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EXISTENCE OF SOLUTIONS FOR THE P-LAPLACIAN INVOLVING A RADON MEASURE

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Abstract. In this article we study the existence of solutions to eigenvalue problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \lambda|u|^{p-2}u\mu = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

where Ω is a bounded domain in $\mathbb{R}^N$ and μ is a nonnegative Radon measure.

1. Introduction

In this article study the existence of weak solutions of the quasilinear elliptic problem

$$\begin{gathered} -\Delta_p u - \lambda|u|^{p-2}u\mu = f(x), \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$, and μ is a nonnegative bounded measure on Ω.

Singular nonlinear problems were studied in [8, 9, 12, 17, 21, 28, 31, 32, 33]. Some recent papers [2, 3, 6, 10, 11, 16, 18] studied functional

$$\frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{|x|^p}dx - \int_\Omega f(x)u(x)dx,$$

where f belongs to $L^{p'}(\Omega)$ and λ is a real positive number sufficiently small. This functional is coercive, and one can expect that there exists a global minimum. Since the Nemitski operator $u(x) \mapsto \frac{u(x)}{|x|}$ from $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$ is continuous but not compact, it is not clear if we can obtain directly the weak lower semicontinuity of the functional on $W_0^{1,p}(\Omega)$ by using De Giorgi Theorem [13, 19], so that it seems that we cannot apply the direct methods of the calculus of variations. In [3], using a critical point technique based on the coercivity and the homogeneity of the functional, it is shown the existence of a global minimum (for λ belonging to the set in which the functional is coercive) without using the direct methods of the calculus of variations. Reference [6] treats more general problems with an interesting nonvariational method, which does not require homogeneity, but only

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coercivity of the quadratic form associated to the equation. In any case, both papers leave open the question of whether the functional is weakly lower semicontinuous. In [25], the author proved that the functionals

$$\mathcal{H}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{|x|^p} dx$$

and

$$\mathcal{S}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\Big(\int_\Omega |u|^{p^\star} dx\Big)^{p/p^*}$$

are weakly lower semicontinuous in $W_0^{1,p}(\Omega)$, provided λ belongs to the set of $\mathbb{R}$ in which the functionals are coercive. Note that both functionals have a nonlinear term which is continuous but not compact on $W_0^{1,p}(\Omega)$. The author showed the following result.

Theorem 1.1. *For all $\lambda \in [0, 1/C]$ and all $f \in L^{p'}(\Omega)$, $1 < p < N$, the problem*

$$-\Delta_p u = \lambda \frac{|u|^{p-2}u}{|x|^p} + f(x), \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

has a weak solution $u \in W_0^{1,p}(\Omega)$, where $C = (\frac{p}{N-p})^p$ is the best constant satisfying

$$\int_\Omega \frac{|u|^p}{|x|^p} dx \le C \int_\Omega |\nabla u|^p dx.$$

For $p = 2$, Dupaigne [14] showed that the problem

$$-\Delta u - \frac{c}{|x|^2} u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

has a unique solution for all $0 < c < (p-2)^2/4$ and $f \in H^{-1}(\Omega)$. Moreover Peral [27] showed that the problem

$$-\Delta_p u - \frac{\lambda}{|x|^p}|u|^{p-2}u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

has at least one solution in $W_0^{1,p}(\Omega)$ for all $0 < \lambda < \frac{1}{C}$ and $f \in W^{-1,p'}(\Omega)$. For the proof of this result, the author used the convergence Theorem by Boccardo and Murat [7].

Remark 1.2. When $p > 2$, the uniqueness is in general not true, see [15]. However, the uniqueness in the case $1 < p < 2$ seems to be an open problem.

In this paper we assume that the measure μ is a nonnegative Radon measure satisfying the following assumptions.

(H0) For each Borel set $A \subset \Omega$, $\mu(A) = 0$ implies $|A| = 0$, where $|\cdot|$ denotes the Lebesgue measure.

(H1) There exists a constant $C > 0$ such that

$$\int_\Omega |u|^p d\mu \le C \int_\Omega |\nabla u|^p dx, \quad \forall u \in C_0^\infty(\Omega).$$

(H2) There exists $(\mu_n)_n \subset \mathcal{M}(\Omega)$ such that for each integer n, the embedding $W_0^{1,p}(\Omega, dx) \hookrightarrow L^p(\Omega, \mu_n)$ is compact, where $\mathcal{M}(\Omega)$ is the set of bounded Radon measures.

(H3) $\mu_n \nearrow \mu$ in $\mathcal{M}(\Omega)$; i.e., $\int_\Omega \varphi d\mu_n \to \int_\Omega \varphi d\mu$, for all $\varphi \in C_0^\infty(\Omega)$.

Remark 1.3. When $d\mu(x) = (1/|x|^p)dx$, (H1) is the classical Hardy inequality for $p > 1$, where the constant $C = (\frac{p}{N-p})^p$ is optimal.

Remark 1.4. Let $d\mu(x) = \frac{1}{(\delta(x))^p}dx$, where $\delta(x)$ is the distance function to the boundary, the following inequality holds true (see [20, 26]).

$$\int_\Omega \frac{|u|^p}{(\delta(x))^p}dx \le C_{n,p}(\Omega)\int_\Omega |\nabla u|^p dx, \quad \forall u \in C_0^\infty(\Omega).$$

Moreover, we will show that $d\mu(x) = \frac{1}{|x|^p}dx$ and $d\mu(x) = \frac{1}{(\delta(x))^p}dx$ are special cases of measures satisfying (H2) and (H3).

Theorem 1.5. *The measure* $d\mu(x) = \frac{1}{|x|^p}dx$ *and* $d\mu(x) = \frac{1}{(\delta(x))^p}dx$ *satisfy conditions* (H2) *and* (H3).

We define the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \lambda |u|^{p-2}u\mu_n = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Let $f \in L^{p'}(\Omega)$. We shall say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.2) (resp. (1.1)) if u satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \lambda\int_\Omega |u|^{p-2}u\varphi d\mu_n = \int_\Omega f\varphi dx, \tag{1.3}$$

respectively,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \lambda\int_\Omega |u|^{p-2}u\varphi d\mu = \int_\Omega f\varphi dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{1.4}$$

Notice that assumption (H3) ensures that the integral $\int_\Omega |u|^{p-2}u\varphi d\mu$ makes sense whenever u and φ are in $W_0^{1,p}(\Omega)$. We prove the following results.

Theorem 1.6. *Let* $f \in L^{p'}(\Omega)$, $1 < p < N$ *and* C *satisfying (H1). Then for all* $0 < \lambda < \frac{1}{C}$, *the problem* (1.1) *has at least a weak solution* $u \in W_0^{1,p}(\Omega)$.

Theorem 1.7. *Consider the Dirichlet problem*

$$\begin{gathered} -\Delta_p u - \lambda |u|^{p-2}u\mu = |u|^{\alpha-2}u \quad \text{in } W_0^{1,p}(\Omega), \\ u = 0 \quad \text{on } \partial(\Omega). \end{gathered} \tag{1.5}$$

For every $0 < \lambda < 1/C$ *and* $p < \alpha < p^\star = Np/(N-p)$, *there exists a nontrivial solution* $u \in W_0^{1,p}(\Omega)$.

Note that problem (1.5) has been studied by Peral [27] when $d\mu(x) = \frac{dx}{|x|^p}$. Next, we prove an auxiliary result.

Theorem 1.8. *For every* $n \in \mathbb{N}$, *the problem*

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u\mu_n \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.6}$$

has a sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$*, such that* $\lim_{k\to\infty}\lambda_k = +\infty$*. Moreover, the first eigenvalue* $\lambda_1(n)$ *is simple, isolated and is defined by*

$$\lambda_1(n) = \inf\Big\{\|\nabla u\|^p_{L^p} : u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |u|^p d\mu_n = 1\Big\}. \tag{1.7}$$

Notation. for $p > 1$, we denote by p' the real number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. As usual $W^{1,p}(\Omega)$ is the Sobolev space equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \big(\|u\|^p_{L^p} + \|\nabla u\|^p_{L^p}\big)^{1/p};$$

$W_0^{1,p}(\Omega)$ is the Sobolev space equipped with the norm

$$\|u\| = \|u\|_{W_0^{1,p}(\Omega)} = \big(\|\nabla u\|^p_{L^p}\big)^{1/p}.$$

For a positive Radon measure, we set

$$L^p(\Omega,\mu) = \{u : u \text{ is measurable and } \int_\Omega |u|^p d\mu < \infty\}.$$

When $d\mu = dx$, we set $L^p = L^p(\Omega, dx)$.

Proof of Theorem 1.5. We start by proving that $d\mu = \frac{1}{|x|^p}dx$ and $d\mu = \frac{1}{(\delta(x))^p}dx$ satisfy conditions (H2) and (H3). For each $n \in \mathbb{N}^\star$, we define $w_n(x) = \min(n, |x|^{-p})$ and $d\mu_n(x) = w_n(x)dx$.

Case $d\mu(x) = \frac{1}{|x|^p}dx$. Since $d\mu_n(x) \le n dx$, (H2) is obvious. To prove (H3), let $f \in C_0^\infty(\Omega)$. Using the fact that $p < N$, we obtain

$$\int_\Omega \frac{f}{|x|^p}dx \le \|f\|_\infty \int_\Omega \frac{1}{|x|^p}dx < +\infty.$$

On the other hand, since $(fw_n)_n$ converges to $f.\frac{1}{|x|^p}$, by Dominated Convergence Theorem we obtain,

$$\lim_{n\to\infty}\int_\Omega f w_n dx = \int_\Omega \frac{f}{|x|^p}dx.$$

Case: $d\mu(x) = \frac{1}{(\delta(x))^p}dx$. For $n \in \mathbb{N}^\star$, we define $w_n(x) = \min(n, (\delta(x))^{-p})$ and $d\mu_n(x) = w_n(x)dx$. As in the example from above, (H2) is obvious. Now using the fact that $|fw_n| \le \frac{|f|}{(\delta(x))^p}$ and

$$\int_\Omega \frac{f}{(\delta(x))^p}dx \le C\int_\Omega |\nabla f|^p dx, \quad \forall f \in C_0^\infty(\Omega),$$

we obtain (H3). □

2. Proof of Theorem 1.8

The proof is rather straightforward adaptation of [20, Theorem 3.2] with $d\mu_n = v(x)dx$, where the weight v is in L^r with $r = r(p,N)$ satisfying the following conditions

$$r \begin{cases} > \frac{N}{p} & \text{if } 1 < p < N,\\ = 1 & \text{if } p > N,\\ > p & \text{if } p = N.\end{cases}$$

Let $n \in \mathbb{N}$ fixed and define $G : W_0^{1,p}(\Omega) \to \mathbb{R}$ and $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$G(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx, \quad F(u) = \frac{1}{p}\int_\Omega |u|^p d\mu_n.$$

In the sequel we consider the functional

$$\begin{aligned}\phi : & W_0^{1,p}(\Omega) \to \mathbb{R} \\ u & \mapsto (G(u))^2 - F(u).\end{aligned}$$

Proposition 2.1. *The functionals G and F are of class C^1 on $W_0^{1,p}(\Omega)$. Moreover*

$$\langle DG(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx,$$

and

$$\langle DF(u), v\rangle = \int_\Omega |u|^{p-2}uv d\mu_n, \quad \forall v \in W_0^{1,p}(\Omega).$$

Proof. We only consider F, the proof for G is similar. Let u and $\varphi \in W_0^{1,p}(\Omega)$.

$$\begin{aligned}\lim_{t\to 0^+} \frac{F(u+t\varphi) - F(u)}{t} &= \frac{1}{p}\frac{d}{dt}F(u+t\varphi)|_{t=0} \\ &= \frac{1}{p}\frac{d}{dt}\int_\Omega |u+t\varphi|^p|_{t=0} d\mu_n \\ &= \frac{1}{p}\int_\Omega \frac{\partial}{\partial t}|u+t\varphi|^p|_{t=0} d\mu_n \\ &= \int_\Omega |u|^{p-2}u\varphi d\mu_n = \langle DF(u), \varphi\rangle.\end{aligned}$$

The differentiation under the integral is allowed since, if $|t| < 1$ then

$$\begin{aligned}\| u+t\varphi|^{p-2}(u+t\varphi)\varphi| &\le (|u| + |t||\varphi|)^{p-1}|\varphi| \\ &\le (|u| + |\varphi|)^{p-1}|\varphi| \in L^1(\Omega, \mu_n).\end{aligned}$$

Next, we show that $DF(u)$ is continuous. Indeed, by Hölder inequality and using hypotheses (H1)–(H3), we obtain

$$\begin{aligned}|\langle DF(u), \varphi\rangle| = | & \int_\Omega |u|^{p-2}u\varphi d\mu_n| \\ & \le \int_\Omega |u|^{p-1}|\varphi| d\mu_n \le \|u\|^{p-1}_{L^p(\Omega,\mu_n)}\|\varphi\|_{L^p(\Omega,\mu_n)} \\ & \le C\|u\|^{p-1}\|\varphi\|.\end{aligned}$$

□

Lemma 2.2. *The eigenvalues and eigenfunctions associated to the problem* (1.6) *are entirely determined by the nontrivial critical values of ϕ.*

Proof. Let $u \not\equiv 0$ be a critical point of ϕ associated with a critical value c, which means that $\phi(u) = c$ and $\phi'(u) = 0$. Hence

$$2G'(u)G(u) = F'(u).$$

With the condition $G'(u) \neq 0$ we obtain $G(u) = \frac{1}{2\lambda} = \lambda F(u)$ thus

$$F(u) = \frac{1}{2\lambda^2}$$

so $c = -G^2(u)$. Therefore,

$$\langle G'(u), v\rangle = \frac{1}{2\sqrt{-c}}\langle F'(u), v\rangle \quad \text{for all } v \in C_c^\infty(\Omega),$$

$$\langle \phi'(u), v\rangle = \frac{1}{2\sqrt{-c}}\langle F'(u), v\rangle \quad \text{for all } v \in C_c^\infty(\Omega).$$

Thus, we deduce that $\lambda = 1/(2\sqrt{-c})$ is a positive eigenvalue of (1.6) and u is its associated eigenfunction. Conversely, let $(u \not\equiv 0, \lambda)$ be a solution of (1.6). Then, for every $\beta \in \mathbb{R}^\star$, βu is also an eigenfunction associated to λ. In particular for $\beta = 1/(2\lambda G(u))^{1/p}$, the function $v = (2\lambda G(u))^{-1/p}u$ is an eigenfunction associated to $\lambda = 1/(2\sqrt{-c})$, which proves that v is a critical point associated to the critical value $c = -1/(4\lambda^2)$. □

Next, we recall the Genus function defined as follows $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$, where $\Sigma = \{A \subset W_0^{1,p}(\Omega) : A \text{ is closed }, A = -A\}$ by

$$\gamma(A) = \min\{i \in \mathbb{N} : \exists\, \varphi \in C(A, \mathbb{R}^i\backslash\{0\}),\ \varphi(x) = -\varphi(-x)\}.$$

Let us now consider the sequence

$$c_k = \inf_{K\in A_k} \sup_{v\in K} \phi(v), \tag{2.1}$$

where for $k \geq 1$, and

$$A_k = \{K \subset W_0^{1,p}(\Omega) : K \text{ is compact symmetric and } \gamma(K) \geq k\}.$$

Proposition 2.3. *The values c_k defined by* (2.1) *are the critical values of ϕ. Moreover $c_k < 0$ for $k \geq 1$ and $\lim_{k\to\infty} c_k = 0$.*

Proof. The proof is based on the fundamental theorem of multiplicity and the approximation of Sobolev imbedding by operators of finite rank. We first show that for all $k \geq 1$, c_k is a critical value of ϕ and $c_k < 0$. Since ϕ is even and is C^1 on $W_0^{1,p}(\Omega)$, then the result follows from the fundamental theorem of multiplicity if ϕ satisfies the following conditions:

(1) ϕ is bounded below.
(2) ϕ verify the Palais-Smale condition (P-S).
(3) For all $k \geq 1$, there exists a compact symmetric subset K such that $\gamma(K) = k$ and $\sup_{v\in K} \phi(v) < 0$.

Let us verify assertion (1). Indeed, condition (H1) implies that

$$\phi(u) \geq \frac{1}{p^2}\|u\|^p(\|u\|^p - Cp), \quad \forall u \in W_0^{1,p}(\Omega),$$

which proves that ϕ is bounded below and $\phi(u) \to +\infty$ as $\|u\| \to +\infty$.

Assertion (2). We show that ϕ verify the Palais-Smale condition. Let $(u_k)_k$ be a sequence in $W_0^{1,p}(\Omega)$ such that $(\phi(u_k))_k$ is bounded and $(\phi'(u_k))_k \to 0$ in $(W_0^{1,p}(\Omega))'$. Since ϕ is coercive then $(u_k)_k$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists a subsequence still denoted by $(u_k)_k$ such that $(\nabla u_k)_k$ converges to ∇u weakly in L^p, and $(u_k)_k$ converges to strongly in L^p. By (H3), we obtain that $(u_k)_k$ converges strongly in $L^p(\Omega, \mu_n)$. Suppose that $(\|u_k\|)_k$ converges to some constant $\alpha \geq 0$. We distinguish two cases.

Case 1: $\alpha = 0$. Since $(u_k)_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\|u_k\| \to 0$, then $(u_k)_k \to 0$ in $W_0^{1,p}(\Omega)$. Consequently, the condition (P-S) is satisfied.

Case 2: $\alpha > 0$. For $k \geq 1$ we have

$$\phi'(u_k) = 2G(u_k)G'(u_k) - F'(u_k)$$

which yields

$$G'(u_k) = \frac{1}{2G(u_k)}(\phi'(u_k) + F'(u_k));$$

i.e.,

$$\frac{p}{2}\frac{(\phi'(u_k) + F'(u_k))}{\|\nabla u_k\|_{L^p}^p} = G'(u_k).$$

Since $u \mapsto |u|^{p-2}u$ is strongly continuous in $L^p(\Omega, \mu_n)$, $\|u_k\| \to \alpha > 0$ and $(\phi'(u_k))_k \to 0$, then the expression

$$V_k = \frac{p}{2}\frac{(\phi'(u_k) + F'(u_k))}{\|\nabla u_k\|_{L^p}^p}$$

converges strongly in $(W_0^{1,p}(\Omega))'$. However, G' is continuous, thus $u_k = (G')^{-1}V_k$ converge strongly in $W_0^{1,p}(\Omega)$, from where the (P-S) condition holds.

Next, we prove (3). Indeed, by (H0), there exists a family of balls $(B_i)_{1\leq i\leq k}$ in Ω such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $\mu_n(\Omega \cap B_i) \neq 0$. We define

$$v_i = \begin{cases} u_i(x)\left(\int_{B_i} |u_i|^p d\mu_n\right)^{-1/p} & \text{if } x \in B_i, \\ 0 & \text{if } x \in \Omega\backslash B_i. \end{cases}$$

Let X_k denote subspace of $W_0^{1,p}(\Omega)$ spanned by $\{v_1, v_2, \dots, v_k\}$. Since the v_i's are linearly independent, we have that $\dim X_k = k$. For each $v \in X_k$, $v = \sum_{i=1}^{i=k}\alpha_i v_i$, we obtain $F(v) = \sum_{i=1}^{i=k}|\alpha_i|^p$.

Thus $u \mapsto (F(u))^{1/p}$ defines a norm on X_k. Then there exists $c > 0$ such that

$$cF(u) \leq G(u) \leq \frac{1}{c}F(u) \quad \forall u \in X_k.$$

Let K be defined as

$$K = \{u \in W_0^{1,p}(\Omega) \text{ such that } \frac{c^2}{3} \leq F(u) \leq \frac{c^2}{2}\}.$$

It is clear that $K_1 = K \cap X_k \neq \emptyset$ and $\sup_{u\in K_1} \phi(u) < -c/12 < 0$. Since X_k is isomorphic to $\mathbb{R}^k$, one can identify K_1 to a crown K_1' of $\mathbb{R}^k$ such that $S^{k-1} \subset K_1' \subset \mathbb{R}^k\backslash\{0\}$ where S^{k-1} is the unit sphere of $\mathbb{R}^k$. Then $\gamma(K_1) = k$ and the result follows.

Finally, we shall prove that $\lim_{k\to+\infty} c_k = 0$. Consider $\{E_i\}$ sequence of linear subspaces in $W_0^{1,p}(\Omega)$, such that

- $E_i \subset E_{i+1}$,
- $\overline{\cup E_i} = W_0^{1,p}(\Omega)$,
- $\dim(E_i) = i$.

Define

$$\widetilde{c_k} = \inf_{K\in A_k} \sup_{v\in K\cap E_{i-1}^c} \phi(v)$$

where E_i^c is the linear topological complementary of E_i. Obviously $\widetilde{c_k} \leq c_k < 0$. So, it is sufficient to prove that

$$\lim_{k\to+\infty} \widetilde{c_k} = 0.$$

Assume, by contradiction, that there exists a constant $\alpha < 0$ such that $\widetilde{c_k} < \alpha < 0$ for all $k \in \mathbb{N}$, then for each $k \in \mathbb{N}$, there exists K_k such that $\widetilde{c_k} <$

$\sup_{u\in K_k\cap E_{i-1}^c}\phi(u) < \alpha$ and there exists $u_k \in K_k \cap E_{i-1}^c$ such that $\widetilde{c_k} < \phi(u_k) < \alpha$. In this way, ϕ is bounded, hence for some subsequence still denoted (u_k),

$$\begin{gathered} u_k \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \\ u_k \to u \quad \text{in } L^p(\Omega, \mu_n). \end{gathered}$$

Hence $\phi(u) < \alpha < 0$, which is a contradiction with the fact that $u \equiv 0$ because $u_k \in E_{i-1}^c$. $\square$

Remark 2.4. It is clear that the sequence $(\lambda_k)_k$ defined by the formula $\lambda_k = \frac{1}{2\sqrt{-c_k}} \to +\infty$ as $k \to +\infty$.

Remark 2.5. We consider $\lambda_k = \inf_{K\in\Gamma_k}\sup_{u\in K} G(u)$, where Γ_k is define by

$$\{K \subset W_0^{1,p}(\Omega)\backslash\{0\} : K \text{ is compact, symmetric } \gamma(K) \geq k, \|u\|_{L^p(\Omega,\mu_n)} = 1\}.$$

Particulary

$$\lambda_1(n) = \inf\big\{\|\nabla u\|_{L^p}^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_{L^p(\Omega,\mu_n)} = 1\big\}.$$

Moreover, using [23, Theorem 4.11], we obtain the following result.

Theorem 2.6. *If $u \in W_0^{1,p}(\Omega)$ is an eigenfunction of* (1.3)*, then u is continuous in Ω.*

In what follows we will use the so-called Picone's identity proved in [1]. We recall it here for completeness.

Theorem 2.7 (Picone's identity)**.** *Let $u > 0$, $v > 0$ be two continuous functions in Ω, differentiable a.e.. Denote*

$$\begin{gathered} L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2} - \nabla u \nabla v, \\ R(u,v) = |\nabla u|^p - |\nabla v|^{p-2}\nabla\Big(\frac{u^p}{v^{p-1}}\Big)\nabla v. \end{gathered}$$

Then

- (i) $L(u,v) = R(u,v)$,
- (ii) $L(u,v) \geq 0$ *a.e.,*
- (iii) $L(u,v) = 0$ *a.e. in Ω if and only if $u = kv$ for some $k \in \mathbb{R}$.*

We will show that the first eigenvalue $\lambda_1(n)$ of (1.6) defined by (1.7) is simple and isolated, and only eigenfunctions associated with $\lambda_1(n)$ do not change sign.

Proposition 2.8. *The first eigenvalue $\lambda_1(n)$ is simple.*

Proof. Let u, v be two eigenfunctions associated to $\lambda_1(n)$ and fixed $\epsilon > 0$. We can assume without restriction that u and v are positive in Ω. From Picone's identity we have

$$\begin{aligned} \int_\Omega L(u, v+\epsilon)dx &= \int_\Omega R(u, v+\epsilon)dx \\ &= \lambda_1(n)\int_\Omega u^p d\mu_n - \int_\Omega |\nabla v|^{p-2}\nabla\big(\frac{u^p}{(v+\epsilon)^{p-1}}\big)\nabla v dx. \end{aligned}$$

The functional $u^p/(v+\epsilon)^{p-1}$ belongs to $W_0^{1,p}(\Omega)$ and then it is admissible for the weak formulation of $-\Delta_p u = \lambda_1(n)|u|^{p-2}u\mu_n$. It follows that

$$0 \le \int_\Omega L(u, v+\epsilon)dx = \lambda_1(n)\int_\Omega u^p(1-\frac{v^{p-1}}{(v+\epsilon)^{p-1}})d\mu_n.$$

Letting $\epsilon \to 0$, we obtain $L(u,v)=0$, a.e. in Ω, and therefore using (iii), we obtain $u = kv$. □

Proposition 2.9. *Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative weak solution of* (1.6)*, then either $u \equiv 0$ or $u(x) > 0$ for all $x \in \Omega$.*

The proof of the above proposition is a direct consequence of Harnack's inequality, see [35, 36].

Theorem 2.10. *Let $(u,\lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R}_+$ be an eigensolution of* (1.1)*. Then $u \in L^\infty(\Omega,\mu_n)$.*

The proof of the above theorem is rather a straightforward adaptation of [22, Theorem 4.1] with $d\mu_n = dx$.

Theorem 2.11. *Let u be an eigenfunction of* (1.6) *associated to an eigenvalue $\lambda \neq \lambda_1(n)$ and $1 \le q < p$. We define*

$$I = \min\Big\{\int_\Omega |u|^q d\mu_n, u \in L^p(\Omega,\mu_n), \int_\Omega |u|^p d\mu_n = 1\Big\}.$$

Then

$$\min(\mu_n(\Omega^-),\mu_n(\Omega^+)) \ge ((C\lambda)^{-1/p}I)^{\frac{pq}{p-q}}, \tag{2.2}$$

where $\Omega^+ = \{x \in \Omega,\, u(x) > 0\}$ and $\Omega^- = \{x \in \Omega,\, u(x) < 0\}$.

Proof. Let u be an eigenfunction associated to λ, then

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega |u|^{p-2}uv d\mu_n, \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.3}$$

For $\lambda \neq \lambda_1(n)$, u changes sign i.e., $u^+ \neq 0$ and $u^- \neq 0$. Since $u^+ \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u^+|^p dx = \lambda \int_\Omega |u^+|^p d\mu_n.$$

For $1 \le q < p$, we have:

$$\begin{aligned}\int_\Omega |u^+|^q d\mu_n &\le \Big(\int_\Omega |u^+|^p d\mu_n\Big)^{q/p}(\mu_n(\Omega^+))^{1-\frac{q}{p}}\\ &\le C^{q/p}\mu_n(\Omega^+)^{1-\frac{q}{p}}\Big(\int_\Omega |\nabla u^+|^p dx\Big)^{q/p}\\ &\le (\lambda C)^{q/p}\mu_n(\Omega^+)^{1-\frac{q}{p}}\Big(\int_\Omega |u^+|^p d\mu_n\Big)^{q/p}\end{aligned}$$

and

$$\|u^+\|^q_{L^q(\Omega,\mu_n)} \le (\lambda C)^{q/p}\mu_n(\Omega^+)^{\frac{p-q}{pp}}\|u^+\|_{L^p(\Omega,\mu_n)}.$$

Finally

$$\mu_n(\Omega^+) \ge ((C\lambda)^{-1/p}I)^{\frac{pq}{p-q}}. \tag{2.4}$$

□

Now, we establish the isolation of the first eigenvalue.

Theorem 2.12. *Let $\lambda_1(n)$ be the first eigenvalue of the problem* (1.6)*. Then $\lambda_1(n)$ is isolated.*

Proof. Our approach is related to the method of [4, 5]. Let $\lambda > 0$ be an eigenvalue of (1.6) and let v be the corresponding eigenfunction. By (1.7), it follows that $\lambda_1(n) < \lambda$ and so $\lambda_1(n)$ is left-isolated. To prove that $\lambda_1(n)$ is right-isolated, we argue by contradiction. We suppose that there exists a sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$, such that $\lambda_k \neq \lambda_1(n)$ and $\lambda_k \to \lambda_1(n)$. Let $(u_k)_{k\in\mathbb{N}}$ be the corresponding sequence of eigenfunctions such that

$$\int_\Omega |\nabla u_k|^p dx = 1, \quad \forall\, k \in \mathbb{N}. \tag{2.5}$$

There exists a subsequence, denoted again by $(u_k)_k$ and a function $u \in W_0^{1;p}(\Omega)$ such that

$$\begin{gathered} u_k \rightharpoonup u \quad \text{on } W_0^{1,p}(\Omega) \\ u_k \to u \quad \text{on } L^p(\Omega, \mu_n). \end{gathered}$$

Our next aim is to show that u is the eigenfunction corresponding to $\lambda_1(n)$. First, since $-\Delta_p$ is a continuous and one-to-one operator from $W_0^{1,p}(\Omega)$ into $W_0^{-1,p'}(\Omega)$ and so is its inverse operator $(-\Delta_p)^{-1}$ defined from $W_0^{-1,p'}(\Omega)$ into $W_0^{1,p}(\Omega)$ (see [27]). Thus,

$$u_k = (-\Delta_p)^{-1}(\lambda_k |u_k|^{p-2} u\mu_n).$$

By Vitali's Theorem, we have

$$\lambda_k u_k^{p-2} u_k \to \lambda |u|^{p-2} u \quad \text{strongly in } L^{\frac{p}{p-1}}(\Omega, \mu_n) \hookrightarrow W^{-1,p'}(\Omega).$$

The continuity property of $(-\Delta_p)^{-1}$ implies that

$$u_k \to u \quad \text{strongly in } W_0^{1,p}(\Omega).$$

Hence, u is an eigenfunction of (1.6), corresponding to $\lambda_1(n)$. Using Vitali's Theorem, again, we have

$$|\nabla u_k|^{p-2}\nabla u_k \to |\nabla u|^{p-2}\nabla u \quad \text{strongly in } L^1.$$

$$\int_\Omega |u_k|^{p-2} u_k v d\mu_n \le \Big(\int_\Omega |u_k|^p d\mu_n\Big)^{\frac{p-1}{p}} . \Big(\int_\Omega |v|^p d\mu_n\Big)^{1/p} \le \|v\|.$$

It should appear a constant $\eta_\epsilon > 0$ for every $\epsilon > 0$ and $\Omega_\epsilon \subset \Omega$ such that

$$\mu_n(\Omega \backslash \Omega_\epsilon) \le \frac{\epsilon}{2} \quad \text{and} \quad u(x) \ge 2\eta_\epsilon \quad \text{for every } x \in \Omega_\epsilon. \tag{2.6}$$

Let us denote

$$\Omega_k^+ = \{x \in \Omega,\, u_k(x) > 0\}, \tag{2.7}$$

$$\Omega_k^- = \{x \in \Omega,\, u_k(x) < 0\}. \tag{2.8}$$

Moreover, by Egorov's Theorem, there exists $\Omega'_\epsilon \subset \Omega$ such that

$$\mu_n(\Omega \backslash \Omega'_\epsilon) \le \frac{\epsilon}{2}$$

and u_k converges uniformly to u. On the other hand, there exists $N_\epsilon > 0$ such that for every $k > N_\epsilon$, we have

$$\Omega_\epsilon \cap \Omega'_\epsilon \subset \Omega_k^+$$

and then

$$\mu_n(\Omega_k^+) \geq \mu_n(\Omega_\epsilon \cap \Omega'_\epsilon) \geq \mu_n(\Omega) - (\mu_n(\Omega\backslash\Omega'_\epsilon) + \mu_n(\Omega\backslash\Omega_\epsilon)) \geq \mu_n(\Omega) - \epsilon.$$

Hence, it follows that $\mu_n(\Omega_k^+)$ and $\mu_n(\Omega_k^-) \geq K$, where $K = ((C\lambda)^{-1/p} I)^{\frac{pq}{p-q}}$. If we choose $\epsilon = \frac{K}{2}$, we obtain

$$\mu_n(\Omega) = \mu_n(\Omega_k^-) + \mu_n(\Omega_k^+) \geq \mu_n(\Omega) - \epsilon + K = \mu_n(\Omega) + \epsilon > \mu_n(\Omega),$$

which is a contradiction. Therefore $\lambda_1(n)$ is isolated. □

3. Proof of Theorem 1.6

Lemma 3.1. *Let $\lambda_1(n)$ be the first eigenvalue associated to* (1.6). *Then, $\lambda_1(n) \geq \frac{1}{C}$ and $\lim_{n\to\infty}\lambda_1(n) = \frac{1}{C}$.*

Proof. Notice that

$$\lambda_1(n) = \inf_\Omega \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p d\mu_n} \geq \inf_\Omega \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p d\mu} \geq \frac{1}{C}.$$

Since $(\lambda_1(n))_n$ is a non increasing sequence, we have to prove that the limit can not be larger than $\frac{1}{C}$. Assume by contradiction that $\lim_{n\to\infty} \lambda_1(n) = \frac{1}{C} + \delta$, for some $\delta > 0$. Then we can choose $\phi \in C_0^\infty(\Omega)$ such that

$$\frac{\int_\Omega |\nabla\phi|^p dx}{\int_\Omega |\phi|^p d\mu} < \frac{1}{C} + \frac{\delta}{2}.$$

Which gives us

$$\lambda_1(n) \leq \frac{\int_\Omega |\nabla\phi|^p dx}{\int_\Omega |\phi|^p d\mu_n} \leq \frac{1}{C} + \frac{\delta}{2}$$

for n large enough. □

In the sequel for $\lambda > 0$ let us denote by $\mathfrak{L}_\lambda^{\mu_n}$ the operator defined on $W_0^{1,p}(\Omega)$ by

$$\mathfrak{L}_\lambda^{\mu_n} u = -\Delta_p u - \lambda |u|^{p-2} u \mu_n.$$

The first result in this section is an easy consequence of the Hardy's inequality.

Lemma 3.2. *If $0 < \lambda < \frac{1}{C}$, then $\mathfrak{L}_\lambda^{\mu_n}$ is a positive operator.*

Proof. From assumption (H1) we have

$$\langle \mathfrak{L}_\lambda^{\mu_n} u, u\rangle \geq (1 - \lambda C)\|u\|^p \geq 0$$

whenever $0 < \lambda < 1/C$. □

Next we recall a formula from [34].

Lemma 3.3. *Let $a, b \in \mathbb{R}^N$ and $\langle ., .\rangle$ be the standard scalar product in $\mathbb{R}^N$. Then*

$$\langle |a|^{p-2}a - |b|^{p-2}b, (a-b)\rangle \geq \begin{cases} C_p|a-b|^p & \text{if } p \geq 2 \\ C_p \frac{|a-b|^2}{(|a|+|b|)^{2-p}} & \text{if } 1 < p < 2. \end{cases}$$

Lemma 3.4. *The operator $\mathfrak{L}_\lambda^{\mu_n} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is uniformly continuous on bounded sets.*

Proof. Assume $p > 2$ and consider $K \subset W_0^{1,p}(\Omega)$ be a bounded set; i.e., there exists $M > 0$ such that
$$\|u\| \leq M, \quad \forall u \in K.$$
Then, using Lemma 3.3 and Hölder inequality, for $u, v \in K$ and $\phi \in W_0^{1,p}(\Omega)$, we obtain
$$\begin{aligned}
&|\langle \mathfrak{L}_\lambda^{\mu_n} u - \mathfrak{L}_\lambda^{\mu_n} v, \phi\rangle| \\
&\leq \int_\Omega (|\nabla u|^{p-2} + |\nabla v|^{p-2})|\nabla u - \nabla v||\nabla \phi| dx + \lambda \int_\Omega (|u|^{p-2} + |v|^{p-2})|u - v||\phi| d\mu_n \\
&\leq 2c_p M^{p-2}\|\nabla u - \nabla v\|_{L^p} + 2\lambda c_p M^{p-2}\|u - v\|_{L^p(\Omega,\mu_n)} \\
&\leq 2c_p M^{p-2}(\min\{1, C\lambda\})\|u - v\|.
\end{aligned}$$
The same process is applied for $1 < p < 2$. □

Lemma 3.5. *The operator $\mathfrak{L}_\lambda^{\mu_n} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is pseudo-monotone.*

Proof. Let $(u_k)_{k\geq 1} \subset W_0^{1,p}(\Omega)$ such that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and
$$\limsup_{k\to\infty} < \mathfrak{L}_\lambda^{\mu_n} u_k, u_k - u > \leq 0.$$
We want to prove that
$$\liminf \langle \mathfrak{L}_\lambda^{\mu_n} u_k, u_k - v\rangle \geq \langle \mathfrak{L}_\lambda^{\mu_n} u, u - v\rangle \quad \text{for all } v \in W_0^{1,p}(\Omega).$$
Since $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, it follows that
$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u_k - u) dx \to 0, \quad \text{as } k \to +\infty. \tag{3.1}$$
We estimate
$$\int_\Omega |u_k|^{p-2} u_k (u_k - u) d\mu_n \leq \|u_k\|_{L^p(\Omega,\mu_n)}^{p-1}\|u_k - u\|_{L^p(\Omega,\mu_n)}.$$
Through $u_k \to u$ in $L^p(\Omega, \mu_n)$ and u_k is bounded in $L^p(\Omega, \mu_n)$ then
$$\int_\Omega |u_k|^{p-2} u_k (u_k - u) d\mu_n \to 0, \quad \text{as } k \to +\infty.$$
So
$$\limsup_{k\to\infty} \Big(\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k - u) dx + (-\lambda \int_\Omega |u_k|^{p-2} u_k (u_k - u) d\mu_n) \Big) \leq 0,$$
which yields
$$\limsup_{k\to\infty} \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k - u) dx \leq 0. \tag{3.2}$$
Combining (3.1), (3.2) and Lemma 3.3 we obtain that $(u_k)_k \to u$ in $W_0^{1,p}(\Omega)$. The proof is complete. □

Proposition 3.6. *For every $0 < \lambda < 1/C$, the operator $\mathfrak{L}_\lambda^{\mu_n}$ is coercive.*

Proof. Using (H1)–($\ddot{3}$), we have
$$\langle \mathfrak{L}_\lambda^{\mu_n} u, u\rangle = \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u|^p d\mu_n \geq (1 - \lambda C) \int_\Omega |\nabla u|^p dx,$$
which implies that $\mathfrak{L}_\lambda^{\mu_n}$ is coercive, whenever $0 < \lambda < \frac{1}{C}$. □

By Proposition 3.6 and Lemma 3.5 the operator $\mathfrak{L}_\lambda^{\mu_n} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is coercive, bounded from below and pseudo-monotone. Hence, by [23, Theorem 4.11], it is onto. Thus we have the following result.

Theorem 3.7. *For every $f \in L^{p'}$, there exists $u_n \in W_0^{1,p}(\Omega)$ which is a solution of* (1.2).

Lemma 3.8. *For each $n \in \mathbb{N}$, let u_n be a solution of the Dirichlet problem* (1.2). *Then the sequence $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Since

$$\int_\Omega |\nabla u_n|^p dx - \lambda \int_\Omega |u_n|^p d\mu_n = \int_\Omega f u_n dx,$$

and using (H1), we obtain

$$(1 - \lambda C)\|u_n\|^p \le \|f\|_{L^{p'}} \|u_n\|_{L^p},$$
$$\|u_n\| \le \Big(C_1 \frac{\|f\|_{L^{p'}}}{(1 - \lambda C)}\Big)^{\frac{1}{p-1}},$$

where C_1 is the positive constant of the continuous of Sobolev embedding satisfied. □

Lemma 3.9. *Let $(u_n)_n$ be the sequence as defined in Theorem 3.7. Then $(u_n)_n$ converges to a weak solution u of* (1.1).

Proof. By Lemma 3.8, since u_n is bounded in $W_0^{1,p}(\Omega)$, we have

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \\ u_n \rightharpoonup u \quad \text{in } L^p(\Omega, \mu), \\ u_n \to u \quad \text{in } L^p. \end{gathered} \tag{3.3}$$

Then

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx \to |\nabla u|^{p-2} \nabla u \nabla \varphi dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

Next, we show that

$$\int_\Omega |u_n|^{p-2} u_n \varphi d\mu_n \to \int_\Omega |u|^{p-2} u \varphi d\mu, \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Indeed, we have

$$\begin{aligned} &| \int_\Omega |u_n|^{p-2} u_n \varphi d\mu_n - \int_\Omega |u|^{p-2} u \varphi d\mu | \\ &= | \int_\Omega |u|^{p-2} u \varphi d(\mu - \mu_n) - \int_\Omega (|u_n|^{p-2} u_n - |u|^{p-2} u) \varphi d\mu_n | \\ &\le \|u\|_{L^p(\Omega, \mu - \mu_n)}^{p-1} \|\varphi\|_{L^p(\Omega, \mu - \mu_n)} + | \int_\Omega (|u_n|^{p-2} u_n - |u|^{p-2} u|) \varphi | d\mu_n |. \end{aligned}$$

So, by (H3) the first integral converges to 0, as $n \to \infty$; respectively by the weak convergence in (3.3), the second integral converges to 0, as $n \to \infty$. Therefore, u is a solution of our problem in the sense of distributions. Moreover by density argument and taking into account that $u \in W_0^{1,p}(\Omega)$, we conclude that u is solution in the sense of $W_0^{1,p}(\Omega)$. □

4. Proof of Theorem 1.7

Some recent papers [17, 21, 28, 30, 32, 29] considered a class of functionals with the minimax method. We will use again the variational approach to study the case of unbounded functionals, more precisely the existence of solution via the mountain-pass theorem. For instance the following result holds. For $0 < \lambda < \frac{1}{C}$, let

$$J(u) = \frac{1}{p}\langle \mathfrak{L}_\lambda^{\mu_n} u, u\rangle - \frac{1}{\alpha}\|u\|_{L^\alpha}^\alpha, \quad u \in W_0^{1,p}(\Omega).$$

To obtain a nontrivial critical point of the functional J, we apply the following version of the mountain-pass theorem from [24] with the usual Palais-Smale compactness condition. So the critical points of the functional J are a weak solutions for (1.5).

Theorem 4.1. *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfying Palais-Smale condition. Suppose that $J(0) = 0$ and for some $\sigma, \rho > 0$ and $e \in E$, with $\|e\| > \rho$, one has $\sigma \le \inf_{\|u\|=\rho} J(u)$ and $J(e) < 0$. Then J has a critical value $c \ge \sigma$ characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0,\, \gamma(1) = e\}.$$

The proof of the above theorem follows from the following lemma.

Lemma 4.2. *The functional J satisfies the Palais-Smale condition.*

Proof. Let $(u_k)_k \in W_0^{1,p}(\Omega)$ be a Palais-Smale sequence. Set

$$c = \lim_{k\to\infty} J(u_k), \quad J'(u_k) = \epsilon_k$$

such that $(\epsilon_k)_k \to 0$. Thus

$$|J'(u_k)w| \le \epsilon_k\|w\|, \text{for all } \ w \in W_0^{1,p}(\Omega).$$

For k large enough, we will have

$$\begin{aligned} c + 1 &\ge J(u_k) - \frac{1}{\alpha}\langle J'(u_k), u_k\rangle + \frac{1}{\alpha}\langle J'(u_k), u_k\rangle, \\ &\ge (\frac{1}{p} - \frac{1}{\alpha})(1 - \lambda C)\|u_k\|^p - \frac{1}{\alpha}\|u_k\|\epsilon_k, \\ &\ge (\frac{1}{p} - \frac{1}{\alpha})(1 - \lambda C)\|u_k\|^p - \frac{1}{\alpha}\|u_k\|. \end{aligned}$$

Hence, the sequence $(u_k)_k$ is bounded in $W_0^{1,p}(\Omega)$. By compactness argument we can assume that

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega),$$
$$u_k \to u \quad \text{in } L^\alpha(\Omega), \quad \text{for } p < \alpha < p^\star.$$

Using (H2), we obtain that $(u_k)_k$ converges to u in $L^p(\Omega, \mu_n)$. It follows that $|u_k|^{p-2}u_k \to |u|^{p-2}u$ in $L^{p'}(\Omega, \mu_n)$, hence in $W^{-1,p'}(\Omega)$. Let us denote by $V_k = |u_k|^{p-2}u_k\mu_n - |u_k|^{\alpha-2}u_k$ and $V = |u|^{p-2}u\mu_n - |u|^{\alpha-2}u$. Since $(-\Delta_p)^{-1}$ is continuous, we conclude that

$$u_k = (-\Delta_p)^{-1}(V_k) \text{ converges to } \ (-\Delta_p)^{-1}(V) = u.$$

Therefore, $|u_k|^{p-2}u_k$ converges to $|u|^{p-2}u$ in $W_0^{1,p}(\Omega)$. ∎

Lemma 4.3. *The functional J satisfies the conditions for the mountain-pass theorem.*

Proof. Let $\delta_1 = \alpha\delta$. First, we show that there exist positive constants ρ and α_1 such that

$$J(u) \geq \alpha, \quad \text{if } \|u\| = \rho,$$

and there exists $\varphi \in W_0^{1,p}(\Omega)$ such that $J(t\varphi) \to -\infty$, as $t \to \infty$. Indeed, for $u \in W_0^{1,p}(\Omega)$, we have

$$J(u) = \frac{1}{p}\langle \mathfrak{L}_\lambda^{\mu_n} u, u\rangle - \frac{1}{\alpha}\|u\|_{L^\alpha}^\alpha \geq \frac{1}{p}(1-\lambda C)\|u\|^p - \frac{\delta_1}{\alpha}\|u\|^\alpha.$$

Since $\lambda < 1/C$ and $p < \alpha$, we can set

$$\rho = \Big(\frac{(1-\lambda C)\alpha S^{\alpha/p}}{|\Omega|^{1-\frac{\alpha}{p^\star}}}\Big)^{1/\alpha - p)}, \quad \alpha_1 = \Big(\frac{(1-\lambda C)^\alpha}{(\delta_1)^p}\Big)^{1/(\alpha-p)}\big(\frac{1}{p} - \frac{1}{\alpha}\big),$$

such that $J(u) \geq \alpha_1$ if $\|u\| = \rho$.

Let us prove the second assertion. Let $t > 0$ large enough, and choose $\varphi \in W_0^{1,p}(\Omega)\backslash\{0\}$ satisfying

$$J(t\varphi) = \frac{1}{p}t^p\langle \mathfrak{L}_\lambda^{\mu_n}\varphi, \varphi\rangle - \frac{1}{\alpha}t^\alpha\|\varphi\|_{L^\alpha}^\alpha \to -\infty \quad \text{as } t \to +\infty.$$

Thus, we have $J(t\varphi) < 0$, for sufficiently large t.

So, we can conclude that J has a critical value $c \geq \alpha_1$, which can be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)),\ \gamma(0) = 0,\ \gamma(1) = e\}.$$

Next, we shall prove the positivity of the solution. Multiply the equation $-\Delta_p u - \lambda|u|^{p-2}u\mu_n = |u|^{\alpha-2}u$ by u^- and integrate over Ω, we find $\|u^-\| = 0$ and so u is a positive solution of $(P_{\alpha,\lambda}^{\mu_n})$ the proof is complete. □

For the proof of Theorem 1.7, we need the following results.

Lemma 4.4. *Let $(u_n)_n$ be a sequence of weak solutions of* (1.5) *with μ_n instead of μ. Then, $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. As u_n is a weak solution of (1.5) with μ_n instead of μ, then u_n is a critical point of the functional J. Since J satisfies the Palais-Smale condition, then $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. □

Lemma 4.5. *Let $(u_n)_n$ be a sequence of weak solutions of the problem* (1.5) *with μ_n instead of μ. Then $(u_n)_n$ converges to a weak solutions u of* (1.5)*.*

Proof. By Lemma 3.8, since $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$, it follows that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \\ u_n \rightharpoonup u \quad \text{in } L^p(\Omega,\mu), \\ u_n \to u \quad \text{in } L^\alpha \quad p < \alpha < p^\star. \end{gathered} \tag{4.1}$$

Hence

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi dx \to |\nabla u|^{p-2}\nabla u\nabla\varphi dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

By the compactness of Sobolev embedding, we obtain

$$\int_\Omega |u_n|^{\alpha-2}u_n\varphi dx \to \int_\Omega |u|^{\alpha-2}u\varphi dx, \quad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

Next, we show that

$$\int_\Omega |u_n|^{p-2}u_n\varphi d\mu_n \to \int_\Omega |u|^{p-2}u\varphi d\mu, \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Indeed, we have

$$\begin{aligned} &\Big|\int_\Omega |u_n|^{p-2}u_n\varphi d\mu_n - \int_\Omega |u|^{p-2}u\varphi d\mu\Big| \\ &= \Big|\int_\Omega |u|^{p-2}u\varphi d(\mu-\mu_n) - \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)\varphi d\mu_n\Big| \\ &\le \|u\|^{p-1}_{L^p(\Omega,\mu-\mu_n)}\|\varphi\|_{L^p(\Omega,\mu-\mu_n)} + \Big|\int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u|)\varphi|d\mu_n\Big|. \end{aligned}$$

So, using (H3) the first integral converges to 0, as $n \to \infty$ respectively by the weak convergence in (4.1), the second integral converges to 0, as $n \to \infty$. Therefore, u is a solution of our problem in the sense of distribution. Moreover by density argument and taking into account that $u \in W_0^{1,p}(\Omega)$, we conclude that u is solution in the sense of $W_0^{1,p}(\Omega)$. □

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