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SINGULAR ELLIPTIC SYSTEMS INVOLVING CONCAVE TERMS AND CRITICAL CAFFARELLI-KOHN-NIRENBERG EXPONENTS

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Abstract. In this article, we establish the existence of at least four solutions to a singular system with a concave term, a critical Caffarelli-Kohn-Nirenberg exponent, and sign-changing weight functions. Our main tools are the Nehari manifold and the mountain pass theorem.

1. Introduction

In this article, we consider the existence of multiple nontrivial nonnegative solutions of the

$$\begin{gathered}-L_{\mu,a}u = (\alpha+1)|x|^{-2_*b}h|u|^{\alpha-1}u|v|^{\beta+1} + \lambda_1|x|^{-c}f_1|u|^{q-2}u \quad \text{in } \Omega\backslash\{0\}\\ -L_{\mu,a}v = (\beta+1)|x|^{-2_*b}h|u|^{\alpha+1}|v|^{\beta-1}v + \lambda_2|x|^{-c}f_2|v|^{q-2}v \quad \text{in } \Omega\backslash\{0\}\\ u = v = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $L_{\mu,a}w := \operatorname{div}(|x|^{-2a}\nabla w) - \mu|x|^{-2(a+1)}w$, Ω is a bounded regular domain in $\mathbb{R}^N$ ($N \geq 3$) containing 0 in its interior, $-\infty < a < (N-2)/2$, $a \leq b < a+1$, $1 < q < 2$, $2_* = 2N/(N-2+2(b-a))$ is the critical Caffarelli-Kohn-Nirenberg exponent, $0 < c = q(a+1) + N(1-q/2)$, $-\infty < \mu < \bar{\mu}_a := ((N-2(a+1))/2)^2$, α, β are positive reals such that $\alpha+\beta = 2_*-2$, λ_1, λ_2 are real parameters, f_1, f_2 and h are functions defined on $\bar{\Omega}$.

Elliptic systems have been widely studied in recent years, we refer the readers to [1, 7] for regular systems which derive from potential. However, only a few results for singular systems, we can cite [3, 7]. As noticed, when $a = b = c = 0$, $h \equiv 1$, $q = 2$ and $f_1 \equiv f_2 \equiv 1$, Liu and Han [11] studied (1.1). By applying the mountain pass theorem, they proved that, if $0 < \mu \leq \bar{\mu}_0 - 1$ then, system (1.1) admits one positive solution for all $\lambda_1, \lambda_2 \in (0, \eta_1(\mu))$. Here, $\eta_1(\mu)$ denote the first eigenvalue of the positive operator $-\Delta - \mu|x|^{-2}$ with Dirichlet boundary condition. Wu [13] proved that the system (1.1) with $\mu = 0$, has at least two nontrivial nonnegative solutions when $a = b = c = 0$, the pair of the parameters (λ_1, λ_2) belong to a certain subset of $\mathbb{R}^2$ and under some conditions on the weight functions f_1, f_2 and h. For $c = 0$, $q = 1$ and $h \equiv 1$, system (1.1) has been studied by Bouchekif and El

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Mokhtar [2]. By using the Nehari manifold, they proved that there exists a positive constant Λ_0 such that (1.1) admits two nontrivial solutions when λ_1, λ_2 satisfy $0 < |\lambda_1|\|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2|\|f_2\|_{\mathcal{H}'_\mu} < (1/2)\Lambda_0$.

The starting point of the variational approach to our problem is the following Caffarelli-Kohn-Nirenberg inequality [5], which ensures the existence of a positive constant $C_{a,b}$ such that

$$(\int_{\mathbb{R}^N} |x|^{-2_* b}|v|^{2_*}dx)^{2/2_*} \le C_{a,b}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla v|^2 dx, \quad \text{for all } v \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{1.2}$$

In this equation, if $b = a+1$, then $2_* = 2$ and we have the weighted Hardy inequality [6]:

$$\int_{\mathbb{R}^N} |x|^{-2(a+1)}v^2 dx \le (1/\bar{\mu}_a)\int_{\mathbb{R}^N} |x|^{-2a}|\nabla v|^2 dx, \quad \text{for all } v \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{1.3}$$

We introduce a weighted Sobolev spaces $\mathcal{D}_a^{1,2}(\Omega)$ and $\mathcal{H}_\mu := \mathcal{H}_\mu(\Omega)$ which are the completion of the space $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norms

$$\|u\|_{0,a} = \Big(\int_\Omega |x|^{-2a}|\nabla u|^2 dx\Big)^{1/2},$$
$$\|u\|_{\mu,a} = \Big(\int_\Omega (|x|^{-2a}|\nabla u|^2 - \mu|y|^{-2(a+1)}|u|^2)dx\Big)^{1/2}$$

for $-\infty < \mu < \bar{\mu}_a$, respectively.

It is known that by weighted Hardy inequality, the norm $\|u\|_{\mu,a}$ is equivalent to $\|u\|_{0,a}$. More explicitly, we have

$$(1 - (1/\bar{\mu}_a)\max(\mu,0))^{1/2}\|u\|_{0,a} \le \|u\|_{\mu,a} \le (1 - (1/\bar{\mu}_a)\min(\mu,0))^{1/2}\|u\|_{0,a},$$

for all $u \in \mathcal{H}_\mu$.

Define the space $\mathcal{H} := \mathcal{H}_\mu \times \mathcal{H}_\mu$ which is endowed with the norm

$$\|(u,v)\|_{\mu,a} = \big(\|u\|_{\mu,a}^2 + \|v\|_{\mu,a}^2\big)^{1/2}.$$

From the boundlessness of Ω and the standard approximation arguments, it is easy to see that (1.2) hold for any $u \in \mathcal{H}_\mu$ in the sense

$$(\int_\Omega |x|^{-c}|u|^p dx)^{1/p} \le C_{a,p}(\int_\Omega |x|^{-2a}|\nabla v|^2 dx)^{1/2}, \tag{1.4}$$

where $C_{a,p}$ positive constant, $1 \le p \le 2N/(N-2)$, $c \le p(a+1) + N(1-p/2)$, and in [13], if $p < 2N/(N-2)$ the embedding $\mathcal{H}_\mu \hookrightarrow L_p(\Omega, |x|^{-c})$ is compact, where $L_p(\Omega, |x|^{-c})$ is the weighted L_p space with norm

$$|u|_{p,c} = (\int_\Omega |x|^{-c}|u|^p dx)^{1/p}.$$

Since our approach is variational, we define the functional $J := J_{\lambda_1,\lambda_2,\mu}$ on $\mathcal{H}$ by

$$J(u,v) := (1/2)\|(u,v)\|_{\mu,a}^2 - P(u,v) - Q(u,v),$$

with

$$P(u,v) := \int_\Omega |x|^{-2_* b}h|u|^{\alpha+1}|v|^{\beta+1}dx,$$
$$Q(u,v) := (1/q)\int_\Omega |x|^{-c}(\lambda_1 f_1|u|^q + \lambda_2 f_2|v|^q)dx.$$

A couple $(u,v)\in\mathcal{H}$ is a weak solution of the system (1.1) if it satisfies

$$\langle J'(u,v),(\varphi,\psi)\rangle := R(u,v)(\varphi,\psi)-S(u,v)(\varphi,\psi)-T(u,v)(\varphi,\psi)=0$$

for all $(\varphi,\psi)\in\mathcal{H}$ with

$$\begin{gathered}R(u,v)(\varphi,\psi):=\int_\Omega\big(|x|^{-2a}(\nabla u\nabla\varphi+\nabla v\nabla\psi)-\mu|x|^{-2(a+1)}(u\varphi+v\psi)\big)\\ S(u,v)(\varphi,\psi):=\int_\Omega|x|^{-2_*b}h((\alpha+1)|u|^{\alpha}|v|^{\beta+1}\varphi+(\beta+1)|u|^{\alpha+1}|v|^{\beta}\psi)\\ T(u,v)(\varphi,\psi):=\int_\Omega|x|^{-c}(\lambda_1f_1|u|^{q-1}\varphi+\lambda_2f_2|v|^{q-1}\psi).\end{gathered}$$

Here $\langle .,.\rangle$ denotes the product in the duality $\mathcal{H}'$, $\mathcal{H}$, where $\mathcal{H}'$ is the dual of $\mathcal{H}$). Let

$$\begin{gathered}S_\mu:=\inf_{u\in\mathcal{H}_\mu\setminus\{0\}}\frac{\|u\|^2_{\mu,a}}{(\int_\Omega|x|^{-2_*b}|u|^{2_*}dx)^{2/2_*}},\\ \tilde S_\mu:=\inf_{(u,v)\in(\mathcal{H}\setminus\{(0,0)\})^2}\frac{\|(u,v)\|^2_{\mu,a}}{(\int_\Omega|x|^{-2_*b}|u|^{\alpha+1}|v|^{\beta+1}dx)^{2/2_*}}.\end{gathered}$$

From [10], it is known that S_μ is achieved.

Lemma 1.1. *Let Ω be a domain (not necessarily bounded), $-\infty<\mu<\bar\mu_a$ and $\alpha+\beta\le 2_*-2$. Then we have*

$$\tilde S_\mu:=\big[(\frac{\alpha+1}{\beta+1})^{(\beta+1)/2_*}+(\frac{\alpha+1}{\beta+1})^{-(\alpha+1)/2_*}\big]S_\mu.$$

With $[(\frac{\alpha+1}{\beta+1})^{(\beta+1)/2_*}+(\frac{\alpha+1}{\beta+1})^{-(\alpha+1)/2_*}]$ simply written as $K(\alpha,\beta)$.

Proof. The proof is essentially the same as in [1], with minor modifications. □

We put assumptions on h which is somewhere positive but which may change sign in $\bar\Omega$

- (H1) $h\in C(\bar\Omega)$ and $h^+=\max\{h,0\}\not\equiv 0$ in Ω
- (H2) There exists ϱ_0 positive such that $|h^+|_\infty=h(0)=\max_{x\in\bar\Omega}h(x)>\varrho_0$.

In our work, we research for critical points as the minimizers of the energy functional associated with (1.1) with the constraint defined by the Nehari manifold, which are solutions of our system.

Let Λ_0 be positive number and f_1, f_2 be continuous functions such that

$$\Lambda_0:=(C_{a,q})^{-q}(|h^+|_\infty)^{-1/(2_*-2)}[(S_\mu)K(\alpha,\beta)]^{2_*/2(2_*-2)}L(q)$$

and $|f_i(x)|_\infty=\sup_{x\in\bar\Omega}|f_i(x)|$ for $i=1,2$, where

$$L(q):=(\frac{2_*-2}{2_*-q})^{1/(2-q)}[(\frac{2-q}{2_*(2_*-q)})]^{1/(2_*-2)}.$$

Now we state our main results as follows.

Theorem 1.2. *Let f_1, $f_2\in L^\infty(\Omega)$. Assume that $-\infty<a<(N-2)/2$, $0<c=q(a+1)+N(1-q/2)$, $\alpha+\beta+2=2_*$, $-\infty<\mu<\bar\mu_a$,* (H1) *satisfied and λ_1, λ_2 satisfying $0<(|\lambda_1||f_1|_\infty)^{1/(2-q)}+(|\lambda_2||f_2|_\infty)^{1/(2-q)}<\Lambda_0$, then* (1.1) *has at least one positive solution.*

Theorem 1.3. *In addition to the assumptions of the Theorem 1.2, if* (H2) *holds and* λ_1, λ_2 *satisfy*

$$0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < (1/2)\Lambda_0,$$

then (1.1) *has at least two positive solutions.*

Theorem 1.4. *In addition to the assumptions of the Theorem 1.3, assuming* $N \geq \max(3, 6(a-b+1))$*, there exists a positive real* Λ_1 *such that, if* λ_1, λ_2 *satisfy*

$$0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < \min((1/2)\Lambda_0, \Lambda_1),$$

then (1.1) *has at least two positive solution and two opposite solutions.*

This article is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1.2 and 1.3. In the last Section, we prove the Theorem 1.4.

2. Preliminaries

Definition 2.1. Let $c \in \mathbb{R}$, E a Banach space and $I \in C^1(E, \mathbb{R})$.

(i) $(u_n, v_n)_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for I if

$$I(u_n, v_n) = c + o_n(1) \text{ and } I'(u_n, v_n) = o_n(1),$$

where $o_n(1)$ tends to 0 as n approaches infinity.

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

Lemma 2.2. *Let* X *Banach space, and* $J \in C^1(X, \mathbb{R})$ *satisfying the Palais -Smale condition. Suppose that* $J(0,0) = 0$ *and that:*

(i) *there exist* $R > 0$, $r > 0$ *such that if* $\|(u,v)\| = R$*, then* $J(u,v) \geq r$*;*
(ii) *there exist* $(u_0, v_0) \in X$ *such that* $\|(u_0, v_0)\| > R$ *and* $J(u_0, v_0) \leq 0$.

Let $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]}(J(\gamma(t)))$ *where*

$$\Gamma = \{\gamma \in C([0,1]; X) \text{ such that } \gamma(0) = (0,0) \text{ and } \gamma(1) = (u_0, v_0)\},$$

then c *is a critical value of* J *such that* $c \geq r$.

2.1. Nehari manifold.

It is well known that J is of class C^1 in $\mathcal{H}$ and the solutions of (1.1) are the critical points of J which is not bounded below on $\mathcal{H}$. Consider the Nehari manifold

$$\mathcal{N} = \{(u,v) \in \mathcal{H} \backslash \{0,0\} : \langle J'(u,v), (u,v) \rangle = 0\},$$

Thus, $(u,v) \in \mathcal{N}$ if and only if

$$\|(u,v)\|^2_{\mu,a} - 2_* P(u,v) - Q(u,v) = 0. \tag{2.1}$$

Note that $\mathcal{N}$ contains every nontrivial solution of (1.1). Moreover, we have the following results.

Lemma 2.3. *J is coercive and bounded from below on* $\mathcal{N}$.

Proof. If $(u,v) \in \mathcal{N}$, then by (2.1) and the Hölder inequality, we deduce that

$$\begin{aligned} J(u,v) &= ((2_* - 2)/2_* 2)\|(u,v)\|^2_{\mu,a} - ((2_* - q)/2_* q)Q(u,v) \\ &\geq ((2_* - 2)/2_* 2)\|(u,v)\|^2_{\mu,a} - (\frac{(2_* - q)}{2_* q})\Big((|\lambda_1||f_1|_\infty)^{1/(2-q)} \\ &\quad + (|\lambda_2||f_2|_\infty)^{1/(2-q)}\Big)(C_{a,p})^q \|(u,v)\|^q_{\mu,a}. \end{aligned} \tag{2.2}$$

Thus, J is coercive and bounded from below on $\mathcal{N}$. □

Define
$$\phi(u,v) = \langle J'(u,v),(u,v)\rangle.$$
Then, for $(u,v)\in\mathcal{N}$,
$$\begin{aligned}\langle \phi'(u,v),(u,v)\rangle &= 2\|(u,v)\|^2_{\mu,a} - (2_*)^2 P(u,v) - qQ(u,v)\\ &= (2-q)\|(u,v)\|^2_{\mu,a} - 2_*(2_*-q)P(u,v)\\ &= (2_*-q)Q(u,v) - (2_*-2)\|(u,v)\|^2_{\mu,a}.\end{aligned} \tag{2.3}$$
Now, we split $\mathcal{N}$ into three parts:
$$\begin{aligned}\mathcal{N}^+ &= \{(u,v)\in\mathcal{N} : \langle \phi'(u,v),(u,v)\rangle > 0\}\\ \mathcal{N}^0 &= \{(u,v)\in\mathcal{N} : \langle \phi'(u,v),(u,v)\rangle = 0\}\\ \mathcal{N}^- &= \{(u,v)\in\mathcal{N} : \langle \phi'(u,v),(u,v)\rangle < 0\}.\end{aligned}$$
We have the following results.

Lemma 2.4. *Suppose that (u_0,v_0) is a local minimizer for J on $\mathcal{N}$. Then, if $(u_0,v_0)\notin\mathcal{N}^0$, (u_0,v_0) is a critical point of J.*

Proof. If (u_0,v_0) is a local minimizer for J on $\mathcal{N}$, then (u_0,v_0) is a solution of the optimization problem
$$\min_{\{(u,v):\phi(u,v)=0\}} J(u,v).$$
Hence, there exists a Lagrange multipliers $\theta\in\mathbb{R}$ such that
$$J'(u_0,v_0) = \theta\phi'(u_0,v_0) \text{ in } \mathcal{H}'$$
Thus,
$$\langle J'(u_0,v_0),(u_0,v_0)\rangle = \theta\langle \phi'(u_0,v_0),(u_0,v_0)\rangle.$$
But $\langle \phi'(u_0,v_0),(u_0,v_0)\rangle \neq 0$, since $(u_0,v_0)\notin\mathcal{N}^0$. Hence $\theta=0$. This completes the proof. □

Lemma 2.5. *There exists a positive number Λ_0 such that for all λ_1, λ_2 satisfying*
$$0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < \Lambda_0,$$
we have $\mathcal{N}^0=\emptyset$.

Proof. By contradiction, suppose $\mathcal{N}^0\neq\emptyset$ and that
$$0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < \Lambda_0.$$
Then, by (2.3) and for $(u,v)\in\mathcal{N}^0$, we have
$$\|(u,v)\|^2_{\mu,a} = 2_*(2_*-q)/(2-q)P(u,v) = ((2_*-q)/(2_*-2))Q(u,v) \tag{2.4}$$
Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain
$$\|(u,v)\|_{\mu,a} \geq [(S_\mu)K(\alpha,\beta)]^{2_*/2(2_*-2)}[(2-q)/2_*(2_*-q)|h^+|_\infty]^{-1/(2_*-2)} \tag{2.5}$$
and
$$\|(u,v)\|_{\mu,a} \leq [(\frac{2_*-q}{2_*-2})^{-1/(2-q)}((|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)})(C_{a,q})^q]. \tag{2.6}$$
From (2.5) and (2.6), we obtain $(|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} \geq \Lambda_0$, which contradicts an hypothesis. □

Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$. Define

$$c := \inf_{u\in\mathcal{N}} J(u,v), \quad c^+ := \inf_{u\in\mathcal{N}^+} J(u,v), \quad c^- := \inf_{u\in\mathcal{N}^-} J(u,v).$$

In the sequel, we need the following Lemma.

Lemma 2.6. *(i) For all λ_1, λ_2 with $0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < \Lambda_0$, one has $c \le c^+ < 0$.*

(ii) For all λ_1, λ_2 such that $0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < (1/2)\Lambda_0$, one has

$$\begin{aligned} c^- > C_0 &= C_0(\lambda_1,\lambda_2,S_\mu,\|f_1\|_{\mathcal{H}'_\mu},\|f_2\|_{\mathcal{H}'_\mu}) \\ &= (\frac{(2_*-2)}{2_*2})[\frac{(2-q)}{2_*(2_*-q)|h^+|_\infty}]^{2/(2_*-2)}[K(\alpha,\beta)]^{2_*/(2_*-2)}(S_\mu)^{2_*/(2_*-2)} \\ &\quad - (\frac{(2_*-q)}{2_*q})((|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)})(C_{a,q})^q. \end{aligned}$$

Proof. (i) Let $(u,v) \in \mathcal{N}^+$. By (2.3), we have

$$[(2-q)/2_*(2_*-1)]\|(u,v)\|^2_{\mu,a} > P(u,v)$$

and so

$$\begin{aligned} J(u,v) &= (-1/2)\|(u,v)\|^2_{\mu,a} + (2_*-1)P(u,v) \\ &< -[\frac{2_*(2_*-q) - 2(2_*-1)(2-q)}{2_*2(2_*-q)}]\|(u,v)\|^2_{\mu,a}. \end{aligned}$$

We conclude that $c \le c^+ < 0$.

(ii) Let $(u,v) \in \mathcal{N}^-$. By (2.3), we obtain

$$[(2-q)/2_*(2_*-q)]\|(u,v)\|^2_{\mu,a} < P(u,v).$$

Moreover, by (H1) and Sobolev embedding theorem, we have

$$P(u,v) \le [K(\alpha,\beta)]^{-2_*/2}(S_\mu)^{-2_*/2}|h^+|_\infty\|(u,v)\|^{2_*}_{\mu,a}.$$

This implies

$$\|(u,v)\|_{\mu,a} > [(S_\mu)K(\alpha,\beta)]^{2_*/2(2_*-2)}[\frac{(2-q)}{2_*(2_*-q)|h^+|_\infty}]^{-1/(2_*-2)} \tag{2.7}$$

for all $u \in \mathcal{N}^-$. By (2.2), we obtain

$$\begin{aligned} J(u,v) \ge ((2_*-2)/2_*2)\|(u,v)\|^2_{\mu,a} - (\frac{(2_*-q)}{2_*q})\Big((|\lambda_1||f_1|_\infty)^{1/(2-q)} \\ + (|\lambda_2||f_2|_\infty)^{1/(2-q)}\Big)(C_{a,p})^q\|(u,v)\|^q_{\mu,a}. \end{aligned}$$

Thus, for all (λ_1,λ_2) such that $0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < (1/2)\Lambda_0$, we have $J(u,v) \ge C_0$. ☐

For each $(u,v) \in \mathcal{H}$ with $\int_\Omega |x|^{-2_*b}h|u|^{\alpha+1}|v|^{\beta+1}dx > 0$, we write

$$t_m := t_{\max}(u,v) = [\frac{(2-q)\|(u,v)\|^2_{\mu,a}}{2_*(2_*-q)\int_\Omega |x|^{-2_*b}h|u|^{\alpha+1}|v|^{\beta+1}dx}]^{(2-q)/2_*(2_*-q)} > 0.$$

Lemma 2.7. *Let λ_1, λ_2 real parameters such that $0 < |\lambda_1| \|f_1\|_{\mathcal{H}'_\mu} + |\lambda_2| \|f_2\|_{\mathcal{H}'_\mu} < \Lambda_0$. For each $(u,v) \in \mathcal{H}$ with $\int_\Omega |x|^{-2_* b} h |u|^{\alpha+1} |v|^{\beta+1} dx > 0$, one has the following:*

(i) If $Q(u,v) \le 0$, then there exists a unique $t^- > t_m$ such that $(t^- u, t^- v) \in \mathcal{N}^-$ and

$$J(t^- u, t^- v) = \sup_{t \ge 0} (tu, tv).$$

(ii) If $Q(u,v) > 0$, then there exist unique t^+ and t^- such that $0 < t^+ < t_m < t^-$, $(t^+ u, t^+ v) \in \mathcal{N}^+$, $(t^- u, t^- v) \in \mathcal{N}^-$,

$$J(t^+ u, t^+ v) = \inf_{0 \le t \le t_m} J(tu, tv) \text{ and } J(t^- u, t^- v) = \sup_{t \ge 0} J(tu, tv).$$

The proof of the above lemma is the same as in [4], with minor modifications.

Proposition 2.8 ([4])**.** *(i) For all λ_1, λ_2 such that*

$$0 < (|\lambda_1| |f_1|_\infty)^{1/(2-q)} + (|\lambda_2| |f_2|_\infty)^{1/(2-q)} < \Lambda_0,$$

there exists a $(PS)_{c^+}$ sequence in $\mathcal{N}^+$.

(ii) For all λ_1, λ_2 such that $0 < (|\lambda_1| |f_1|_\infty)^{1/(2-q)} + (|\lambda_2| |f_2|_\infty)^{1/(2-q)} < (1/2)\Lambda_0$, there exists a $(PS)_{c^-}$ sequence in $\mathcal{N}^-$.

3. Proof of Theorem 1.2

Now, taking as a starting point the work of Tarantello [12], we establish the existence of a local minimum for J on $\mathcal{N}^+$.

Proposition 3.1. *For all λ_1, λ_2 with $0 < (|\lambda_1| |f_1|_\infty)^{1/(2-q)} + (|\lambda_2| |f_2|_\infty)^{1/(2-q)} < \Lambda_0$, the functional J has a minimizer $(u_0^+, v_0^+) \in \mathcal{N}^+$ and it satisfies:*

(i) $J(u_0^+, v_0^+) = c = c^+$,

(ii) (u_0^+, v_0^+) *is a nontrivial solution of* (1.1).

Proof. If $0 < (|\lambda_1| |f_1|_\infty)^{1/(2-q)} + (|\lambda_2| |f_2|_\infty)^{1/(2-q)} < \Lambda_0$, then by Proposition 2.8 (i), there exists a $(u_n, v_n)_n$ $(PS)_{c^+}$ sequence in $\mathcal{N}^+$, thus it bounded by Lemma 2.3. Then, there exists $(u_0^+, v_0^+) \in \mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n, v_n)_n$ such that

$$\begin{gathered} (u_n, v_n) \rightharpoonup (u_0^+, v_0^+) \quad \text{weakly in } \mathcal{H} \\ (u_n, v_n) \rightharpoonup (u_0^+, v_0^+) \quad \text{weakly in } (L^{2_*}(\Omega, |x|^{-2_* b}))^2 \\ (u_n, v_n) \to (u_0^+, v_0^+) \quad \text{strongly in } (L^q(\Omega, |x|^{-c}))^2 \\ u_n \to u_0^+ \quad \text{a.e in } \Omega, \\ v_n \to v_0^+ \quad \text{a.e in } \Omega. \end{gathered} \tag{3.1}$$

Thus, by (3.1), (u_0^+, v_0^+) is a weak nontrivial solution of (1.1). Now, we show that (u_n, v_n) converges to (u_0^+, v_0^+) strongly in $\mathcal{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\|u_0^+\|_{\mu,a} < \liminf_{n\to\infty} \|u_n\|_{\mu,a}$ or $\|v_0^+\|_{\mu,a} < \liminf_{n\to\infty} \|v_n\|_{\mu,a}$ and we obtain

$$\begin{aligned} c \le J(u_0^+, v_0^+) &= ((2_* - 2)/2_* 2) \|(u_0^+, v_0^+)\|_{\mu,a}^2 - ((2_* - q)/2_* q) Q(u_0^+, v_0^+) \\ &< \liminf_{n\to\infty} J(u_n, v_n) = c. \end{aligned}$$

We obtain a contradiction. Therefore, (u_n, v_n) converge to (u_0^+, v_0^+) strongly in $\mathcal{H}$. Moreover, we have $(u_0^+, v_0^+) \in \mathcal{N}^+$. If not, then by Lemma 2.7, there are two numbers t_0^+ and t_0^-, uniquely defined so that $(t_0^+ u_0^+, t_0^+ v_0^+) \in \mathcal{N}^+$ and $(t^- u_0^+, t^- v_0^+) \in \mathcal{N}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J(tu_0^+, tv_0^+)_{\downarrow t=t_0^+} = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J(tu_0^+, tv_0^+)_{\downarrow t=t_0^+} > 0,$$

there exists $t_0^+ < t^- \leq t_0^-$ such that $J(t_0^+ u_0^+, t_0^+ v_0^+) < J(t^- u_0^+, t^- v_0^+)$. By Lemma 2.7, we obtain

$$J(t_0^+ u_0^+, t_0^+ v_0^+) < J(t^- u_0^+, t^- v_0^+) < J(t_0^- u_0^+, t_0^- v_0^+) = J(u_0^+, v_0^+),$$

which contradicts the fact that $J(u_0^+, v_0^+) = c^+$. Since $J(u_0^+, v_0^+) = J(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in \mathcal{N}^+$, then by Lemma 2.4, we may assume that (u_0^+, v_0^+) is a nontrivial nonnegative solution of (1.1). By the Harnack inequality, we conclude that $u_0^+ > 0$ and $v_0^+ > 0$, see for example [9]. □

4. Proof of Theorem 1.3

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$. For this, we require the following Lemma.

Lemma 4.1. *For all λ_1, λ_2 such that $0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < (1/2)\Lambda_0$, the functional J has a minimizer (u_0^-, v_0^-) in $\mathcal{N}^-$ and it satisfies:*
- (i) $J(u_0^-, v_0^-) = c^- > 0$,
- (ii) (u_0^-, v_0^-) *is a nontrivial solution of* (1.1) *in* $\mathcal{H}$.

Proof. If $0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < (1/2)\Lambda_0$, then by Proposition 2.8 (ii) there exists a $(u_n, v_n)_n$, $(PS)_{c^-}$ sequence in $\mathcal{N}^-$, thus it bounded by Lemma 2.3. Then, there exists $(u_0^-, v_0^-) \in \mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n, v_n)_n$ such that

$$\begin{gathered}
(u_n, v_n) \rightharpoonup (u_0^-, v_0^-) \quad \text{weakly in } \mathcal{H} \\
(u_n, v_n) \rightharpoonup (u_0^-, v_0^-) \quad \text{weakly in } (L^{2_*}(\Omega, |y|^{-2_* b}))^2 \\
(u_n, v_n) \to (u_0^-, v_0^-) \quad \text{strongly in } (L^q(\Omega, |x|^{-c}))^2 \\
u_n \to u_0^- \quad \text{a.e in } \Omega, \\
v_n \to v_0^- \quad \text{a.e in } \Omega.
\end{gathered}$$

This implies $P(u_n, v_n) \to P(u_0^-, v_0^-)$, as $n \to \infty$. Moreover, by (H2) and (2.3) we obtain

$$P(u_n, v_n) > A(q)\|(u_n, v_n)\|_{\mu,a}^2, \tag{4.1}$$

where, $A(q) := (2-q)/2_*(2_* - q)$. By (2.5) and (4.1) there exists a positive number

$$C_1 := [A(q)K(\alpha, \beta)]^{2_*/(2_*-2)} (S_\mu)^{2_*/(2_*-2)},$$

such that

$$P(u_n, v_n) > C_1. \tag{4.2}$$

This implies $P(u_0^-, v_0^-) \geq C_1$.

Now, we prove that $(u_n, v_n)_n$ converges to (u_0^-, v_0^-) strongly in $\mathcal{H}$. Suppose otherwise. Then, either $\|u_0^-\|_{\mu,a} < \liminf_{n\to\infty} \|u_n\|_{\mu,a}$ or $\|v_0^-\|_{\mu,a} < \liminf_{n\to\infty} \|v_n\|_{\mu,a}$. By Lemma 2.7 there is a unique t_0^- such that $(t_0^- u_0^-, t_0^- v_0^-) \in \mathcal{N}^-$. Since

$$(u_n, v_n) \in \mathcal{N}^-, J(u_n, v_n) \geq J(tu_n, tv_n), \quad \text{for all } t \geq 0,$$

we have

$$J(t_0^- u_0^-, t_0^- v_0^-) < \lim_{n\to\infty} J(t_0^- u_n, t_0^- v_n) \le \lim_{n\to\infty} J(u_n, v_n) = c^-,$$

and this is a contradiction. Hence, $(u_n, v_n)_n \to (u_0^-, v_0^-)$ strongly in $\mathcal{H}$. Thus,

$$J(u_n, v_n) \text{ converges to } J(u_0^-, v_0^-) = c^- \text{ as } n \to +\infty.$$

Since $J(u_0^-, v_0^-) = J(|u_0^-|, |v_0^-|)$ and $(u_0^-, v_0^-) \in \mathcal{N}^-$, then by (4.2) and Lemma 2.4, we may assume that (u_0^-, v_0^-) is a nontrivial nonnegative solution of (1.1). By the maximum principle, we conclude that $u_0^- > 0$ and $v_0^- > 0$. □

Now, we complete the proof of Theorem 1.3. By Propositions 3.1 and Lemma 4.1, we obtain that (1.1) has two positive solutions $(u_0^+, v_0^+) \in \mathcal{N}^+$ and $(u_0^-, v_0^-) \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct.

5. Proof of Theorem 1.4

In this section, we consider the Nehari submanifold of $\mathcal{N}$

$$\mathcal{N}_\varrho = \{(u, v) \in \mathcal{H}\backslash\{0, 0\} : \langle J'(u, v), (u, v)\rangle = 0 \text{ and } \|(u, v)\|_{\mu,a} \ge \varrho > 0\}.$$

Thus, $(u, v) \in \mathcal{N}_\varrho$ if and only if

$$\|(u, v)\|_{\mu,a}^2 - 2_* P(u, v) - Q(u, v) = 0 \text{ and } \|(u, v)\|_{\mu,a} \ge \varrho > 0.$$

Firstly, we need the following Lemmas.

Lemma 5.1. *Under the hypothesis of theorem 1.4, there exist ϱ_0, $\Lambda_2 > 0$ such that $\mathcal{N}_\varrho$ is nonempty for any $\lambda \in (0, \Lambda_2)$ and $\varrho \in (0, \varrho_0)$.*

Proof. Fix $(u_0, v_0) \in \mathcal{H}\backslash\{0, 0\}$ and let

$$\begin{aligned} g(t) &= \langle J'(tu_0, tv_0), (tu_0, tv_0)\rangle \\ &= t^2\|(u_0, v_0)\|_{\mu,a}^2 - 2_* t^{2_*} P(u_0, v_0) - tQ(u_0, v_0). \end{aligned}$$

Clearly $g(0) = 0$ and $g(t) \to -\infty$ as $n \to +\infty$. Moreover, we have

$$\begin{aligned} g(1) &= \|(u_0, v_0)\|_{\mu,a}^2 - 2_* P(u_0, v_0) - Q(u_0, v_0) \\ &\ge [\|(u_0, v_0)\|_{\mu,a}^2 - 2_*[K(\alpha, \beta)]^{-2_*/2}(S_\mu)^{-2_*/2}|h^+|_\infty \|(u_0, v_0)\|_{\mu,a}^{2_*}] \\ &\quad - ((|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)})\|(u_0, v_0)\|_{\mu,a}. \end{aligned}$$

If $\|(u_0, v_0)\|_{\mu,a} \ge \varrho > 0$ for

$$0 < \varrho < \varrho_0 = (|h^+|_\infty 2_*(2_* - 1))^{-1/(2_*-2)}([K(\alpha, \beta)]S_\mu)^{2_*/2(2_*-2)},$$

$|h^+|_\infty \in (0, \alpha_0)$ for $\alpha_0 = ([K(\alpha, \beta)]S_\mu)^{2_*/2}/(2_*(2_* - 1))^{(2_*-1)/2_*}$, then there exists

$$\Lambda_2 := [(|h^+|_\infty 2_*(2_* - 1))([K(\alpha, \beta)]S_\mu)^{-2_*/2}]^{-1/(2_*-2)} - \Theta \times \Phi,$$

where

$$\begin{aligned} \Theta &:= (2_*(2_* - 1))^{2_*-1}((|h^+|_\infty)^{2_*/2}[K(\alpha, \beta)]S_\mu)^{-(2_*)^2/2}, \\ \Phi &:= [(|h^+|_\infty 2_*(2_* - 1))([K(\alpha, \beta)]S_\mu)^{-2_*/2}]^{-1/(2_*-2)} \end{aligned}$$

and there exists $t_0 > 0$ such that $g(t_0) = 0$. Thus, $(t_0u_0, t_0v_0) \in \mathcal{N}_\varrho$ and $\mathcal{N}_\varrho$ is nonempty for any $\lambda \in (0, \Lambda_2)$. □

Lemma 5.2. *There exist M, Λ_1 positive reals such that*

$$\langle \phi'(u,v),(u,v)\rangle < -M < 0$$

for $(u,v) \in \mathcal{N}_\varrho$ and any λ_1, λ_2 satisfying

$$0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < \min((1/2)\Lambda_0, \Lambda_1).$$

Proof. Let $(u,v) \in \mathcal{N}_\varrho$, then by (2.1), (2.3) and the Holder inequality, allows us to write

$$\begin{aligned}&\langle \phi'(u,v),(u,v)\rangle \\ &\le \|(u_n,v_n)\|^2_{\mu,a}[((|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)})B(\varrho,q) - (2_* - 2)],\end{aligned}$$

where $B(\varrho,q) := (2_* - 1)(C_{a,p})^q \varrho^{q-2}$. Thus, if

$$0 < (|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < \Lambda_3 = [(2_* - 2)/B(\varrho,q)],$$

and choosing $\Lambda_1 := \min(\Lambda_2, \Lambda_3)$ with Λ_2 defined in Lemma 5.1, then we obtain that

$$\langle \phi'(u,v),(u,v)\rangle < 0, \text{ for any } (u,v) \in \mathcal{N}_\varrho. \tag{5.1}$$

□

Lemma 5.3. *Suppose $N \ge \max(3, 6(a-b+1))$ and $\int_\Omega |x|^{-2_* b} h|u|^{\alpha+1}|v|^{\beta+1}dx > 0$. Then, there exist r and η positive constants such that*

(i) *we have*

$$J(u,v) \ge \eta > 0 \quad \text{for } \|(u,v)\|_{\mu,a} = r.$$

(ii) *there exists $(\sigma,\omega) \in \mathcal{N}_\varrho$ when $\|(\sigma,\omega)\|_{\mu,a} > r$, with $r = \|(u,v)\|_{\mu,a}$, such that $J(\sigma,\omega) \le 0$.*

Proof. We assume that the minima of J are realized by (u_0^+, v_0^+) and (u_0^-, v_0^-). The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have

(i) By (2.3), (5.1) and $P(u,v) \le [K(\alpha,\beta)]^{-2_*/2}(S_\mu)^{-2_*/2}|h^+|_\infty \|(u,v)\|^{2_*}_{\mu,a}$, we obtain

$$J(u,v) \ge [(1/2) - (2_* - 2)/(2_* - q)q]\|(u,v)\|^2_{\mu,a} - C_2\|(u,v)\|^{2_*}_{\mu,a},$$

where $C_2 = [K(\alpha,\beta)]^{-2_*/2}(S_\mu)^{-2_*/2}|h^+|_\infty$ Using the function $l(x) = x(2_* - x)$ and if $N \ge \max(3, 6(a-b+1))$, we obtain that $[(1/2) - (2_* - 2)/(2_* - q)q] > 0$ for $1 < q < 2$. Thus, there exist η, $r > 0$ such that

$$J(u,v) \ge \eta > 0 \quad \text{when } r = \|(u,v)\|_{\mu,a} \text{ is small.}$$

(ii) Let $t > 0$. Then for all $(\phi,\psi) \in \mathcal{N}_\varrho$

$$J(t\phi, t\psi) := (t^2/2)\|(\phi,\psi)\|^2_\mu - (t^{2_*})P(\phi,\psi) - (t^q/q)Q(\phi,\psi).$$

Letting $(\sigma,\omega) = (t\phi, t\psi)$ for t large enough. Since

$$P(\phi,\psi) := \int_\Omega |x|^{-2_* b} h|\phi|^{\alpha+1}|\psi|^{\beta+1}dx > 0,$$

we obtain $J(\sigma,\omega) \le 0$. For t large enough we can ensure $\|(\sigma,\omega)\|_{\mu,a} > r$. □

Let Γ and c defined by

$$\Gamma := \{\gamma : [0,1] \to \mathcal{N}_\varrho : \gamma(0) = (u_0^-, v_0^-), \quad \gamma(1) = (u_0^+, v_0^+)\},$$
$$c := \inf_{\gamma\in\Pi} \max_{t\in[0,1]} (J(\gamma(t))).$$

Proof of Theorem 1.4. If

$$(|\lambda_1||f_1|_\infty)^{1/(2-q)} + (|\lambda_2||f_2|_\infty)^{1/(2-q)} < \min((1/2)\Lambda_0, \Lambda_1),$$

then, by the Lemma 2.3 and Proposition 2.8 (ii), the function J satisfying the Palais-Smale condition on $\mathcal{N}_\varrho$. Moreover, from the Lemmas 2.4, 5.2 and 5.3, there exists (u_c, v_c) such that

$$J(u_c, v_c) = c \quad \text{and} \quad (u_c, v_c) \in \mathcal{N}_\varrho.$$

Thus (u_c, v_c) is the third solution of our system such that $(u_c, v_c) \neq (u_0^+, v_0^+)$ and $(u_c, v_c) \neq (u_0^-, v_0^-)$. Since $(\mathcal{S}_{\lambda_1,\lambda_2,\mu})$ is odd with respect (u, v), we obtain that $(-u_c, -v_c)$ is also a solution of (1.1). □

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