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NORMAL EXTENSIONS OF A SINGULAR MULTIPOINT DIFFERENTIAL OPERATOR OF FIRST ORDER

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Abstract. In this work, we describe all normal extensions of the minimal operator generated by linear singular multipoint formally normal differential expression $l = (l_1, l_2, l_3)$, $l_k = \frac{d}{dt} + A_k$ with selfadjoint operator coefficients A_k in a Hilbert space. This is done as a direct sum of Hilbert spaces of vector-functions

$$L_2(H, (-\infty, a_1)) \oplus L_2(H, (a_2, b_2)) \oplus L_2(H, (a_3, +\infty))$$

where $-\infty < a_1 < a_2 < b_2 < a_3 < +\infty$. Also, we study the structure of the spectrum of these extensions.

1. Introduction

Many problems arising in modeling processes in multi-particle quantum mechanics, in quantum field theory, in multipoint boundary value problems for differential equations, and in the physics of rigid bodies use normal extensions of formaly normal differential operators as a direct sum of Hilbert spaces [1, 19, 20]. The general theory of these normal extensions of formally normal operators in Hilbert spaces has been investigated by many mathematicians; see for example [2, 3, 4, 12, 13, 14]. Applications of this theory to two-point differential operators in Hilbert space of functions can be found in [7, 8, 9, 10, 11, 15, 16, 17].

It is clear that the minimal operators $L_0(1, 0, 0) = L_{10} \oplus 0 \oplus 0$ and $L_0(0, 0, 1) = 0 \oplus 0 \oplus L_{30}$ generated by differential expressions for the forms $(\frac{d}{dt} + A_1, 0, 0)$ and $(0, 0, \frac{d}{dt} + A_3)$ in Hilbert spaces of vector-functions $L_2(1, 0, 0) = L_2(H, (-\infty, a_1)) \oplus 0 \oplus 0$, $L_2(0, 0, 1) = 0 \oplus 0 \oplus L_2(H, (a_3, \infty))$ respectively, where $A_1 = A_1^* \le 0$, $A_3 = A_3^* \ge 0$, $-\infty < a < b < \infty$, are maximal formally normal. Consequently they do not have normal extensions. But the minimal operator $L_0(0, 1, 0) = 0 \oplus L_{20} \oplus 0$ generated by differential expression of the form $(0, \frac{d}{dt} + A_2, 0)$ in the Hilbert spaces of vector-functions $L_2(0, 1, 0) = 0 \oplus L_2(H, (a_2, b_2)) \oplus 0$ is formally normal and not maximal.

Unfortunately, multipoint situations may occur in different tables in the following sense. Let B_1 , B_2 and B_3 be minimal operators generated by the linear differential expression $\frac{d}{dt}$ in the Hilbert space of functions $L_2(-\infty, a_1)$, $L_2(a_2, b_2)$

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and $L_2(a_3,+\infty)$ where $-\infty < a_1 < a_2 < b_2 < a_3 < \infty$ respectively. Consequently, B_1 and B_3 are maximal formally normal operators, but are not normal extensions. However, the direct sum $B_1 \oplus B_2 \oplus B_3$ of operators B_1, B_2 *and* B_3 in a direct sum $L_2(-\infty,a_1) \oplus L_2(a_2,b_2) \oplus L_2(a_3,+\infty)$ has a normal extension. For example, in case $H = \mathbb{C}$ it can be easily shown that an extension of the minimal operator $B_1 \oplus B_2 \oplus B_3$ with the boundary conditions

$$\begin{gathered} u_3(a_3) = e^{i\varphi} u_1(a_1), \quad \varphi \in [0,2\pi), \\ u_2(\ b_2) = e^{i\psi} u_2(a_2\ ), \psi \in [0,2\pi), \\ u = (u_1,u_2,u_3), \quad u_1 \in D(B_1^*), \quad u_2 \in D(B_2^*), u_3 \in D(B_3^*) \end{gathered}$$

is normal in $L_2(-\infty,a_1) \oplus L_2(a_2,b_2) \oplus L_2(a_3,+\infty)$.

In the general case of H being the direct sum $L_0(1,1,1) = L_{10} \oplus L_{20} \oplus L_{30}$ of operators L_{10}, L_{20} and L_{30} is formally normal, is not maximal. Moreover it has normal extensions in the direct sum

$$L_2(1,1,1) = L_2(H,(-\infty,a_1)) \ \oplus L_2(H,(a_2,b_2)) \oplus L_2(H,(a_3,\infty)).$$

In singular cases, however, there has been no investigation so far. But the physical and technical process for many of the problems resulting from the examination of the solution is of great importance for the singular cases.

In this article, it will be considered as a linear multipoint differential-operator expression

$$l = (l_1,l_2,l_3), \quad l_k = \frac{d}{dt} + A_k, \quad k = 1,2,3$$

in the direct sum of Hilbert spaces of vector-functions $L_2(1,1,1)$, where $A_1 = A_1^* \leq 0$, $A_2 = A_2^* \geq 0$, $A_3 = A_3^* \geq 0$, $-\infty < a_1 < a_2 < b_2 < a_3 < \infty$.

In the second section, by the method of Calkin-Gorbachuk theory, we describe all normal extensions of the minimal operator generated by singular multipoint formally normal differential expression for first order $l(.\ )$ in the direct sum of Hilbert space $L_2(1,1,1)$ in terms of boundary values. In the third section the spectrum of such extensions is studied.

2. Description of normal extensions

Let H be a separable Hilbert space and $a_1,a_2,b_2,a_3 \in \mathbb{R}$, $a_1 < a_2 < b_2 < a_3$. In the Hilbert space $L_2(1,1,1)$ of vector-functions let us consider the linear multipoint differential expression

$$l(u) = (l_1(u_1), l_2(u_2), l_3(u_3)) = (u_1' + A_1u_1, u_2' + A_2u_2, u_3' + A_3u_3),$$

where $u = (u_1,u_2,u_3)$, $A_k : D(A_k) \subset H \to H$, $k = 1,2,3$ are linear selfadjoint operators in H and $A_1 = A_1^* \leq 0$, $A_2 = A_2^* \geq 0$, $A_3 = A_3^* \geq 0$. In the linear manifold $D(A_k) \subset H$ introduce the inner product

$$(f,g)_{k,+} := (A_k f A_k, g)_H + (f,g)_H, \quad f,g \in D(A_k),\ k = 1,2,3.$$

For $k = 1,2,3$, $D(A_k)$ is a Hilbert space under the positive norm $\|\cdot\|_{k,+}$ with respect to Hilbert space H. It is denoted by $H_{k,+}$. Denote the $H_{k,-}$ a Hilbert space with the negative norm. It is clear that an operator A_k is continuous from $H_{k,+}$ to H and that its adjoint operator $\tilde{A}_k : H \to H_{k,-}$ is a extension of the operator A_k. On the other hand, the operator $\tilde{A}_k : D(\tilde{A}_k) = H \subset H_{k,-1} \to H_{k,-1}$ is a linear selfadjoint.

In the direct sum, $L_2(1,1,1)$ is defined by

$$\tilde{l}(u) = (\tilde{l}_1(u_1), \tilde{l}_2(u_2), \tilde{l}_3(u_3)), \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ and $\tilde{l}_1(u_1) = u_1' + \tilde{A}_1 u_1$, $\tilde{l}_2(u_2) = u_2' + \tilde{A}_2 u_2$, $\tilde{l}_3(u_3) = u_3' + \tilde{A}_3 u_3$.

The operators $L_0(1,1,1) = L_{10} \oplus L_{20} \oplus L_{30}$ and $L(1,1,1) = L_1 \oplus L_2 \oplus L_3$ in the space $L_2(1,1,1)$ are called minimal (multipoint) and maximal (multipoint) operators generated by the differential expression (2.1), respectively.

Here all normal extensions of the minimal operator $L_0(1,1,1)$ in $L_2(1,1,1)$ in terms of the boundary values are described.

Note that a space of boundary values has an important role in the theory extensions of the linear symmetric differential operators [6] and will be used in the last investigation.

Let $B : D(B) \subset \mathcal{H} \to \mathcal{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$, having equal finite or infinite deficiency indices. A triplet $(\mathfrak{H}, \gamma_1, \gamma_2)$, where $\mathfrak{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings of $D(B^*)$ into $\mathfrak{H}$, is called a space of boundary values for the operator B if for any $f, g \in D(B^*)$

$$(B^* f, g)_{\mathcal{H}} - (f, B^* g)_{\mathcal{H}} = (\gamma_1(f), \gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f), \gamma_1(g))_{\mathfrak{H}},$$

while for any $F, G \in \mathfrak{H}$, there exists an element $f \in D(B^*)$, such that $\gamma_1(f) = F$ and $\gamma_2(f) = G$.

Note that any symmetric operator with equal deficiency indices has at least one space of boundary values [6].

Now let us construct a space of boundary values for the minimal operators $M_0\ (1,0,1)$ and $M_0(0,1,0)$ generated by linear singular differential expressions of first order in the form

$$(m_1(u_1), 0, m_3(u_3)) = (-i\frac{du_1}{dt}, 0, -i\frac{du_3}{dt}),$$
$$(0, m_2(u_2), 0) = (0,\ -i\frac{du_2}{dt}, 0)$$

in the direct sum $L_2(1,0,1)$ and $L_2(0,1,0)$, respectively. Note that the minimal operators $M_0(1,0,1)$ and $M_0(0,1,0)$ are closed symmetric operators in $L_2(1,0,1)$ and $L_2(0,1,0)$ with deficiency indices $(\dim H, \dim H)$.

Lemma 2.1. *The triplet (H, γ_1, γ_2), where*

$$\gamma_1 : D(M_0^*) \to H, \quad \gamma_1(u) = \frac{1}{i\sqrt{2}}(u_3(a_3) + u_1(a_1)),$$
$$\gamma_2 : D(M_0^*\ ) \to H, \quad \gamma_2(u) = \frac{1}{\sqrt{2}}(u_3(a_3) - u_1(a_1)),\ u = (u_1, 0, u_3) \in D(M_0^*)$$

is a space of boundary values of the minimal operator $M_0(1,0,1)$ in $L_2(1,0,1)$.

Proof. For arbitrary $u = (u_1, , 0, u_3)$ and $v = (v_1, 0, v_3)$ from $D(M_0^*(1,0,1))$ the validity of the equality

$$(M_0^*(1,0,1)u, v)_{L_2(1,0,1)} - (u, M_0^*(1,0,1)v)_{L_2(1,0,1)}$$
$$= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H$$

can be easily verified. Now for any given elements $f, g \in H$, we will find the function $u = (u_1, 0, u_3) \in D(M_0^*(1,0,1))$ such that

$$\gamma_1(u) = \frac{1}{i\sqrt{2}}(u_3(a_3) + u_1(a_1)) = f \quad \text{and} \quad \gamma_2(u) = \frac{1}{\sqrt{2}}(u_3(a_3) - u_1(a_1)) = g;$$

that is,

$$u_1(a_1) = (if - g)/\sqrt{2} \quad \text{and} \quad u_3(a_3) = (if + g)/\sqrt{2}.$$

If we choose the functions $u_1(t), u_2(t)$ in the form $u_1(t) = \int_{-\infty}^{t} e^{s-a} ds (if - g)/\sqrt{2}$ with $t < a_1$ and $u_3(t) = \int_t^{\infty} e^{a_3 - t} ds (if + g)/\sqrt{2}$ with $t > a_3$, then it is clear that $(u_1, u_2) \in D(M_0^*)$ and $\gamma_1(u) = f$, $\gamma_2(u) = g$. □

Lemma 2.2. *The triplet* (H, Γ_1, Γ_2),

$$\Gamma_1 : D(M_0^*(0,1,0)) \to H, \quad \Gamma_1(u) = \frac{1}{i\sqrt{2}}(u_2(b_2) + u_2(a_2)),$$
$$\Gamma_2 : D(M_0^*(0,1,0)) \to H, \quad \Gamma_2(u) = \frac{1}{\sqrt{2}}(u_2(b_2) - u_2(a_2)),$$
$$u = (0, u_2, 0) \in D(M_0^*(0,1,0))$$

is a space of boundary values of the minimal operator $M_0(0,1,0)$ *in the direct sum* $L_2(0,1,0)$.

Theorem 2.3. *If the minimal operators* L_{10}, L_{20} *and* L_{30} *are formally normal then*

$$D(L_{10}) \subset W_2^1(H, (-\infty, a_1)), \quad A_1 D(L_{10}) \subset L_2(H, (-\infty, a_1)),$$
$$D(L_{20}) \subset W_2^1(H, (a_2, b_2)), \quad A_2 D(L_{20}) \subset L_2(H, (a_2, b_2)),$$
$$D(L_{30}) \subset W_2^1(H, (a_3, \infty)), \quad A_3 D(L_{30}) \subset L_2(H, (a_3, \infty)).$$

Proof. Indeed, in this case for each $u_1 \in D(L_{10}) \subset D(L_{10}^*)$ is true

$$u_1' + A_1 u_1 \in L_2(H, (-\infty, a)) \ \text{ and } \ u_1' - A_1 u_1 \in L_2(H, (-\infty, a)),$$

hence

$$u_1' \in L_2(H, (-\infty, a)) \quad \text{and} \quad A_1 u_1 \in L_2(H, (-\infty, a));$$

i.e.,

$$D(L_{10}) \subset W_2^1(H, (-\infty, a)) \ \text{ and } \ A_1 D(L_{10}) \subset L_2(H, (-\infty, a)).$$

The second and third parts of theorem can be proved in a similar way. □

The following result can be easily established.

Lemma 2.4. *Every normal extension of* $L_0(1,1,1)$ *in* $L_2(1,1,1)$ *is a direct sum of normal extensions of the minimal operator* $L_0(1,0,1) = L_{10} \oplus 0 \oplus L_{30}$ *in*

$$L_2(1,0,1) = L_2(H, (-\infty, a_1)) \oplus 0 \oplus L_2(H, (a_3, \infty))$$

and minimal operator $L_0(0,1,0) = 0 \oplus L_{20} \oplus 0$ *in* $L_2(0,1,0) = 0 \oplus L_2(H, (a_2, b_2)) \oplus 0$.

Finally, using the method in [6, 7, 8, 9, 10, 11, 15, 16, 17] and Lemmas 2.1 and 2.2 the following result can be deduced.

Theorem 2.5. *Let* $(-A_1)^{1/2}W_2^1(H,(-\infty,a_1)) \subset W_2^1(H,(-\infty,a_1))$,

$$A_2^{1/2}W_2^1(H,(a_2,b_2)) \subset W_2^1(H,(a_2,b_2)),$$
$$A_3^{1/2}W_2^1(H,(a_3,\infty)) \subset W_2^1(H,(a_3,\infty)).$$

Each normal extension $\tilde{L}$ *of the minimal operator* L_0 *in the Hilbert space* $L_2(1,1,1)$ *is generated by differential expression* (2.1) *and boundary conditions*

$$u_3(a_3) = W_1u_1(a_1), \quad u_1(a_1) \in \ker(-A_1)^{1/2}\ , \quad u_3(a_3) \in \ker A_3^{1/2}, \tag{2.2}$$
$$u_2(b_2) = W_2u_1(a_2), \tag{2.3}$$

where $W_1, W_2 : H \to H$ *is a unitary operators. Moreover, the unitary operators* W_1, W_2 *in* H *are determined by the extension* $\tilde{L}$*; i.e.,* $\widetilde{L} = L_{W_1W_2}$ *and vice versa.*

Corollary 2.6. *If at least one of the operators* A_1 *and* A_3 *is one-to-one mapping in* H*, then minimal operator* $L_0(1,1,1)$ *is maximally formal normal in* $L_2(1,1,1)$*.*

Corollary 2.7. *If there exists at least one normal extension of the minimal operator* $L_0(1,1,1)$*, then*

$$\dim\ker(-A_1)^{1/2} = \dim\ker A_3^{1/2} > 0.$$

3. The spectrum of the normal extensions

In this section the structure of the spectrum of the normal extension $L_{W_1W_2}$ in $L_2(1,1,1)$ will be investigated. In this case by the Lemma 2.4 it is clear that

$$L_{W_1W_2} = L_{W_1} \oplus L_{W_2},$$

where L_{W_1} and L_{W_2} are normal extensions of the minimal operators $L_0(1,0,1)$ and $L_0(0,1,0)$ in the Hilbert spaces $L_2(1,0,1)$ and $L_2(0,1,0)$ respectively. Later, it will be assumed that $A_1 = A_1^* \le 0$, $A_2 = A_2^* \ge 0, A_3 = A_3^* \ge 0$ and $0 \in \sigma_p((-A_1)^{1/2}) \cap \sigma_p(A_3^{1/2})$. First, we have to prove the following result.

Theorem 3.1. *The point spectrum of any normal extension* L_{W_1} *of the minimal operator* $L_0(1,0,1)$ *in the Hilbert space* $L_2(1,0,1)$ *is empty; i.e.,* $\sigma_p(L_{W_1}) = \emptyset$*.*

Proof. Let us consider the following problem for the spectrum of the normal extension L_{W_1} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$,

$$L_{W_1}u = \lambda u, \quad \lambda = \lambda_r + i\lambda_i \in \mathbb{C}, \quad u = (u_1,0,u_3) \in L_2(1,0,1);$$

that is,

$$\tilde{l}_1(u_1) = u_1' + \tilde{A}_1u_1 = \lambda u_1, \quad u_1 \in L_2(H,(-\infty,a_1)),$$
$$\tilde{l}_3(u_3) = u_3' + \tilde{A}_3u_3 = \lambda u_3, \quad u_3 \in L_2(H,(a_3,+\infty)), \quad \lambda \in \mathbb{R},$$
$$u_3(a_3) = W_1u_1(a_1), \quad u_1(a_1) \in \ker(-A_1)^{1/2}, \quad u_3(a_3) \in \ker A_3^{1/2}.$$

The general solution of this problem is

$$u_1(\lambda;t) = e^{-(\tilde{A}_1-\lambda)(t-a_1)}f_1^*, \quad t < a_1,\ f_1^* \in H_{-1/2}(-A_1),$$
$$u_3(\lambda;t) = e^{-(\tilde{A}_3-\lambda)(t-a_3)}f_3^*, \quad t > a_3,\ f_3^* \in H_{-1/2}(A_3),$$
$$f_3^* = W_1f_1^*, \quad f_1^*, f_3^* \in H, \quad f_1^* = u_1(\lambda;a_1), \quad f_3^* = u_3(\lambda;a_3).$$

Since $0 \in \sigma_p((-A_1)^{1/2}) \cap \sigma_p(A_2^{1/2})$ and $(-A_1)^{1/2}f_1^* = 0$, $A_2^{1/2}f_3^* = 0$, we have

$$u_1(\lambda;t) = e^{\lambda(t-a)}f_1^*, \quad t < a,\ f_1^* \in H_{-1/2}((-A_1)),$$

$$u_2(\lambda;t) = e^{\lambda(t-b)} f_3^*, \quad t > b,\ f_3^* \in H_{-1/2}(A_3),$$
$$f_3^* = W_1 f_1^*, \quad f_1^* = u_1(\lambda;a_1), \quad f_3^* = u_3(\lambda;a_3).$$

In order for $u_1(\lambda;.) \in L_2(H,(-\infty,a_1))$ and $u_2(\lambda;.) \in L_2(H,(a_3,\infty))$ necessary and sufficient condition is that $\lambda_r \geq 0$ and $\lambda_r \leq 0$ respectively. Hence $\lambda_r = 0$. Consequently,

$$u_1(\lambda;t) = e^{i\lambda_i(t-a_1)} f_1^*, \quad t < a_1,$$
$$u_2(\lambda;t) = e^{i\lambda_i(t-a_3)} f_3^*, \quad t > a_3,\ f_3^* = W_1 f_1^*.$$

In this case for $u_1(\lambda;.) \in L_2(H,(-\infty,a_1))$ and $u_2(\lambda;.) \in L_2(H,(a_3,\infty))$, clearly, necessary and sufficient conditions are that $f_1^* = 0$, $f_3^* = 0$. This implies that $u_1 = 0$ and $u_2 = 0$ in L^2. Therefore $\sigma_p(L_{W_1}) = \emptyset$. □

Since residual spectrum of any normal operators in any Hilbert space is empty, it is sufficient to investigate the continuous spectrum of the normal extensions L_{W_1} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$.

Theorem 3.2. *The continuous spectrum of any normal extension L_{W_1} of the minimal operator $L_0(1,0,1)$ in the Hilbert space $L_2(1,0,1)$ is $\sigma_c(L_{W_1}) = i\mathbb{R}$.*

Proof. Assume that $\lambda \in \sigma_c(L_{W_1})$. Then by the theorem for the spectrum of normal operators [5],

$$\sigma(L_{W_1}) \subset \sigma(\operatorname{Re} L_{W_1}) + i\sigma(\operatorname{Im} L_{W_1}),$$

we obtain that

$$\lambda_r \in \sigma(\operatorname{Re} L_{W_1}), \quad \lambda_i \in \sigma(\operatorname{Im} L_{W_1}).$$

This implies that $\lambda_r \in \sigma(A_1)$ and $\lambda_r \in \sigma(A_3)$, hence by the conditions to the operators A_1 and A_3 we have $\lambda_r = 0$. On the other hand from the proof of previous theorem we see that $\ker(L_{W_1} - \lambda) = \{0\}$ for any $\lambda \in \mathbb{C}$. Consequently, $\sigma_c(L_{W_1}) \subset i\mathbb{R}$. Furthermore, it is clear that for the $\lambda = i\lambda_i \in \mathbb{C}$ the general solution of the boundary value problem

$$u_1' + A_1 u_1 = i\lambda_i u_1 + f_1, \quad u_1, f_1 \in L_2(H,(-\infty,a_1)),$$
$$u_3' + A_3 u_3 = i\lambda_i u_3 + f_3, \quad u_3, f_3 \in L_2(H,(a_3,\infty)),\ \lambda_i \in \mathbb{R},$$
$$u_3(a_3) = W_1 u_1(a_1),\ u_1(a_1) \in \ker(-A_1)^{1/2},\ u_3(a_3) \in \ker A_3^{1/2}$$

will be of the form

$$u_1(i\lambda_i;t) = e^{-(A_1 - i\lambda_i)(t-a_1)} f_{i\lambda_i} - \int_t^{a_1} e^{-(A_1 - i\lambda_i)(t-s)} f_1(s)ds, \quad t < a_1,$$
$$u_3(i\lambda_i;t) = \mathrm{e}^{-(\mathrm{A}_3 - i\lambda_i)(t-a_3)} g_{i\lambda_i} + \int_{a_3}^t \mathrm{e}^{-(\mathrm{A}_3 - i\lambda_i)(t-s)} f_3(s)\mathrm{ds}, \quad \mathrm{t} > a_3,$$
$$g_{i\lambda_i} = W_1 f_{i\lambda_i}.$$

In this case,

$$e^{-(A_1 - i\lambda_i)(t-a_1)} f_{i\lambda_i} \in L_2(H,(-\infty,a_1)), \quad \mathrm{e}^{-(\mathrm{A}_2 - i\lambda_i)(t-a_3)} g_{i\lambda_i} \in L_2(H,(a_3,\infty))$$

for any $g_{i\lambda_i}, f_{i\lambda_i} \in H$. If choose $f_1(t) = e^{i\lambda_i t} e^{-(t-a_1)} f^*$, $f^* \in \ker(-A_1)^{1/2}$, $t < a_1$, then

$$\int_t^{a_1} e^{-(A_1 - i\lambda_i)(t-s)} f_1(s)ds = e^{-i\lambda_i t} \int_t^{a_1} e^{-(s-a_1)} f^* ds$$
$$= e^{-i\lambda_i t}(e^{-(t-a_1)} - 1) f^*, \quad t < a_1.$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{a_1} \|e^{-i\lambda_i t}(e^{-(t-a_1)}-1)f^*\|^2 dt &= \int_{-\infty}^{a_1} \|e^{-i\lambda_i t}(e^{-(t-a_1)}-1)f^*\|^2 dt \\ &= \int_{-\infty}^{a_1} (e^{-2(t-a_1)}-2e^{-(t-a_1)}+1)dt\|f^*\|^2 = \infty\end{aligned}$$

Consequently, we have $f_1(t) \in L_2(H,(-\infty,a_1))$, $u_1(i\lambda_i;t) \notin L_2(H,(-\infty,a_1))$. This implies that for any $\lambda \in \mathbb{C}$, an operator $L_{W_1}-\lambda$ is one-to-one in $L_2(1,0,1)$, but it is not an onto transformation. On the other hand, since the residual spectrum $\sigma_r(L_{W_1})$ is empty, we have $\sigma(L_{W_1}) = \sigma_c(L_{W_1}) = i\mathbb{R}$. □

Now, we investigate the spectrum of normal extensions L_{W_2} of the minimal operator $L_0(0,1,0)$ in $L_2(1,0,1)$.

Theorem 3.3. *The spectrum of the normal extension L_{W_2} of the minimal operator $L_0(0,1,0)$ in the Hilbert space $L_2(0,1,0)$ is of the form*

$$\begin{gathered}\sigma(L_{W_2}) = \big\{\lambda \in \mathbb{C}\colon \lambda = \frac{1}{a_2-b_2}(\ln|\mu| + i\ \arg\mu + 2n\pi i), n \in \mathbb{Z}, \\ \mu \in \sigma(W_2^* e^{-\tilde{A}_2(b_2-a_2)}),\ 0 \le \arg\mu < 2\pi\big\}\end{gathered}$$

Proof. The general solution of the problem spectrum of the normal extension L_{W_2},

$$\begin{gathered}\tilde{l}_2(u_2) = u_2' + \tilde{A}_2 u_2 = \lambda u_2 + f_2, \quad u_2, f_2 \in L_2(H,(a_2,b_2)) \\ u_2(b_2) = W_2 u_2(a_2), \quad \lambda \in \mathbb{C}\end{gathered}$$

is of the form

$$\begin{gathered}u_2(t) = e^{-(\tilde{A}_2-\lambda)(t-a_2)} f_2^* + \int_{a_2}^{t} e^{-(\tilde{A}_2-\lambda)(t-s)}\ f_2(s)ds, \\ a_2 < t < b_2,\ f_2^* \in H_{-1/2}(A_2) \\ (e^{-\lambda(b_2-a_2)} - W_2^* e^{-\tilde{A}_2(b_2-a_2)}) f_2^* = W_2^* e^{-\lambda(b_2-a_2)} \int_{a_2}^{b_2} e^{-(\tilde{A}_2-\lambda)(b_2-s)} f_2(s)ds\end{gathered}$$

This implies that $\lambda \in \sigma(L_{W_2})$ if and only if λ is a solution of the equation $e^{-\lambda(b_2-a_2)} = \mu$, where $\mu \in \sigma(W_2^* e^{-\tilde{A}_2(b_2-a_2)})$. We obtain that

$$\lambda = \frac{1}{a_2-b_2}(\ln|\mu| + i\arg\mu + 2n\pi i), \quad n \in \mathbb{Z},\ \mu \in \sigma(W_2^* e^{-\tilde{A}_2(b_2-a_2)}).$$

□

Theorem 3.4. *For the spectrum $\sigma(L_{W_1W_2})$ of any normal extension $L_{W_1W_2} = L_{W_1} \oplus L_{W_2}$, it is true that*

$$\sigma_p(L_{W_1W_2}) = \sigma_p(L_{W_2}), \quad \sigma_c(L_{W_1W_2}) = \{[\sigma_p(L_{W_2})]^c \cap [i\mathbb{R}]\} \cup \sigma_c(L_{W_2})$$

Proof. The validity of this assertion is a simple result of the following claim. If S_1 and S_2 are linear closed operators in any Hilbert spaces H_1 and H_2 respectively, then we have

$$\begin{gathered}\sigma_p(S_1 \oplus S_2) = \sigma_p(S_1) \cup \sigma_p(S_2), \\ \sigma_c(S_1 \oplus S_2) = (\sigma_p(S_1) \cup \sigma_p(S_2))^c \cap (\sigma_r(S_1) \cup \sigma_r(S_2))^c \cap (\sigma_c(S_1) \cup \sigma_c(S_2)).\end{gathered}$$

□

Note that for the singular differential operators for n-th order in scalar case in the finite interval has been studied in [18].

Example 3.5. Consider the boundary-value problem for the differential operator $L_{\varphi\psi}$,

$$\begin{gathered}
L_{\varphi\psi}: \frac{\partial u(t,x)}{\partial t} + \operatorname{sgn} t \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), \quad |t|>1,\ x\in[0,1],\\
\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = f(t,x), \quad |t|<1/2,\ x\in[0,1],\\
u(1,x) = e^{i\varphi} u(-1,x), \quad \varphi\in[0,2\pi),\\
u(1/2,x) = e^{i\psi} u(-1/2,x), \quad \psi\in[0,2\pi),\\
u_x(t,0) = u_x(t,1) = 0, \quad |t|>1,\ |t|<1/2
\end{gathered}$$

in the space $L_2((-\infty,-1)\times(0,1)) \oplus L_2((-1/2,1/2)\times(0,1)) \oplus L_2((1,\infty)\times(0,1))$. In this case it is clear that in the space $L_2(0,1)$, for the operators

$$\begin{gathered}
A_1 = \frac{\partial^2 u(.,x)}{\partial x^2}, \quad x\in[0,1],\ u_x(.,0) = u_x(.,1) = 0,\\
A_2 = -\frac{\partial^2 u(.,x)}{\partial x^2} + u(.,x), \quad x\in[0,1],\ u_x(.,0) = u_x(.,1) = 0,\\
A_3 = -\frac{\partial^2 u(.,x)}{\partial x^2}, \quad x\in[0,1],\ u_x(.,0) = u_x(.,1) = 0
\end{gathered}$$

we have

$$\begin{gathered}
A_1 = A_1^* \le 0, \quad A_2 = A_2^* \ge 1, \quad A_3 = A_3^* \ge 0, \quad \ker(-A_1)^{1/2} \ne \{0\},\\
\ker A_3^{1/2} \ne \{0\}, \quad 0 \in \sigma_p((-A_1)^{1/2}) \cap \sigma_p(A_3^{1/2}).
\end{gathered}$$

On the other hand, since $A_2^{-1} \in \sigma_\infty(L_2(0,1))$, $\sigma(L_\psi) = \sigma_p(L_\psi)$, $\sigma_c(L_\psi) = \emptyset$ and

$$\begin{aligned}
\sigma(L_\psi) &= \Big\{\lambda\in\mathbb{C} : \lambda = \ln|\mu| + i\arg\mu + 2n\pi i, n\in\mathbb{Z},\ \mu\in\sigma(e^{i\psi}e^{-\tilde{A}_2(b_2-a_2)}),\\
&\qquad 0\le \arg\mu < 2\pi\Big\}\\
&\subset \big\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \ge 1\big\},
\end{aligned}$$

then $[\sigma_p(L_\psi)]^c \cap [i\mathbb{R}] = i\mathbb{R}$. Therefore, by the Theorem 3.4, we obtain

$$\sigma_p(L_{\varphi\psi}) = \sigma_p(L_\psi), \quad \sigma_c(L_{\varphi\psi}) = i\mathbb{R}.$$

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