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OSCILLATION OF SOLUTIONS TO THIRD-ORDER HALF-LINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the oscillation of solutions to the third-order neutral differential equations

$$\Big(a(t)\big([x(t) \pm p(t)x(\delta(t))]''\big)^{\alpha}\Big)' + q(t)x^{\alpha}(\tau(t)) = 0.$$

Sufficient conditions are established so that every solution is either oscillatory or converges to zero. In particular, we extend the results obtain in [1] for $a(t)$ non-decreasing, to the non-increasing case.

1. Introduction

In recent years, there has been great interest in studying the oscillatory behavior of differential equations; see for example [3, 4, 5, 6, 7] and the references cited therein. Compared to first and second order, third-order neutral differential equations have received less attention, even though such equations arise in many physical problems. Motivated by this observation, we study the oscillation of solutions to the third-order half-linear neutral differential equations

$$\Big(a(t)\big([x(t) + p(t)x(\delta(t))]''\big)^{\alpha}\Big)' + q(t)x^{\alpha}(\tau(t)) = 0, \quad t \ge t_0, \tag{1.1}$$

and

$$\Big(a(t)\big([x(t) - p(t)x(\delta(t))]''\big)^{\alpha}\Big)' + q(t)x^{\alpha}(\tau(t)) = 0, \quad t \ge t_0\,. \tag{1.2}$$

We assume the following conditions:

- (H1) $a(t)$, $p(t)$, $q(t)$, $\tau(t)$, $\delta(t)$ are in $C([0,\infty))$; $a(t)$, $q(t)$, $\tau(t)$, $\delta(t)$ are positive functions; α is the quotient of two odd positive integers.
- (H2) There is constant p such that $0 \le p(t) \le p < 1$; the delay arguments satisfy $\tau(t) \le t$, $\delta(t) \le t$, $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$.
- (H3) $a(t)$ is positive and non-increasing; $A(t) := \int_{t_0}^{t} a^{-1/\alpha}(s)\,ds \to \infty$ as $t \to \infty$.

By a solution to (1.1), we mean a function $x(t)$ in $\mathcal{C}^2[T_x,\infty)$ for which $a(t)(z''(t))^{\alpha}$ is in $\mathcal{C}^1[T_x,\infty)$ and (1.1) is satisfied on some interval $[T_x,\infty)$, where $T_x \ge t_0$, and $z(t) = x(t) + p(t)x(\delta(t))$. The same concept of a solution applies to (1.2).

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Dzurina [1] obtained sufficient conditions for the oscillation of solutions to (1.1) and to (1.2), under the assumption that $a(t)$ is non-decreasing. Here we establish similar results when $a(t)$ is non-increasing. We follow the same strategy as in [1], but with new estimates in Lemmas 2.3, 2.4, 2.5.

We consider only solutions $x(t)$ for which $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We say that a solution is oscillatory if it has arbitrarily large zeros, and non-oscillatory otherwise. All functional inequalities are assumed to hold eventually; that is, for all t large enough. Note that if $x(t)$ is a solution so is $-x(t)$; so our proofs are done only for positive solutions.

In Section 2, we present oscillation results for (1.1), while in Section 3 we present similar results for (1.2). In both section we give examples to illustrate our results.

2. Oscillation results for (1.1)

For a solution $x(t)$ of (1.1), we define the corresponding function

$$z(t) = x(t) + p(t)x(\delta(t)). \tag{2.1}$$

To obtain sufficient conditions for the oscillation of solutions to (1.1), we need the the following lemmas.

Lemma 2.1 ([1, Lemma 1]). *Let $x(t)$ be a positive solution of (1.1). Then there are only two possible cases:*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $(a(t)(z''(t))^\alpha)' < 0$;
(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, $(a(t)(z''(t))^\alpha)' < 0$.

Lemma 2.2 ([1, Lemma 2]). *Let $x(t)$ be a positive solution of (1.1), and let the corresponding function $z(t)$ satisfy Case (II) of Lemma 2.1. If*

$$\int_{t_0}^\infty \int_v^\infty \Big[\frac{1}{a(u)}\int_u^\infty q(s)ds\Big]^{1/\alpha} du\, dv = \infty, \tag{2.2}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Lemma 2.3. *Assume that $u(t) > 0$, $u'(t) > 0$, $(a(t)(u'(t))^\alpha)' \leq 0$ on $[t_0, \infty)$. Then for each $\ell \in (0,1)$ there exists $T_\ell \geq t_0$ such that*

$$\frac{u(\tau(t))}{A(\tau(t))} \geq \ell \frac{u(t)}{A(t)} \quad \text{for } t \geq T_\ell.$$

Proof. Since $a(t)(u'(t))^\alpha$ is non-increasing, so is $a^{1/\alpha}(t)(u'(t))$. Then by the definition of $A(t)$, we have

$$\begin{aligned} u(t) - u(\tau(t)) &= \int_{\tau(t)}^t a^{1/\alpha}(s)(u'(s))\frac{1}{a^{1/\alpha}(s)}\,ds \\ &\leq a^{1/\alpha}(\tau(t))u'(\tau(t))\big(A(t) - A(\tau(t))\big). \end{aligned} \tag{2.3}$$

Also

$$u(\tau(t)) \geq u(\tau(t)) - u(t_0) \geq a^{1/\alpha}(\tau(t))u'(\tau(t))\big(A(\tau(t)) - A(t_0)\big).$$

Since $\lim_{t\to\infty} \frac{A(\tau)-A(t_0)}{A(\tau)} = 1$, for each $\ell \in (0,1)$ there exists $T_\ell \geq t_0$ such that $\big(A(\tau(t)) - A(t_0)\big) > \ell A(\tau(t))$ for $t \geq T_\ell$. From the above inequality,

$$\frac{u(\tau(t))}{u'(\tau(t))} \geq \ell a^{1/\alpha}(\tau(t))A(\tau(t)), \quad t \geq T_\ell. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\frac{u(t)}{u(\tau(t))} \le 1 + \frac{A(t) - A(\tau(t))}{\ell A(\tau(t))} \le \frac{A(t)}{\ell A(\tau(t))},$$

which completes the proof. □

Lemma 2.4. *Assume that $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $\big(a(t)(z''(t))^\alpha\big)' \le 0$ on (T_ℓ, ∞). Then*

$$\frac{z(t)}{z'(t)} \ge \frac{a^{1/\alpha}(t)A(t)}{2} \quad \textit{for } t \ge T_\ell.$$

Proof. Since $a(t)(z''(t))^\alpha$ is positive and non-increasing, so is $a^{1/\alpha}(t)z''(t)$. From $z'(t) > 0$, $a(t) > 0$, we have

$$z'(t) \ge z'(t) - z'(\tau(t)) \ge \int_{T_\ell}^t \frac{a^{1/\alpha}(s)z''(s)}{a^{1/\alpha}(s)} ds \ge a^{1/\alpha}(t)A(t)z''(t). \tag{2.5}$$

Since $A'(t) = a^{-1/\alpha}(t)$,

$$A'(t)z'(t) \ge A(t)z''(t), \quad t \ge T_\ell. \tag{2.6}$$

Integrating both sides of the above inequality, and using that $A(T_\ell)z'(T_\ell) > 0$, we obtain

$$\int_{T_\ell}^t A'(s)z'(s)ds \ge A(t)z'(t) - \int_{T_\ell}^t A'(s)z'(s)ds.$$

Therefore,

$$\int_{T_\ell}^t A'(s)z'(s)ds \ge \frac{1}{2}A(t)z'(t). \tag{2.7}$$

Since $a(t)$ is non-increasing, we have $A(t) > 0$, $A'(t) > 0$, $A''(t) \ge 0$. and

$$(A'(t)z(t))' = A'(t)z'(t) + A''(t)z(t) \ge A'(t)z'(t). \tag{2.8}$$

Integrating on both sides of the above equality, then using that $A'(T_\ell)z(T_\ell) > 0$ and (2.7), we obtain

$$A'(t)z(t) \ge \frac{1}{2}A(t)z'(t), \quad t \ge T_\ell,$$

which implies the desired result. □

The next lemma follows from (2.6).

Lemma 2.5. *Assume that $z'(t) > 0$, $z''(t) > 0$, $\big(a(t)(z''(t))^\alpha\big)' \le 0$ on (T_ℓ, ∞). Then*

$$\frac{A(t)z''(t)}{A'(t)z'(t)} \le 1, \quad \textit{for} \quad t \ge T_\ell.$$

For simplicity of notation, we introduce

$$P_\ell(t) = \ell^\alpha (1-p)^\alpha q(t) a(\tau(t)) \Big(\frac{A(\tau(t))}{A(t)}\Big)^\alpha \Big(\frac{A(\tau(t))}{2}\Big)^\alpha$$

with $\ell \in (0,1)$ and $t \ge T_\ell$;

$$P = \liminf_{t\to\infty} A^\alpha(t) \int_t^\infty P_\ell(s)ds, \quad Q = \limsup_{t\to\infty} \frac{1}{A(t)} \int_{t_0}^t A^{\alpha+1}(s)P_\ell(s)ds. \tag{2.9}$$

Further, for $z(t)$ satisfying Case (I) of Lemma 2.1, we define

$$w(t) = a(t)\Big(\frac{z''(t)}{z'(t)}\Big)^{\alpha}, \tag{2.10}$$

$$r = \liminf_{t\to\infty} A^{\alpha}(t)w(t), \quad R = \limsup_{t\to\infty} A^{\alpha}(t)w(t). \tag{2.11}$$

Lemma 2.6. *Let $x(t)$ be a positive solution of* (1.1).

(a) *Let $P < \infty$, $Q < \infty$ and $z(t)$ satisfy Case (I) of Lemma 2.1. Then $P \le r - r^{1+\frac{1}{\alpha}}$ and $P + Q \le 1$.*

(b) *If $P = \infty$ or $Q = \infty$, then $z(t)$ does not satisfy Case (I) of Lemma 2.1.*

Proof. Part (a). Assume that $x(t)$ is a positive solution of (1.1) and the corresponding function $z(t)$ satisfies Case(I) of Lemma 2.1. From the definition of $z(t)$, we have

$$x(t) = z(t) - p(t)x(\delta(t)) > z(t) - p(t)z(\delta(t)) \ge (1-p)z(t).$$

Using this inequality in (1.1), we obtain

$$(a(t)(z''(t))^{\alpha})' \le -(1-p)^{\alpha} q(t) z^{\alpha}(\tau(t)) \le 0. \tag{2.12}$$

Then from its definition, $w(t)$ is positive and satisfies

$$\begin{aligned} w'(t) &= \frac{1}{(z'(t))^{\alpha}}\big(a(t)(z''(t))^{\alpha}\big)' - \alpha a(t)\Big(\frac{z''(t)}{z'(t)}\Big)^{\alpha+1} \\ &\le -q(t)(1-p)^{\alpha}\frac{z^{\alpha}(\tau(t))}{(z'(t))^{\alpha}} - \frac{\alpha}{a^{1/\alpha}(t)} w^{1+\frac{1}{\alpha}}(t). \end{aligned} \tag{2.13}$$

From Lemma 2.3 with $u(t) = z'(t)$, we have

$$\frac{1}{z'(t)} \ge \ell \frac{A(\tau(t))}{A(t)} \frac{1}{z'(\tau(t))}, \quad t \ge T_{\ell},$$

where ℓ is the same as in P_{ℓ}. Now (2.13) becomes

$$w'(t) \le -\ell^{\alpha} q(t)(1-p)^{\alpha}\Big(\frac{A(\tau(t))}{A(t)}\Big)^{\alpha} \frac{z^{\alpha}(\tau(t))}{(z'(\tau(t)))^{\alpha}} - \frac{\alpha}{a^{1/\alpha}(t)} w^{1+\frac{1}{\alpha}}(t).$$

From Lemma 2.4, we have $z(t) \ge \frac{a^{1/\alpha}(t)A(t)}{2} z'(t)$, so that

$$w'(t) + P_{\ell}(t) + \frac{\alpha}{a^{1/\alpha}(t)} w^{1+\frac{1}{\alpha}}(t) \le 0. \tag{2.14}$$

Since $P_{\ell}(t) > 0$ and $w(t) > 0$ for $t \ge T_{\ell}$. It follows that $w'(t) \le 0$ and $-w'(t) \ge \alpha w^{1+(1/\alpha)}(t)/a^{1/\alpha}(t)$; thus

$$\Big(\frac{1}{w^{1/\alpha}(t)}\Big)' > \frac{1}{a^{1/\alpha}(t)}.$$

Integrating the above inequality from T_{ℓ} to t, and using that $w^{-1/\alpha}(T_{\ell}) > 0$, we obtain

$$w(t) < \frac{1}{\big(\int_{T_{\ell}}^{t} a^{-1/\alpha}(s)\,ds\big)^{\alpha}},$$

which in view of (H3) implies that $\lim_{t\to\infty} w(t) = 0$.

On the other hand, from the definition of $w(t)$ and Lemma 2.5,

$$A^{\alpha}(t)w(t) = a(t)\Big(\frac{A(t)z''(t)}{z'(t)}\Big)^{\alpha} = \Big(\frac{A(t)z''(t)}{A'(t)z'(t)}\Big)^{\alpha} \le 1^{\alpha}.$$

Then

$$0 \le r \le R \le 1. \tag{2.15}$$

Next we prove the first inequality in (a). Let $\epsilon > 0$. Then from the definition of P and r, we can choose $t_2 \ge T_\ell$, sufficiently large such that

$$A^\alpha(t)\int_t^\infty P_\ell(s)ds \ge P-\epsilon \quad \text{and} \quad A^\alpha(t)w(t) \ge r-\epsilon \quad \text{for } t \ge t_2.$$

Integrating (2.14) from t to ∞ and using that $\lim_{t\to\infty} w(t) = 0$, we have

$$w(t) \ge \int_t^\infty P_\ell(s)ds + \alpha\int_t^\infty \frac{w^{1+\frac{1}{\alpha}}(s)}{a^{1/\alpha}(s)}ds \quad \text{for } t \ge t_2. \tag{2.16}$$

Multiplying the above inequality by $A^\alpha(t)$ and simplifying, we obtain

$$\begin{aligned} A^\alpha(t)w(t) &\ge A^\alpha(t)\int_t^\infty P_\ell(s)ds + \alpha A^\alpha(t)\int_t^\infty \frac{A^{\alpha+1}(s)w^{1+\frac{1}{\alpha}}(s)}{A^{\alpha+1}(s)a^{1/\alpha}(s)}ds \\ &\ge (P-\epsilon) + (r-\epsilon)^{1+\frac{1}{\alpha}}A^\alpha(t)\int_t^\infty \frac{\alpha A'(s)}{A^{\alpha+1}(s)}ds, \end{aligned}$$

and so

$$A^\alpha(t)w(t) \ge (P-\epsilon) + (r-\epsilon)^{1+\frac{1}{\alpha}}.$$

Taking the limit inferior on both sides as $t \to \infty$, we obtain

$$r \ge (P-\epsilon) + (r-\epsilon)^{1+\frac{1}{\alpha}}.$$

Since $\epsilon > 0$ is arbitrary, we obtain the desired result

$$P \le r - r^{1+\frac{1}{\alpha}}.$$

Next, we prove the second inequality in (a). Multiplying (2.14) by $A^{\alpha+1}(t)$ and integrating it from t_2 to t, we obtain

$$\int_{t_2}^t A^{\alpha+1}(s)w'(s)ds \le -\int_{t_2}^t A^{\alpha+1}(s)P_\ell(s)ds - \alpha\int_{t_2}^t \frac{(A^\alpha(s)w(s))^{(\alpha+1)/\alpha}}{a^{1/\alpha}(s)}ds.$$

Integrating by parts,

$$\begin{aligned} A^{\alpha+1}(t)w(t) \le A^{\alpha+1}(t_2)w(t_2) &- \int_{t_2}^t A^{\alpha+1}(s)P_\ell(s)ds \\ &- \alpha\int_{t_2}^t \frac{(A^\alpha(s)w(s))^{(\alpha+1)/\alpha}}{a^{1/\alpha}(s)}ds + \int_{t_2}^t w(s)\left(A^{\alpha+1}(s)\right)' ds. \end{aligned}$$

Hence,

$$\begin{aligned} A^{\alpha+1}(t)w(t) \le A^{\alpha+1}(t_2)w(t_2) &- \int_{t_2}^t A^{\alpha+1}(s)P_\ell(s)ds \\ &+ \int_{t_2}^t \Big[\frac{(\alpha+1)A^\alpha(s)w(s)}{a^{1/\alpha}(s)} - \frac{\alpha(A^\alpha(s)w(s))^{(\alpha+1)/\alpha}}{a^{1/\alpha}(s)}\Big]ds. \end{aligned}$$

Using the inequality

$$Bu - Du^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{D^\alpha} \tag{2.17}$$

with $u = A^\alpha(t)w(t)$, $D = \frac{\alpha}{a^{1/\alpha}(t)}$, and $B = \frac{\alpha+1}{a^{1/\alpha}(t)}$, we obtain

$$A^{\alpha+1}(t)w(t) \le A^{\alpha+1}(t_2)w(t_2) - \int_{t_2}^{t} A^{\alpha+1}(s)P_\ell(s)ds + A(t) - A(t_2).$$

It follows that

$$A^\alpha(t)w(t) \le \frac{1}{A(t)}A^{\alpha+1}(t_2)w(t_2) - \frac{1}{A(t)}\int_{t_2}^{t} A^{\alpha+1}(s)P_\ell(s)ds + 1 - \frac{A(t_2)}{A(t)}.$$

Taking the limit superior on both sides as $t \to \infty$, we obtain

$$R \le -Q + 1. \tag{2.18}$$

Combining this inequality with (2.15), we have

$$P \le r - r^{1+\frac{1}{\alpha}} \le r \le R \le -Q + 1,$$

which completes the proof of Part (a).

Part (b). Assume that $x(t)$ is a positive solution of (1.1). We shall show that $z(t)$ can not satisfy Case (I) of Lemma 2.1. On the contrary, first, we assume that $P = \infty$. Then (2.16),

$$A^\alpha(t)w(t) \ge A^\alpha(t)\int_t^\infty P_\ell(s)ds.$$

Note that by (2.15), the left-hand side is bounded above by 1. Also note that limit inferior of the right-hand side is $P = \infty$. This leads to a contradiction.

Now, we assume that $Q = \infty$. Then by (2.18), $R = -\infty$, which contradicts $0 \le R \le 1$ in (2.15). The proof is complete. □

Now we present oscillation results whose proofs follow the steps in [1, Theorems 1 and 2].

Theorem 2.7. *Assume that* (2.2) *holds, and let* $x(t)$ *be a solution of* (1.1). *If*

$$P := \liminf_{t\to\infty} A^\alpha(t)\int_t^\infty P_\ell(s)ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{2.19}$$

then $x(t)$ *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose x is a non-oscillatory solution of (1.1). Since $-x$ is also a solution, we can assume without loss of generality that x is positive. If $P = +\infty$, then by Lemma 2.6, $z(t)$ does not have property (I). That is, $z(t)$ satisfies Case (II) of Lemma 2.1. Therefore, n from Lemma 2.2, we have $\lim_{t\to\infty} x(t) = 0$.

Now assume that $z(t)$ satisfies Case (I) of Lemma 2.1. Let $w(t)$ and r be defined by (2.10) and (2.11), respectively. Then from Lemma 2.6, we have $P \le r - r^{(\alpha+1)/\alpha}$. Using (2.17) with $B = D = 1$, we have

$$P \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}},$$

which contradicts (2.19). The proof is complete. □

Theorem 2.8. *Assume that* (2.2) *holds, and let* $x(t)$ *be a solution of* (1.1). *If*

$$P + Q > 1, \tag{2.20}$$

then $x(t)$ *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose x is a non-oscillatory solution of (1.1). Since $-x$ is also a solution, we can assume without loss of generality that x is positive. If P or Q equal $infty$, then by Lemma 2.6, $z(t)$ does not satisfy Case (I), and $z(t)$ must satisfy Case (II). Then from Lemma 2.2, $\lim_{t\to\infty} x(t) = 0$.

Now assume that Case (I) holds. Let $w(t)$ and r be defined as above. Then from Lemma 2.6, $P + Q \le 1$. which contradicts (2.20). The proof is complete. □

As a consequence of Theorem 2.8, we have the following results.

Corollary 2.9. *Assume that* (2.2) *holds. If*

$$\lim_{t\to\infty} \inf A^{\alpha}(t) \int_t^{\infty} q(s)a(\tau(s))\frac{(A(\tau(s)))^{2\alpha}}{A^{\alpha}} ds > \frac{(2\alpha)^{\alpha}}{\ell^{\alpha}(1-p)^{\alpha}} \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}},$$

then every solution $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Corollary 2.10. *Assume that* (2.2) *holds. If*

$$Q = \lim_{t\to\infty} \sup \frac{1}{A(t)} \int_{t_0}^{t} A^{\alpha+1}(s)P_{\ell}(s)ds > 1,$$

then $x(t)$ *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

We conclude this section with an example. Consider the third-order neutral differential equation

$$\Big[\frac{1}{t^3}([x(t) + \frac{1}{3}x(\frac{t}{2})]'')^3\Big]' + \frac{\lambda}{t^{10}}x^3(\frac{t}{2}) = 0, \quad \lambda > 0, \quad t \ge 1. \tag{2.21}$$

Here $a(t) = 1/t^3$, $p = 1/3$, $\alpha = 3$, $\tau(t) = \delta(t) = t/2$, $q(t) = \lambda/t^{10}$. It is easy to see that (2.2) holds. Hence by Corollary 2.9, every non-oscillatory solution of (2.21) converges to zero provided that $\lambda > 3^6 \times 4^5$.

3. Oscillation results for (1.2)

For each solution $x(t)$ of (1.2), we define the associated function

$$z(t) = x(t) - p(t)x(\tau(t)). \tag{3.1}$$

Lemma 3.1 ([1, Lemma 7])**.** *Let* $x(t)$ *be a positive solution of equation*(1.2)*. Then there are the following four cases for* $z(t)$*:*

- (I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $(a(t)(z''(t))^{\alpha})' < 0$;
- (II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, $(a(t)(z''(t))^{\alpha})' < 0$;
- (III) $z(t) < 0$, $z'(t) < 0$, $z''(t) > 0$, $(a(t)(z''(t))^{\alpha})' < 0$;
- (IV) $z(t) < 0$, $z'(t) < 0$, $z''(t) < 0$, $(a(t)(z''(t))^{\alpha})' < 0$.

Lemma 3.2 ([1, Lemma 8])**.** *Let* $x(t)$ *be a positive solution of* (1.2) *and* $z(t)$ *satisfy Case (II) of Lemma* (3.1)*. If* (2.2) *holds, then* $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

For simplicity of notation, we introduce

$$\overline{P_{\ell}}(t) = \ell^{\alpha} q(t)a(\tau(t))\Big(\frac{A(\tau(t))}{A(t)}\Big)^{\alpha}\Big(\frac{A(\tau(t))}{2}\Big)^{\alpha}$$

with $\ell \in (0, 1)$;

$$\overline{P} = \liminf_{t\to\infty} A^{\alpha}(t) \int_t^{\infty} \overline{P_{\ell}}(s)ds, \quad \overline{Q} = \limsup_{t\to\infty} \frac{1}{A(t)} \int_{t_0}^{t} A^{\alpha+1}(s)\overline{P_{\ell}}(s)ds.$$

Also $w(t), r, R$ are defined as in (2.10) and (2.11).

Lemma 3.3. *Let $x(t)$ be a positive solution of* (1.2).

(a) *Let $\overline{P} < \infty$ and $\overline{Q} < \infty$. Assume that $z(t)$ satisfies Case (I) of Lemma 3.1. Then $\overline{P} \le r - r^{1+\frac{1}{\alpha}}$ and $\overline{P} + \overline{Q} \le 1$.*

(b) *If $\overline{P} = \infty$ or $\overline{Q} = \infty$, then $z(t)$ can not satisfy Case (I) of Lemma 3.1.*

Proof. Assume that $x(t)$ is a positive solution of (1.2) and the associated function $z(t)$ satisfies Case (I) of Lemma 3.1. Since $0 < z(t) < x(t)$, equation (1.2) can be written as

$$\big(a(t)(z''(t))^{\alpha}\big)' < -q(t)z^{\alpha}(\tau(t)) < 0.$$

The rest of the proof is similar to that of Lemma 2.6 and hence it is omitted. □

The following theorem presents an oscillation criterion for equation (1.2).

Theorem 3.4. *Assume that* (2.2) *holds. If*

$$\liminf_{t\to\infty} A^{\alpha}(t)\int_t^{\infty} \overline{P_\ell}(s)ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{3.2}$$

then every solution $x(t)$ of (1.2) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

The proof of the above theorem is similar to that of [1, Theorem 3]; hence it is omitted. From the above theorem we have a simplified criterion as follows.

Corollary 3.5. *Assume that* (2.2) *holds. If*

$$\liminf_{t\to\infty} A^{\alpha}(t)\int_t^{\infty} q(s)a(\tau(s))\frac{(A(\tau(s)))^{2\alpha}}{A^{\alpha}(s)}ds > \frac{(2\alpha)^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{3.3}$$

then every solution $x(t)$ of (1.2) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Theorem 3.6. *Assume that* (2.2) *holds. Let $x(t)$ be a solution of* (1.2). *If*

$$\overline{P} + \overline{Q} > 1, \tag{3.4}$$

then every solution of (1.2) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

The proof of the above theorem is similar to that of Theorem 2.8; hence it is omitted.

Corollary 3.7. *Assume that* (2.2) *holds. If*

$$\limsup_{t\to\infty} \frac{1}{A(t)}\int_{t_0}^{t} A^{\alpha+1}(s)\overline{P_\ell}(s)ds > 1, \tag{3.5}$$

then every solution $x(t)$ of (1.2) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

As an example, consider the third-order neutral differential equation

$$\Big(\frac{1}{t^3}([x(t) - \frac{1}{3}x(\frac{t}{2})]'')^3\Big)' + \frac{\lambda}{t^{10}}x^3(\frac{t}{2}) = 0, \quad \lambda > 0, \quad t \ge 1. \tag{3.6}$$

Corollary 3.5 implies that every solution of (3.6) is either oscillatory or approaches zero as $t \to \infty$, provided $\lambda > 6^4 \times 2^7$.

We conclude this article with by remarking that when $a(t)$ is constant, our results coincide with the results in [1].

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