)F(s,x(s),x'(s))ds + \lambda \sum_{k=1}^m W(t,t_k)U_k(x(t_k),x'(t_k)) \\ &\quad + \lambda \sum_{k=1}^m G(t,t_k)V_k(x(t_k),x'(t_k)), \quad 0 \le \lambda \le 1. \end{aligned}$$

Then $H(\lambda,\cdot) : \bar{\Omega} \to PC^1(J)$ is compact and has no fixed point on $\partial\Omega$ (see [6]). It is an admissible homotopy between the constant map $H(0,\cdot) \equiv 0$ and $H(1,\cdot) \equiv \omega$. Since $H(0,\cdot)$ is essential then $H(1,\cdot)$ is essential which implies that $\omega \equiv H(1,\cdot)$ has a fixed point in Ω. This fixed point is a solution of our problem. $\square$

The following assumptions are used in the next theorem.

(H7) there exists $g \in L^1(J)$ such that

$$|F(t,x,y)| \le g(t) \quad \text{for almost } t \in J, \quad x, y \in \mathbb{R}.$$

(H8) $U_k : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists $\alpha_k > 0$ such that

$$|U_k(x(t_k), y(t_k))| \le \alpha_k(\|x\|_0 + \|y\|_0), \; k = 1, 2, \dots, m.$$

(H9) $V_k : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists $\beta_k > 0$ such that

$$|V_k(x(t_k), y(t_k))| \le \beta_k(\|x\|_0 + \|y\|_0), \quad k = 1, 2, \dots, m.$$

(H10) $2PR\sum_{k=1}^m(\alpha_k + \beta_k) < 1$.

Theorem 3.3. *Under assumptions* (H0), (H7)–(H10), *equation* (2.4) *has at least one solution.*

Proof. The proof is given in two steps.

Step1. A priori bound on solutions. We have

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)F(s,x(s),x'(s))ds - \sum_{k=1}^m W(t,t_k)p(t_k)U_k(x(t_k),x'(t_k)) \\ &\quad + \sum_{k=1}^m G(t,t_k)p(t_k)V_k(x(t_k),x'(t_k)). \end{aligned}$$

and

$$\begin{aligned} x'(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t} F(s,x(s),x'(s),(Sx)(s))ds + \sum_{k=1}^m \frac{\partial W(t,t_k)}{\partial t} p(t_k)U_k(x(t_k),x'(t_k)) \\ &\quad + \sum_{k=1}^m \frac{\partial G(t,t_k)}{\partial t} p(t_k)V_k(x(t_k),x'(t_k)). \end{aligned}$$

It is easy to see that

$$\begin{aligned} |x(t)| &\le K \int_0^1 |F(s,x(s),x'(s))|ds + RL \sum_{k=1}^m |U_k(x(t_k),x'(t_k))| \\ &\quad + RK \sum_{k=1}^m |V_k(x(t_k),x'(t_k))|, \end{aligned}$$

and

$$|x'(t)| \leq M \int_0^1 |F(s,x(s),x'(s))|ds + RN \sum_{k=1}^{m} |U_k(x(t_k),x'(t_k))| \\ + RM \sum_{k=1}^{m} |V_k(x(t_k),x'(t_k))|.$$

From (H7), (H8) and (H9), we obtain

$$\|x\|_0 + \|x'\|_0 \leq (K+M)\|g\|_{L^1} + \sum_{k=1}^{m} R(L+N)\alpha_k(\|x\|_0 + \|x'\|_0) \\ + \sum_{k=1}^{m} R(K+M)\beta_k(\|x\|_0 + \|x'\|_0).$$

Setting $\mu = 2PR\sum_{k=1}^{m}(\alpha_k+\beta_k)$, we obtain

$$\|x\|_1 \leq 2P\|g\|_{L^1} + \mu\|x\|_1.$$

Then $(1-\mu)\|x\|_1 \leq 2P\|g\|_{L^1}$. Using condition (H10) we obtain

$$\|x\|_1 \leq (\frac{2P}{1-\mu})\|g\|_{L^1} := r_1.$$

Step 2. Existence of solutions. Let $\Omega_1 = \{x \in PC^1(J) : \|x\|_1 < r_1 + 1\}$. The rest of the proof is similar to that of Theorem 3.2, and it is omitted. $\square$

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References

[1] B. Ahmad; *Existence of solutions for second-order nonlinear impulsive boundary-value problems*, Elect. J. Diff. Equ. 2009 (2009) no. 68, 1–7.

[2] D. d. Bainov, P. S. Simeonov; *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical, Essex, England, 1993.

[3] A. Boucherif, J Henderson; *Topological methods in nonlinear boundary value problems*, Nonlinear Times and Digest 1 (1994), 149-168.

[4] L. Elsgolts; *Differential Equations and Calculus of Variations*, MIR Publishers, Moscow, 1970.

[5] L. H. Erbe, Xinzhi Liu; *Existence results for boundary value problems of second order impulsive differential equations*, J. Math. Anal. Appl. 149(1990), 56-69.

[6] A. Lakmeche, A. Boucherif; *Boundary value problems for impulsive second order differential equations*, Dynam. Cont. Discr. Impul. Syst. Series A: Math. Anal. 9 (2002), 313-319.

[7] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov; *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[8] Y. Lee, X. Liu; *Study of singular boundary value problems for second order impulsive differential equations*, J. Math. Anal. Appl. 331 (2007), 159-176.

[9] J. J. Nieto; *Periodic boundary value problems for first-order impulsive ordinary differential equations*, NonlinearAnal. 51 (2002), 1223–1232.

[10] J. J. Nieto; *Basic theory for nonresonance impulsive periodic problems of first order*, J. Math. Anal. Appl. 205(1997), 423–433.

[11] S. G. Pandit, S. G. Deo; *Differential Systems Involving Impulses*, Lecture Notes in Math. Vol. 954, Springer-Verlag, Berlin, 1982.

[12] I. Rachůnková,J Tomeček; *Singular Dirichlet problem for ordinary differential equations with impulses*, Non. Anal. 65 (2006), 210-229.

[13] Y. V. Rogovchenko; *Impulsive evolution systems:Main results and new trends*, Dyn. Contin. Discrete Impuls.Syst. 3 (1997) 57–88.

[14] A. M. Samoilenko, N. A. Perestyuk; *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[15] J. Tomeček; *Nonlinear boundary value problem for nonlinear second order differential equations with impulses*, Electronic J. Qual. Theo. Diff. Equ. Vol 2005 (2005), No. 10, 1-22.

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