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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS FOR THE RADIAL P-LAPLACIAN EQUATION

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Abstract. We study the existence, uniqueness and asymptotic behavior of positive solutions to the nonlinear problem

$$\frac{1}{A}(A\Phi_p(u'))' + q(x)u^\alpha = 0, \quad \text{in } (0,1),$$
$$\lim_{x\to 0} A\Phi_p(u')(x) = 0, \quad u(1) = 0,$$

where $\alpha < p-1$, $\Phi_p(t) = t|t|^{p-2}$, A is a positive differentiable function and q is a positive measurable function in $(0,1)$ such that for some $c > 0$,

$$\frac{1}{c} \le q(x)(1-x)^\beta \exp\Big(-\int_{1-x}^{\eta} \frac{z(s)}{s} ds\Big) \le c.$$

Our arguments combine monotonicity methods with Karamata regular variation theory.

1. Introduction

Let $p > 1$ and $\alpha < p-1$. We consider the boundary-value problem

$$\begin{gathered} -\frac{1}{A}(A\Phi_p(u'))' + q(x)u^\alpha = 0, \quad \text{in } (0,1) \\ A\Phi_p(u')(0) := \lim_{x\to 0} A\Phi_p(u')(x) = 0, \quad u(1) = 0. \end{gathered} \tag{1.1}$$

Here, A is a continuous function in $[0,1)$, differentiable and positive on $(0,1)$ and for all $t \in \mathbb{R}$, $\Phi_p(t) = t|t|^{p-2}$. Our goal in this paper is to study problem (1.1) under appropriate conditions on q. We obtain the existence of a unique positive continuous solution to (1.1) and establish estimates on such solution.

Several articles have been devoted to the study of the differential equation

$$-\frac{1}{A}(A\Phi_p(u'))' + q(x)u^\alpha = 0, \quad \text{in } (0,1)$$

with various boundary conditions, especially for the one-dimensional p-Laplacian equation (see [1, 2, 3, 4, 5, 11, 13, 14, 15]). For $\alpha < 0$, problem (1.1) has been studied in [4], where the existence and uniqueness of positive solutions and some estimates for the solutions have been obtained. Thus, it is interesting to know the

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exact asymptotic behavior of such solution as $x \to 1$ and to extend the study of (1.1) to $0 \leq \alpha < p-1$.

Asymptotic behavior of solutions of the semilinear elliptic equation

$$-\Delta u = q(x)u^{\alpha}, \quad \alpha < 1,\ x \in \Omega, \tag{1.2}$$

for Ω bounded or an unbounded in $\mathbb{R}^n$ ($n \geq 2$), with homogeneous Dirichlet boundary conditions, has been investigated by several authors; see for example [6, 7, 8, 9, 10, 12, 16, 17, 20] and the references therein. Applying Karamata regular variation theory, Mâagli [16] studied (1.2), when Ω is a bounded $C^{1,1}$-domain. He showed that (1.2) has a unique positive classical solution that satisfies homogeneous Dirichlet boundary conditions and gave sharp estimates on such solution. This studied extended the estimates stated in [12, 17, 20]. In this work, we extend the result established in [16] to the radial case associated to problem (1.1).

To simplify our statements, we need to fix some notation and make some assumptions. Throughout this paper, we shall use $\mathcal{K}$ to denote the set of Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c \exp\Big(\int_t^{\eta} \frac{z(s)}{s} ds\Big),$$

for some positive constants η, c, and a function $z \in C([0,\eta])$ such that $z(0) = 0$. Recall that $L \in \mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$, for some $\eta > 0$, such that

$$\lim_{t\to 0} \frac{tL'(t)}{L(t)} = 0. \tag{1.3}$$

For two nonnegative functions f and g defined on a set S, we write $f(x) \approx g(x)$, if there exists a constant $c > 0$ such that $\frac{1}{c}g(x) \leq f(x) \leq cg(x)$, for each $x \in S$. Furthermore, we refer to $G_p f$, as the function defined on $(0,1)$ by

$$G_p f(x) := \int_x^1 \Big(\frac{1}{A(t)} \int_0^t A(s)f(s)ds\Big)^{\frac{1}{p-1}} dt,$$

where f is a nonnegative measurable function in $(0,1)$. We point out that if f is a nonnegative continuous function such that the mapping $x \mapsto A(x)f(x)$ is integrable in a neighborhood of 0, then $G_p f$ is the solution of the problem

$$\begin{gathered} -\frac{1}{A}(A\Phi_p(u'))' = f, \quad \text{in } (0,1), \\ A\Phi_p(u')(0) = 0, \quad u(1) = 0. \end{gathered} \tag{1.4}$$

As it is mentioned above, our main purpose in this paper is to establish existence and global behavior of a positive solution for problem (1.1). Let us introduce our hypotheses.

The function A is continuous in $[0,1)$, differentiable and positive in $(0,1)$ such that

$$A(x) \approx x^{\lambda}(1-x)^{\mu}$$

with $\lambda \geq 0$ and $\mu < p-1$.

The function q is required to satisfy

(H1) q is a positive measurable function on $(0,1)$ such that

$$q(x) \approx (1-x)^{-\beta} L(1-x),$$

with $\beta \le p$ and $L \in \mathcal{K}$ defined on $(0,\eta]$ $(\eta > 1)$ such that

$$\int_0^\eta t^{\frac{1-\beta}{p-1}} (L(t))^{\frac{1}{p-1}} dt < +\infty.$$

We need to verify the condition

$$\int_0^\eta t^{\frac{1-\beta}{p-1}} (L(t))^{\frac{1}{p-1}} dt < +\infty$$

in hypothesis (H1), only if $\beta = p$ (See Lemma 2.2 below).

As a typical example of function q satisfying (H1), we have

$$q(x) := (1-x)^{-\beta} (\log \frac{2}{1-x})^{-\nu}, \quad x \in [0,1).$$

Then for $\beta < p$ and $\nu \in \mathbb{R}$ or $\beta = p$ and $\nu > p-1$, the function q satisfies (H1).

Our main result is as follows.

Theorem 1.1. *Assume* (H1). *Then problem* (1.1) *has a unique positive and continuous solution* u *satisfying, for* $x \in (0,1)$,

$$u(x) \approx \theta_\beta(x), \tag{1.5}$$

where θ_β *is the function defined on* $[0,1)$ *by*

$$\theta_\beta(x) := \begin{cases} \Big(\int_0^{1-x} \frac{(L(s))^{\frac{1}{p-1}}}{s} ds\Big)^{\frac{p-1}{p-1-\alpha}}, & \text{if } \beta = p \\ (1-x)^{\frac{p-\beta}{p-1-\alpha}} (L(1-x))^{\frac{1}{p-1-\alpha}}, & \text{if } \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1} < \beta < p, \\ (1-x)^{\frac{p-1-\mu}{p-1}}, & \text{if } \beta < \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1} \\ (1-x)^{\frac{p-1-\mu}{p-1}} (\int_{1-x}^\eta \frac{L(s)}{s} ds)^{\frac{1}{p-1-\alpha}}, & \text{if } \beta = \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1}. \end{cases} \tag{1.6}$$

The article is organized as follows. In Section 2, we prove some basic estimates and recall some results on functions belonging to $\mathcal{K}$. In Section 3, we prove Theorem 1.1. In the last section, we present some applications.

2. Estimates

In what follows, we give estimates on the functions $G_p q$ and $G_p(q\theta_\beta^\alpha)$, where q is a function satisfying (H1) and θ_β is the function given by (1.6). To this end, we recall some fundamental properties of functions belonging to the class $\mathcal{K}$, taken from [7, 18, 19].

Lemma 2.1 ([18, 19])**.** *Let* $L_1, L_2 \in \mathcal{K}$, $m \in \mathbb{R}$ *and* $\epsilon > 0$. *Then* $L_1 L_2 \in \mathcal{K}$, $L_1^m \in \mathcal{K}$, *and* $\lim_{t\to 0^+} t^\epsilon L_1(t) = 0$.

Lemma 2.2 ([18, 19])**.** *Let* $L \in \mathcal{K}$ *and* $\delta \in \mathbb{R}$. *Then we have the following:*

(i) *If* $\delta < 2$, *then* $\int_0^\eta t^{1-\delta} L(t) dt$ *converges and*

$$\int_0^s t^{1-\delta} L(t) dt \sim \frac{s^{2-\delta} L(s)}{2-\delta} \quad \text{as } s \to 0^+.$$

(ii) *If* $\delta > 2$, *then* $\int_0^\eta t^{1-\delta} L(t) dt$ *diverges and*

$$\int_s^\eta t^{1-\delta} L(t) dt \sim \frac{s^{2-\delta} L(s)}{\delta - 2} \quad \text{as } s \to 0^+.$$

Lemma 2.3 ([7]). *Let $L \in \mathcal{K}$ be defined on $(0,\eta]$, then we have*

$$t \mapsto \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s} ds$ converges, then

$$t \mapsto \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}.$$

Proposition 2.4. *Assume q satisfies (H1). Then for $x \in (0,1)$, we have*

$$G_p q(x) \approx \Psi(1-x),$$

where ψ is the function defined on $(0,1]$ by

$$\Psi(t) = \begin{cases} \int_0^t \frac{(L(s))^{\frac{1}{p-1}}}{s} ds, & \text{if } \beta = p, \\ t^{\frac{p-\beta}{p-1}} (L(t))^{\frac{1}{p-1}}, & \text{if } \mu+1 < \beta < p, \\ t^{\frac{p-1-\mu}{p-1}}, & \text{if } \beta < \mu+1 \\ t^{\frac{p-1-\mu}{p-1}} (\int_t^\eta \frac{L(s)}{s} ds)^{\frac{1}{p-1}}, & \text{if } \beta = \mu+1. \end{cases} \tag{2.1}$$

Proof. For $x \in (0,1)$, we have

$$G_p q(x) \approx \int_x^1 \frac{1}{t^{\frac{\lambda}{p-1}} (1-t)^{\frac{\mu}{p-1}}} \Big(\int_0^t s^\lambda (1-s)^{\mu-\beta} L(1-s) ds \Big)^{\frac{1}{p-1}} dt.$$

Put

$$h(x) := \int_x^1 \frac{1}{t^{\frac{\lambda}{p-1}} (1-t)^{\frac{\mu}{p-1}}} \Big(\int_0^t s^\lambda (1-s)^{\mu-\beta} L(1-s) ds \Big)^{\frac{1}{p-1}} dt, \ x \in (0,1).$$

We shall estimate $h(x)$. Since h is continuous and positive on $[0,1/2]$, it follows that $h(x) \approx 1$, for $x \in [0,1/2]$. Now, assume that $x \in [1/2,1)$. Then

$$h(x) \approx \int_x^1 \frac{1}{(1-t)^{\frac{\mu}{p-1}}} \Big(\int_0^t s^\lambda (1-s)^{\mu-\beta} L(1-s) ds \Big)^{\frac{1}{p-1}} dt.$$

Moreover, for $t \in [x,1)$, we have

$$\begin{aligned} &\int_0^t s^\lambda (1-s)^{\mu-\beta} L(1-s) ds \\ &= \int_0^{1/2} s^\lambda (1-s)^{\mu-\beta} L(1-s) ds + \int_{\frac{1}{2}}^t s^\lambda (1-s)^{\mu-\beta} L(1-s) ds \\ &\approx 1 + \int_{1-t}^{1/2} s^{\mu-\beta} L(s) ds. \end{aligned}$$

Then we distinguish the following cases:

• If $\beta < \mu+1$, then by Lemma 2.2, $\int_0^{1/2} s^{\mu-\beta} L(s) ds < \infty$. So, since $\mu < p-1$, we obtain

$$h(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}}.$$

• If $p > \beta > \mu+1$, then by Lemma 2.2,

$$\int_{1-t}^{1/2} s^{\mu-\beta} L(s) ds \approx (1-t)^{\mu+1-\beta} L(1-t).$$

So,

$$(1+\int_{1-t}^{1/2} s^{\mu-\beta}L(s)ds)^{\frac{1}{p-1}} \approx (1-t)^{\frac{\mu+1-\beta}{p-1}} L^{\frac{1}{p-1}}(1-t).$$

Thus, using the fact that $\beta < p$ and again Lemma 2.2, we obtain that

$$h(x) \approx (1-x)^{\frac{p-\beta}{p-1}} L^{\frac{1}{p-1}}(1-x).$$

• If $\beta = \mu+1$, then

$$h(x) \approx \int_0^{1-x} \frac{1}{t^{\frac{\mu}{p-1}}} \Big(\int_{1-t}^1 \frac{L(s)}{s} ds\Big)^{\frac{1}{p-1}} dt.$$

So, using Lemma 2.3 and the fact that $\mu < p-1$, by Lemma 2.2 it follows that

$$h(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}} (\int_{1-x}^1 \frac{L(s)}{s} ds)^{\frac{1}{p-1}}.$$

• If $\beta = p$, we deduce by Lemma 2.2 that

$$\int_{1-t}^{1/2} s^{\mu-\beta}L(s)ds \approx (1-t)^{\mu+1-p}L(1-t),$$

hence

$$h(x) \approx \int_0^{1-x} \frac{(L(s))^{\frac{1}{p-1}}}{s} ds.$$

This completes the proof. □

The following proposition plays a crucial role in this article.

Proposition 2.5. *Let q satisfy (H1) and let θ_β be the function given in* (1.6)*. Then for $x \in (0,1)$, we have*

$$G_p(q\theta_\beta^\alpha)(x) \approx \theta_\beta(x).$$

Proof. Let $\beta \le p$ and $\mu < p-1$, a straightforward computation shows that for $x \in (0,1)$,

$$q(x)\theta_\beta^\alpha(x) \approx \widetilde{q}(x),$$

where

$$\widetilde{q}(x) := \begin{cases} \frac{L(1-x)}{(1-x)^p}\Big(\int_0^{1-x} \frac{(L(s))^{\frac{1}{p-1}}}{s} ds\Big)^{\frac{\alpha(p-1)}{p-1-\alpha}} & \text{if } \beta = p \\ \frac{(L(1-x))^{\frac{p-1}{p-1-\alpha}}}{(1-x)^{(\beta-\frac{\alpha(p-\beta)}{p-1-\alpha})}} & \text{if } \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1} < \beta < p, \\ \frac{L(1-x)}{(1-x)^{(\beta-\frac{\alpha(p-1-\mu)}{p-1})}} & \text{if } \beta < \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1} \\ \frac{L(1-x)}{(1-x)^{(\mu+1)}}\Big(\int_{1-x}^{\eta} \frac{L(s)}{s} ds\Big)^{\frac{\alpha}{p-1-\alpha}} & \text{if } \beta = \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1}. \end{cases}$$

So, we deduce that

$$\widetilde{q}(x) = (1-x)^{-\delta}\widetilde{L}(1-x),$$

where $\delta \le p$. Then, using Lemmas 2.1 and 2.3, we verify that $\widetilde{L} \in \mathcal{K}$ and $\int_0^\eta t^{\frac{1-\delta}{p-1}}(\widetilde{L}(t))^{\frac{1}{p-1}}dt < +\infty$. Hence, by Proposition 2.4,

$$G_p(q\theta_\beta^\alpha)(x) \approx G_p\widetilde{q}(x) \approx \widetilde{\psi}(1-x), \quad x \in (0,1),$$

where $\widetilde{\psi}$ is the function defined in (2.1) by replacing L by $\widetilde{L}$ and β by δ. This completes the proof. □

3. Proof of Theorem 1.1

3.1. **Existence and asymptotic behavior.** Let q satisfy (H1) and let θ_β be the function given by (1.6). By Proposition 2.5, there exists a constant $m \geq 1$ such that for each $x \in (0,1)$,

$$\frac{1}{m}\theta_\beta(x) \leq G_p(q\theta_\beta^\alpha)(x) \leq m\theta_\beta(x). \tag{3.1}$$

Now we look at the existence of positive solution of problem (1.1) satisfying (1.5). For the case $\alpha < 0$, we refer to [4]. So prove the existence result only for the case $0 \leq \alpha < p-1$, and then give the precise asymptotic behavior of such solution for $\alpha < p-1$. We will split the proof into two cases.

Case 1: $\alpha < 0$**.** Let u be a positive continuous solution of (1.1). To obtain estimates (1.5) on the function u, we need the following comparison result.

Lemma 3.1. *Let $\alpha < 0$ and $u_1, u_2 \in C^1((0,1)) \cap C([0,1])$ be two positive functions such that*

$$\begin{gathered} -\frac{1}{A}(A\Phi_p(u_1'))' \leq q(x)u_1^\alpha, \quad \textit{in } (0,1),\\ A\Phi_p(u_1')(0) = 0, \quad u_1(1) = 0 \end{gathered} \tag{3.2}$$

and

$$\begin{gathered} -\frac{1}{A}(A\Phi_p(u_2'))' \geq q(x)u_2^\alpha, \quad \textit{in } (0,1),\\ A\Phi_p(u_2')(0) = 0, \quad u_2(1) = 0. \end{gathered} \tag{3.3}$$

Then $u_1 \leq u_2$.

Proof. Suppose that $u_1(x_0) > u_2(x_0)$ for some $x_0 \in (0,1)$. Then there exists $x_1, x_2 \in [0,1]$, such that $0 \leq x_1 < x_0 < x_2 \leq 1$ and for $x_1 < x < x_2$, $u_1(x) > u_2(x)$ with $u_1(x_2) = u_2(x_2)$, $u_1(x_1) = u_2(x_1)$ or $x_1 = 0$.

We deduce that

$$A\Phi_p(u_2')(x_1) \leq A\Phi_p(u_1')(x_1). \tag{3.4}$$

On the other hand, since $\alpha < 0$, we have $u_1^\alpha(x) < u_2^\alpha(x)$, for each $x \in (x_1, x_2)$. This yields

$$\frac{1}{A}(A\Phi_p(u_1'))' - \frac{1}{A}(A\Phi_p(u_2'))' \geq q(u_2^\alpha - u_1^\alpha) \geq 0 \quad \text{on } (x_1, x_2).$$

Using further (3.4), we deduce that the function $\omega(x) := (A\Phi_p(u_1') - A\Phi_p(u_2'))(x)$ is nondecreasing on (x_1, x_2) with $\omega(x_1) \geq 0$. Hence, from the monotonicity of Φ_p, we obtain that the function $x \mapsto (u_1 - u_2)(x)$ is nondecreasing on (x_1, x_2) with $(u_1 - u_2)(x_1) \geq 0$ and $(u_1 - u_2)(x_2) = 0$. This yields to a contradiction. The proof is complete. ☐

Now, we are ready to prove (1.5). Put $c = m^{-\frac{\alpha}{p-1-\alpha}}$ and $v := G_p(q\theta_\beta^\alpha)$. It follows from (1.4) that the function v satisfies

$$-\frac{1}{A}(A\Phi_p(v'))' = q\theta_\beta^\alpha, \quad \text{in } (0,1).$$

According to (3.1), we obtain by simple calculation that $\frac{1}{c}v$ and cv satisfy respectively (3.2) and (3.3). Thus, we deduce by Lemma 3.1 that

$$\frac{1}{c}v(x) \leq u(x) \leq cv(x), \ x \in (0,1).$$

This proves the result.

Case 2: $0 \le \alpha < p-1$. Put $c_0 = m^{\frac{p-1}{p-1-\alpha}}$ and let

$$\Lambda := \big\{u \in C([0,1]);\ \frac{1}{c_0}\theta_\beta \le u \le c_0\theta_\beta\big\}.$$

Obviously, the function θ_β belongs to $C([0,1])$ and so Λ is not empty. We consider the integral operator T on Λ defined by

$$Tu(x) := G_p(qu^\alpha)(x), \quad x \in [0,1].$$

We prove that T has a fixed point in Λ, in order to construct a solution of problem (1.1). For this aim, first we observe that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$, then for each $x \in [0,1)$

$$\frac{1}{c_0^\alpha}(q\theta_\beta^\alpha)(x) \le q(x)u^\alpha(x) \le c_0^\alpha(q\theta_\beta^\alpha)(x).$$

This together with (3.1) implies that

$$\frac{1}{mc_0^{\frac{\alpha}{p-1}}}\theta_\beta \le Tu \le mc_0^{\frac{\alpha}{p-1}}\theta_\beta.$$

Since $mc_0^{\frac{\alpha}{p-1}} = c_0$ and $T\Lambda \subset C([0,1])$, then T leaves invariant the convex Λ. Moreover, since $\alpha \ge 0$, then the operator T is nondecreasing on Λ. Now, let $\{u_k\}_k$ be a sequence of functions in $C([0,1])$ defined by

$$u_0 = \frac{1}{c_0}\theta_\beta, \quad u_{k+1} = Tu_k, \quad \text{for } k \in \mathbb{N}.$$

Since $T\Lambda \subset \Lambda$, we deduce from the monotonicity of T that for $k \in \mathbb{N}$, we have

$$u_0 \le u_1 \le \dots \le u_k \le u_{k+1} \le c_0\theta_\beta.$$

Applying the monotone convergence theorem, we deduce that the sequence $\{u_k\}_k$ converges to a function $u \in \Lambda$ which satisfies

$$u(x) = G_p(qu^\alpha)(x),\ x \in [0,1].$$

We conclude that u is a positive continuous solution of problem (1.1) which satisfies (1.5).

3.2. **Uniqueness.** Assume that q satisfies (H1). For $\alpha < 0$, the uniqueness of solution to problem (1.1) follows from Lemma 3.1. Thus, we look at the case $0 \le \alpha < p-1$. Let

$$\Gamma = \{u \in C([0,1]) : u(x) \approx \theta_\beta(x)\}.$$

Let u and v be two positives solutions of problem (1.1) in Γ. Then there exists a constant $k \ge 1$ such that

$$\frac{1}{k} \le \frac{v}{u} \le k.$$

This implies that the set

$$J = \{t \in (1,+\infty) : \frac{1}{t}u \le v \le tu\}$$

is not empty. Now, put $c := \inf J$, then we aim to show that $c = 1$. Suppose that $c > 1$, then

$$\begin{gathered}
-\frac{1}{A}(A\Phi_p(v'))' + \frac{1}{A}(A\Phi_p(c^{\frac{-\alpha}{p-1}}u'))' = q(x)(v^\alpha - c^{-\alpha}u^\alpha), \quad \text{in } (0,1),\\
\lim_{x\to 0^+}(A\Phi_p(v') - A\Phi_p(c^{\frac{-\alpha}{p-1}}u'))(x) = 0,
\end{gathered}$$

$$(v - c^{\frac{-\alpha}{p-1}} u)(1) = 0.$$

So, we have

$$-\frac{1}{A}(A\Phi_p(v'))' + \frac{1}{A}(A\Phi_p(c^{\frac{-\alpha}{p-1}} u'))' \geq 0 \quad \text{in } (0,1),$$

which implies that the function $\theta(x) := (A\Phi_p(c^{\frac{-\alpha}{p-1}} u') - A\Phi_p(v'))(x)$ is nondecreasing on $(0,1)$ with $\lim_{x\to 0^+} \theta(x) = 0$. Hence from the monotonicity of Φ_p, we obtain that the function $x \mapsto (c^{\frac{-\alpha}{p-1}} u - v)(x)$ is nondecreasing on $[0,1)$ with $(c^{-\frac{\alpha}{p-1}} u - v)(1) = 0$. This implies that $c^{\frac{-\alpha}{p-1}} u \leq v$. On the other hand, we deduce by symmetry that $v \leq c^{\frac{\alpha}{p-1}} u$. Hence $c^{\frac{\alpha}{p-1}} \in J$. Now, since $\alpha < p-1$ and $c > 1$, we have $c^{\frac{\alpha}{p-1}} < c$. This yields to a contradiction with the fact that $c := \inf J$. Hence, $c = 1$ and then $u = v$.

4. Applications

First application. Let q be a positive measurable function in $[0,1)$ satisfying for $x \in [0,1)$

$$q(x) \approx (1-x)^{-\beta}\Big(\log \frac{3}{1-x}\Big)^{-\sigma},$$

where the real numbers β and σ satisfy one of the following two conditions:

- $\beta < p$ and $\sigma \in \mathbb{R}$,
- $\beta = p$ and $\sigma > p-1$.

Using Theorem 1.1, we deduce that problem (1.1) has a positive continuous solution u in $[0,1]$ satisfying

(i) If $\beta < \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1}$, then for $x \in (0,1)$,

$$u(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}}.$$

(ii) If $\beta = \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1}$ and $\sigma = 1$, then for $x \in (0,1)$,

$$u(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}}\Big(\log\log \frac{3}{1-x}\Big)^{\frac{1}{p-1-\alpha}}.$$

(iii) If $\beta = \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1}$ and $\sigma < 1$, then for $x \in (0,1)$,

$$u(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}}\Big(\log \frac{3}{1-x}\Big)^{\frac{1-\sigma}{p-1-\alpha}}.$$

(iv) If $\beta = \frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1}$ and $\sigma > 1$, then for $x \in (0,1)$,

$$u(x) \approx (1-x)^{\frac{p-1-\mu}{p-1}}.$$

(v) If $\frac{(\mu+1)(p-1-\alpha)+\alpha p}{p-1} < \beta < p$, then for $x \in (0,1)$,

$$u(x) \approx (1-x)^{\frac{p-\beta}{p-1-\alpha}}\Big(\log \frac{3}{1-x}\Big)^{\frac{-\sigma}{p-1-\alpha}}.$$

(vi) If $\beta = p$ and $\sigma > p-1$, then for $x \in (0,1)$,

$$u(x) \approx \Big(\log \frac{3}{1-x}\Big)^{\frac{p-1-\sigma}{p-1-\alpha}}$$

Second application. Let q be a function satisfying (H1) and let $\alpha, \gamma < p-1$. We are interested in the nonlinear problem

$$\begin{gathered} -\frac{1}{A}(A\Phi_p(u'))' + \frac{\gamma}{u}\Phi_p(u')u' = q(x)u^{\alpha}, \quad \text{in } (0,1), \\ A\Phi_p(u')(0) = 0, \quad u(1) = 0. \end{gathered} \tag{4.1}$$

Put $v = u^{1-\frac{\gamma}{p-1}}$; then v satisfies

$$\begin{gathered} -\frac{1}{A}(A\Phi_p(v'))' = (\frac{p-1-\gamma}{p-1})^{p-1} q(x) v^{\frac{(\alpha-\gamma)(p-1)}{p-1-\gamma}}, \quad \text{in } (0,1), \\ A\Phi_p(v')(0) = 0, \quad v(1) = 0. \end{gathered} \tag{4.2}$$

Using Theorem 1.1, we deduce that (4.2) has a unique solution v such that $v(x) \approx \widetilde{\theta}_\beta(x)$, where

$$\widetilde{\theta}_\beta(x) = \begin{cases} \Big(\int_0^{1-x} \frac{(L(s))^{\frac{1}{p-1}}}{s} ds\Big)^{\frac{p-1-\gamma}{p-1-\alpha}} & \text{if } \beta = p, \\ (1-x)^{\frac{(p-\beta)(p-1-\gamma)}{(p-1)(p-1-\alpha)}} (L(1-x))^{\frac{p-1-\gamma}{(p-1)(p-1-\alpha)}}, & \text{if } \frac{(\mu+1)(p-1-\alpha)+(\alpha-\gamma)p}{p-1-\gamma} \\ & \quad < \beta < p, \\ (1-x)^{\frac{p-1-\mu}{p-1}} & \text{if } \beta < \frac{(\mu+1)(p-1-\alpha)+(\alpha-\gamma)p}{p-1-\gamma}, \\ (1-x)^{\frac{p-1-\mu}{p-1}} (\int_{1-x}^{\eta} \frac{L(s)}{s} ds)^{\frac{p-1-\gamma}{(p-1)(p-1-\alpha)}} & \text{if } \beta = \frac{(\mu+1)(p-1-\alpha)+(\alpha-\gamma)p}{p-1-\gamma}. \end{cases}$$

Consequently, (4.1) has a unique solution u satisfying

$$u(x) \approx \begin{cases} \Big(\int_0^{1-x} \frac{(L(s))^{\frac{1}{p-1}}}{s} ds\Big)^{\frac{p-1}{p-1-\alpha}}, & \text{if } \beta = p \\ (1-x)^{\frac{p-\beta}{p-1-\alpha}} (L(1-x))^{\frac{1}{p-1-\alpha}}, & \text{if } \frac{(\mu+1)(p-1-\alpha)+(\alpha-\gamma)p}{p-1-\gamma} < \beta < p, \\ (1-x)^{\frac{p-1-\mu}{p-1-\gamma}}, & \text{if } \beta < \frac{(\mu+1)(p-1-\alpha)+(\alpha-\gamma)p}{p-1-\gamma} \\ (1-x)^{\frac{p-1-\mu}{p-1-\gamma}} (\int_{1-x}^{\eta} \frac{L(s)}{s} ds)^{\frac{1}{p-1-\alpha}}, & \text{if } \beta = \frac{(\mu+1)(p-1-\alpha)+(\alpha-\gamma)p}{p-1-\gamma}. \end{cases}$$

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