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MEAN VALUE THEOREM FOR HOLOMORPHIC FUNCTIONS

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Abstract. This article presents a generalization of Myers' theorem and when the boundary assumption $f'(a) = f'(b)$ is removed, and to prove this result for holomorphic functions of one complex variable. After that, the equivalence of Rolle's and mean value theorems in the complex plane are proved.

1. Introduction

We know the following two results, usually covered in a first semester calculus course, and used to solve a great variety of problems in optimization, economics. etc. Let f be a continuous function on a closed interval $[a, b]$. The difference between the values of f at the endpoints of $[a, b]$, if derivative $f'(a)$ exists, can be estimated by using $f'(a)$:

$$f(b) - f(a) \approx f'(a)(b - a), \tag{1.1}$$

where the approximation is good if $b - a$ is small. In fact, this is just the tangent line approximation that takes the form

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x, \tag{1.2}$$

with Δx replaced by $b - a$. Actually, the approximation (1.1) can be replaced by the exact formula

$$f(b) - f(a) = f'(c)(b - a), \tag{1.3}$$

where the derivative f' is evaluated at a suitable point c between a and b, rather than at the end point a. Here we assume that f is differentiable at every point between a and b, and the choice of c depends on the particular function f. This result, known as the mean value theorem, is of great importance in mathematical analysis.

Theorem 1.1 (Mean Value Theorem)**.** *Let f be a real continuous function on $[a, b]$ and differentiable in (a, b). Then there exists a point $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a). \tag{1.4}$$

If $f(a) = f(b)$, then the mean value theorem reduces to Rolle's theorem which is also the another most fundamental results in mathematical analysis.

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Theorem 1.2 (Rolle's Theorem)**.** *Let f be a real continuous function on $[a,b]$ and differentiable in (a,b). Furthermore, assume $f(a)=f(b)$. Then there is a point $c\in(a,b)$ such that $f'(c)=0$.*

The equivalence between Rolle's and mean value theorems for real-valued functions has been proved for example in [9]. There are many other types of mean value theorems that are less known. In particular, in 1958 Flett [3] proved the following variation of the mean value theorem.

Theorem 1.3 (Flett's Theorem)**.** *Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ and $f'(a)=f'(b)$. Then there exists a point $c\in(a,b)$ such that*

$$f(c)-f(a)=f'(c)(c-a). \tag{1.5}$$

In 1977, Myers [8] gave the following result which is a slight modification of Flett's theorem.

Theorem 1.4 (Myers' Theorem)**.** *Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ and $f'(a)=f'(b)$. Then there exists a point $c\in(a,b)$ such that*

$$f(b)-f(c)=f'(c)(b-c). \tag{1.6}$$

The geometric interpretation of above theorems can be found in [3, 8, 14]. For other examples of mean value theorems, we refer the reader to the references in this article. In 1998, Sahoo and Riedel [12] gave a generalization of Flett's mean value theorem and removed the boundary assumption on the derivatives of the function f.

Theorem 1.5 (Sahoo and Riedel's Theorem)**.** *Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$. Then there exists a point $c\in(a,b)$ such that*

$$f(c)-f(a)=f'(c)(c-a)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(c-a)^2. \tag{1.7}$$

In general, these results do not immediately extend to holomorphic functions of one complex variable. For the case of Rolle's theorem, the function $f(z)=e^z-1$ has value 0 at $z=0$ and at $z=2\pi i$, but $f'(z)=e^z$ has no zeros in the complex plane. Evard and Jafari [2] went around this difficulty by working with the real and imaginary parts of a holomorphic function. Another approach is taken by Samuelsson [13]. Moreover, Flett's theorem is not valid for complex-valued functions of one complex variable. To see this, consider the function $f(z)=e^z-z$. Then f is holomorphic, and $f'(z)=e^z-1$. Therefore, we have $f'(2k\pi i)=e^{2k\pi i}-1=0$ for all integers k. In particular, $f'(0)=f'(2\pi i)$, that is, the derivatives of f at the endpoints of the closed interval $[0,2\pi i]$ are equal. Nevertheless, the equation $f(z)-f(0)=f'(z)z$ has no solution on the interval $(0,2\pi i)$, as we now show. The equation above gives $1-z=e^{-z}$ and, since $z=iy$, we obtain $1-iy=\cos y-i\sin y$. The comparison of the real and imaginary parts gives the system $\cos y=1$ and $\sin y=y$, which has no solution in the interval $(0,2\pi)$. Thus Flett's theorem fails in the complex domain.

Now, we give some notation. Let $\mathbb{C}$ denote the set of complex numbers. For distinct a and b in $\mathbb{C}$, let $[a,b]$ denote the set $\{a+t(b-a)\colon t\in[0,1]\}$; we will refer to $[a,b]$ as a line segment or a closed interval in $\mathbb{C}$. Similarly, (a,b) denotes the set $\{a+t(b-a):t\in(0,1)\}$.

In 1999, Davitt et al [1] prove a version of Flett's theorem for holomorphic functions of one complex variable. They gave a generalization of Theorem 1.5 for holomorphic functions where

$$\langle u, v\rangle = \mathrm{Re}(u\overline{v}) \tag{1.8}$$

for any two complex numbers u and v.

Theorem 1.6 (Davitt, Powers, Riedel and Sahoo's Theorem)**.** *Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let a and b be two distinct points in D_f. Then there exist $z_1, z_2 \in (a, b)$ such that*

$$\mathrm{Re}(f'(z_1)) = \frac{\langle b-a, f(z_1)-f(a)\rangle}{\langle b-a, z_1-a\rangle} + \frac{1}{2}\frac{\mathrm{Re}(f'(b)-f'(a))}{b-a}(z_1-a) \tag{1.9}$$

and

$$\mathrm{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(z_2)-f(a)]\rangle}{\langle b-a, z_2-a\rangle} + \frac{1}{2}\frac{\mathrm{Im}(f'(b)-f'(a))}{b-a}(z_2-a). \tag{1.10}$$

In this paper, our first aim is to present a generalization of Myers' theorem and removed the boundary assumption on the derivatives of the function f, i.e. $f'(a) = f'(b)$. Our second aim is to provide this result for holomorphic functions of one complex variable. After that the equivalence of Rolle's and mean value theorems in the complex plane are proved.

2. Main Results

Our first goal of this paper is to extend Myers' theorem for real-valued functions to a result that does not depend on the hypothesis $f'(a) = f'(b)$, but reduces to Myers' theorem when this is the case.

Theorem 2.1. *If $f : [a, b] \to \mathbb{R}$ is a differentiable function, then there exists a point $c \in (a, b)$ such that*

$$f(b) - f(c) = f'(c)(b-c) + \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(b-c)^2. \tag{2.1}$$

Proof. Consider the auxiliary function $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) = f(x) - \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(x-a)^2. \tag{2.2}$$

Then h is differentiable on $[a, b]$, and

$$h'(x) = f'(x) - \frac{f'(b)-f'(a)}{b-a}(x-a). \tag{2.3}$$

It follows that $h'(a) = h'(b) = f'(a)$. Applying Myers' theorem to h gives $h(b) - h(c) = h'(c)(b-c)$ for some $c \in (a, b)$. Rewriting h and h' in terms of f gives the asserted result. □

Remark 2.2. It is easy to see that if $f'(a) = f'(b)$, then this result reduces to Theorem 1.4. Furthermore, Theorem 2.1 remains valid if the function h given by (2.2) is replaced by

$$h(x) = f(x) - \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(x-b)^2. \tag{2.4}$$

This shows our that the function h is not unique. So, we can find same result by using different an auxiliary function h.

The second goal of this paper is to prove a version of Myers' theorem for holomorphic functions of one complex variable in the spirit of Evard and Jafari [2].

Theorem 2.3. *Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let a and b be two distinct points in D_f. Then there exist $z_1, z_2 \in (a,b)$ such that*

$$\operatorname{Re}(f'(z_1)) = \frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{\operatorname{Re}(f'(b)-f'(a))}{b-a}(b-z_1) \tag{2.5}$$

and

$$\operatorname{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle} - \frac{1}{2}\frac{\operatorname{Im}(f'(b)-f'(a))}{b-a}(b-z_2). \tag{2.6}$$

Proof. Let $u(z) = \operatorname{Re}(f(z))$ and $v(z) = \operatorname{Im}(f(z))$ for $z \in D_f$. We now define the auxiliary function $\Phi : [0,1] \to \mathbb{R}$ by

$$\Phi(t) = \langle b-a, f(a+t(b-a))\rangle, \tag{2.7}$$

which is

$$\Phi(t) = \operatorname{Re}[(b-a)\ u(a+t(b-a))] + \operatorname{Im}[(b-a)\ v(a+t(b-a))] \tag{2.8}$$

for every $t \in [0,1]$. Therefore, using the Cauchy-Riemann equations, we obtain

$$\begin{aligned}
\Phi'(t) &= \langle b-a, (b-a)f'(a+t(b-a))\rangle \\
&= \operatorname{Re}((b-a)^2)\frac{\partial u(z)}{\partial x} + \operatorname{Im}((b-a)^2)\frac{\partial u(z)}{\partial x} \\
&= |b-a|^2\frac{\partial u(z)}{\partial x} \\
&= |b-a|^2 \operatorname{Re}(f'(z)).
\end{aligned}$$

Applying Theorem 2.1 to Φ on $[0,1]$, we obtain

$$(1-t_1)\Phi'(t_1) = \Phi(1) - \Phi(t_1) - \frac{1}{2}\frac{\Phi'(1)-\Phi'(0)}{1-0}(1-t_1)^2 \tag{2.9}$$

for some $t_1 \in (0,1)$. Thus

$$(1-t_1)|b-a|^2 \operatorname{Re}(f'(z_1)) = \Phi(1) - \Phi(t_1) - \frac{1}{2}[\Phi'(1)-\Phi'(0)](1-t_1)^2, \tag{2.10}$$

where $z_1 = a + t_1(b-a)$. Further, since $z_1 = a+t_1(b-a)$ and $t_1 \in [0,1]$, we have $(1-t_1)|b-a|^2 =< b-a, b-z_1 >$. Hence the equation (2.10) reduces to

$$\operatorname{Re}(f'(z_1)) = \frac{\Phi(1)-\Phi(t_1)}{(1-t_1)|b-a|^2} - \frac{1}{2}\frac{\Phi'(1)-\Phi'(0)}{|b-a|^2}(1-t_1). \tag{2.11}$$

Using (2.7) and the fact that $z_1 = a+t_1(b-a)$ in the above equation, we obtain

$$\operatorname{Re}(f'(z_1)) = \frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{1}{2}\frac{\operatorname{Re}(f'(b)-f'(a))}{b-a}(b-z_1). \tag{2.12}$$

Letting $g = -if$, we have

$$\operatorname{Re}(g'(z)) = \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y} = \operatorname{Im}(f'(z)). \tag{2.13}$$

Now, applying the first part to g, we obtain

$$\operatorname{Re}(g'(z_2)) = \frac{\langle b-a, g(b)-g(z_2)\rangle}{\langle b-a, b-z_2\rangle} - \frac{1}{2}\frac{\operatorname{Re}(g'(b)-g'(a))}{b-a}(b-z_2) \tag{2.14}$$

for some $z_2 \in (a,b)$; i.e. $z_2 = a + t_2(b-a)$ and $t_2 \in [0,1]$. By using (2.12), the above equation yields

$$\mathrm{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle} - \frac{1}{2}\frac{\mathrm{Im}(f'(b)-f'(a))}{b-a}(b-z_2). \tag{2.15}$$

The proof is complete. □

It is easy to see that if $f'(a) = f'(b)$, then this result reduces to the following complex version of Myers' theorem.

Corollary 2.4. *Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let a and b be two distinct points in D_f, and $f'(a) = f'(b)$. Then there exist $z_1, z_2 \in (a,b)$ such that*

$$\mathrm{Re}(f'(z_1)) = \frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} \tag{2.16}$$

and

$$\mathrm{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle}. \tag{2.17}$$

Returning to the example $f(z) = e^z - z$, z_1 and z_2 predicted by Corollary 2.4 have values $z_1 \approx 1.78659i \in (0, 2\pi i)$ and $z_2 \approx 3.94888i \in (0, 2\pi i)$.

The third goal of this paper is to prove the equivalence of Rolle's and mean value theorems in the complex plane, given by Evard and Jafari [2].

Theorem 2.5 (Complex Rolle's Theorem)**.** *Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let $a, b \in D_f$ be such that $f(a) = f(b) = 0$ and $a \neq b$. Then there exist $z_1, z_2 \in (a,b)$ such that* $\mathrm{Re}(f'(z_1)) = 0$ *and* $\mathrm{Im}(f'(z_2)) = 0$.

Theorem 2.6 (Complex Mean Value Theorem)**.** *Let f be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let a and b be two distinct points in D_f. Then there exist $z_1, z_2 \in (a,b)$ such that* $\mathrm{Re}(f'(z_1)) = \mathrm{Re}\left(\frac{f(b)-f(a)}{b-a}\right)$ *and* $\mathrm{Im}(f'(z_2)) = \mathrm{Im}\left(\frac{f(b)-f(a)}{b-a}\right)$.

Proof of the equivalence. It is clear that Theorem 2.5 must hold if Theorem 2.6 does. To show the converse, assume that f satisfy the conditions of Theorem 2.6. Then

$$\begin{aligned} g(z) &:= \frac{1}{a-b}\begin{vmatrix} f(z) & f(a) & f(b) \\ z & a & b \\ 1 & 1 & 1 \end{vmatrix} \\ &= f(z) - f(a)\frac{z-b}{a-b} + f(b)\frac{z-a}{a-b} \end{aligned} \tag{2.18}$$

is also a holomorphic function for every $z \in D_f$. It is easy to see that the function g satisfies the condition $g(a) = g(b) = 0$. Hence, by Theorem 2.5, there exist $z_1, z_2 \in (a,b)$ such that $\mathrm{Re}(g'(z_1)) = 0$ and $\mathrm{Im}(g'(z_2)) = 0$. Thus, by (2.18), we obtain

$$g'(z) = f'(z) - \frac{f(b)-f(a)}{b-a} \tag{2.19}$$

for every $z \in D_f$. Hence,

$$0 = \mathrm{Re}(g'(z_1)) = \mathrm{Re}(f'(z_1)) - \mathrm{Re}\left(\frac{f(b)-f(a)}{b-a}\right), \tag{2.20}$$

$$0 = \mathrm{Im}(g'(z_2)) = \mathrm{Im}(f'(z_2)) - \mathrm{Im}\Big(\frac{f(b) - f(a)}{b - a}\Big) \tag{2.21}$$

which proves that Theorem 2.5 implies Theorem 2.6. Therefore, Theorems 2.5 and 2.6 are equivalent.

There are many ways to generalize the results of this paper due to the papers [4, 7, 10, 13, 17] by using the same method could be used here. We skip further details in this regard.

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