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SOLVABILITY OF SECOND-ORDER BOUNDARY-VALUE PROBLEMS AT RESONANCE INVOLVING INTEGRAL CONDITIONS

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Abstract. This article concerns the second-order differential equation with integral boundary conditions
$$x''(t) = f(t, x(t), x'(t)), \quad t \in (0, 1),$$
$$x(0) = \int_0^1 x(s)d\alpha(s), \quad x(1) = \int_0^1 x(s)d\beta(s).$$
Under the resonance conditions, we construct a projector and then applying coincidence degree theory to establish the existence of solutions.

1. Introduction

We consider the nonlinear second-order differential equation with integral boundary conditions
$$\begin{gathered} x''(t) = f(t, x(t), x'(t)), \quad t \in (0, 1), \\ x(0) = \int_0^1 x(s)d\alpha(s), \quad x(1) = \int_0^1 x(s)d\beta(s), \end{gathered} \tag{1.1}$$
where $f \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$; α and β are right continuous on $[0, 1)$, left continuous at $t = 1$; $\int_0^1 u(s)d\alpha(s)$ and $\int_0^1 u(s)d\beta(s)$ denote the Riemann-Stieltjes integrals of u with respect to α and β, respectively.

The boundary-value problem (1.1) is at resonance in the sense that the associated linear homogeneous boundary-value problem
$$\begin{gathered} x''(t) = 0, \quad t \in (0, 1), \\ x(0) = \int_0^1 x(s)d\alpha(s), \quad x(1) = \int_0^1 x(s)d\beta(s) \end{gathered} \tag{1.2}$$
has nontrivial solutions. The resonance condition is $\kappa_1\kappa_4 - \kappa_2\kappa_3 = 0$, where
$$\kappa_1 = 1 - \int_0^1 (1 - t)d\alpha(t), \quad \kappa_2 = \int_0^1 t d\alpha(t),$$
$$\kappa_3 = \int_0^1 (1 - t)d\beta(t), \quad \kappa_4 = 1 - \int_0^1 t d\beta(t).$$

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Boundary value problems with integral boundary conditions for ordinary differential equations arise in different fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. Moreover, boundary-value problems with Riemann-Stieltjes integral conditions constitute a very interesting and important class of problems. They include two, three, multi-point and integral boundary-value problems as special cases, see [2, 3, 8, 9]. The existence and multiplicity of solutions for such problems have received a great deal of attention in the literature. We refer the reader to [10, 11, 12, 14] for some recent results at non-resonance and to [1, 4, 5, 13, 15] at resonance. Zhang, Feng and Ge [13] obtained some excellent results for certain integral boundary conditions at resonance with dim ker$L = 2$. Zhao and Liang [15] studied the following second-order functional boundary-value problem

$$x''(t) = f(t, x(t), x'(t)), \quad t \in (0, 1),$$
$$\Gamma_1(x) = 0, \quad \Gamma_2(x) = 0,$$

where $\Gamma_1, \Gamma_2 : C^1[0, 1] \to \mathbb{R}$ are continuous linear functionals. We should note that all boundary-value conditions in the work of Zhao and Liang are relied on both x and x'. By using the Mawhin's continuation theorem [6, 7], some existence results were obtained when certain resonance conditions hold. However, integral boundary-value problem is so complex that many problem still remain open. One problem is that all known results about resonance problem were done under special resonance conditions. For example, the known works referred to (1.1), concentrate on the resonance condition that at least three constants of $\{\kappa_i\}_{i=1}^4$ is equals to 0, see [15].

Motivated by all the above works, we give some sufficient conditions for the existence of solutions to (1.1) at resonance. Our method is based upon the coincidence degree theory of Mawhin [6, 7].

Throughout this paper, we suppose that $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ satisfy

(H0) $\kappa_1\kappa_2\kappa_3\kappa_4 \neq 0$; $\kappa_1\kappa_4 - \kappa_2\kappa_3 = 0$.

2. Preliminaries

In this section, we provide some definitions and lemmas used for establishing the existence of solutions in $C^1[0, 1]$.

Definition 2.1. Let Y, Z be real Banach spaces, $L : Y \supset \operatorname{dom} L \to Z$ be a linear operator. L is said to be the Fredholm operator of index zero provided that

(i) Im L is a closed subset of Z,
(ii) $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$.

Let Y, Z be real Banach spaces and $L : Y \supset \operatorname{dom} L \to Z$ be a Fredholm operator of index zero. $P : Y \to Y$, $Q : Z \to Z$ are continuous projectors such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$, $Y = \ker L \oplus \ker P$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is reversible. We denote the inverse of the mapping by K_P (generalized inverse operator of L). If Ω is an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the mapping $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to Y$ is compact.

Our main tools are [6, Theorem 2.4] and [7, Theorem IV.13].

Theorem 2.2. *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$.*
(ii) *$Nx \notin ImL$ for every $x \in \ker L \cap \partial\Omega$.*
(iii) *$\deg(QN|_{\ker L}, \ker L \cap \Omega, 0) \neq 0$, where $Q : Z \to Z$ is a projector as above with $\operatorname{Im} L = \ker Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

We use the classical spaces $C[0,1]$, $C^1[0,1]$ and $L^1[0,1]$. For $x \in C^1[0,1]$, we use the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. And denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We also use the Sobolev space $W^{2,1}(0,1)$ defined by

$$W^{2,1}(0,1) = \{x : [0,1] \to \mathbb{R} \mid x, x' \text{ are absolutely cont.on } [0,1],\ x'' \in L^1[0,1]\}$$

with its usual norm.

Let $Y = C^1[0,1]$, $Z = L^1[0,1]$. Let the linear operator $L : Y \supset \operatorname{dom} L \to Z$ with

$$\operatorname{dom} L = \{x \in W^{2,1}(0,1) : u(0) = \int_0^1 u(s)d\alpha(s),\ u(1) = \int_0^1 u(s)d\beta(s)\}$$

be define by $Lx = x''$. Let the nonlinear operator $N : Y \to Z$ be defined by

$$(Nx)(t) = f(t, x(t), x'(t)).$$

Then (1.1) can be written as

$$Lx = Nx.$$

Lemma 2.3. *Let L be the linear operator defined as above. If* (H0) *holds then*

$$\ker L = \{x \in \operatorname{dom} L : c(1 + (\rho - 1)t),\ c \in \mathbb{R},\ t \in [0,1]\}$$

and

$$\operatorname{Im} L = \{y \in Z : \kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\},$$

where $\rho = \kappa_3/\kappa_4 = \kappa_1/\kappa_2$, and

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Proof. Let $x(t) = 1 + (\rho - 1)t$. Considering $\rho = \kappa_3/\kappa_4 = \kappa_1/\kappa_2$, $\int_0^1 x(t)d\alpha(t) = \int_0^1((1-t) + \rho t)d\alpha(t) = 1 - \kappa_1 + \rho\kappa_2 = 1 = x(0)$ and $\int_0^1 x(t)d\beta(t) = \int_0^1((1-t) + \rho t)d\beta(t) = \kappa_3 + \rho(1-\kappa_4) = \rho = x(1)$. So

$$\{x \in \operatorname{dom} L : c(1 + (\rho - 1)t),\ c \in \mathbb{R},;\ t \in [0,1]\} \subset \ker L.$$

If $Lx = x'' = 0$, then $x(t) = a(1-t) + bt$. Considering $x(0) = \int_0^1 u(t)d\alpha(t)$ and $x(1) = \int_0^1 x(t)d\beta(t)$, we can obtain that $a = \int_0^1 x(t)d\alpha(t) = \int_0^1 (a(1-t)+bt)d\alpha(t) = a(1-\kappa_1) + b\kappa_2$. It yields $a\kappa_1 = b\kappa_2$ and $\ker L \subset \{x \in \text{domL} : c(1 + (\rho - 1)t),\ c \in \mathbb{R},\ t \in [0,1]\}$.

We now show that

$$\operatorname{Im} L = \{y \in Z : \kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\}.$$

If $y \in \operatorname{Im} L$, then there exists $x \in \operatorname{dom} L$ such that $x''(t) = y(t)$. Hence

$$x(t) = -\int_0^1 k(t,s)y(s)ds + x(0)(1-t) + x(1)t.$$

Integrating with respect to $d\alpha(t)$ and $d\beta(t)$ respectively on $[0,1]$ gives

$$\int_0^1 x(t)d\alpha(t) = -\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + x(0)(1-\kappa_1) + x(1)\kappa_2$$

and

$$\int_0^1 x(t)d\beta(t) = -\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t) + x(0)\kappa_3 + x(1)(1-\kappa_4).$$

Therefore,

$$\begin{pmatrix} \kappa_1 & -\kappa_2 \\ -\kappa_3 & \kappa_4 \end{pmatrix}\begin{pmatrix} x(0) \\ x(1) \end{pmatrix} = \begin{pmatrix} -\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) \\ -\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t) \end{pmatrix}$$

and so

$$\kappa_1 : (-\kappa_3) = (-\kappa_2) : \kappa_4 = \int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) : \int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t).$$

It yields

$$\operatorname{Im} L \subset \{y \in Z : \kappa_3\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\}.$$

On the other hand, $y \in Z$ satisfies

$$\kappa_3\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0.$$

Let

$$x(t) = -\int_0^1 k(t,s)y(s)ds + \frac{t}{\kappa_2}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t),$$

then $Lx = x'' = y(t)$, $x(0) = 0$ and $x(1) = \frac{1}{\kappa_2}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t)$. Simple computations yield

$$\int_0^1 x(t)d\alpha(t) = -\int_0^1\int_0^1 k(t,s)y(s)ds\alpha(t) + \int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) = 0$$

and

$$\begin{aligned}\int_0^1 x(t)d\beta(t) &= -\int_0^1\int_0^1 k(t,s)y(s)ds\beta(t) + \frac{1-\kappa_4}{\kappa_2}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t)\\ &= \frac{\kappa_3}{\kappa_1}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \frac{1-\kappa_4}{\kappa_2}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t)\\ &= \frac{\kappa_4}{\kappa_2}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \frac{1-\kappa_4}{\kappa_2}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t)\\ &= \frac{1}{\kappa_2}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) = x(1).\end{aligned}$$

Therefore,

$$\{y \in Z : \kappa_3\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\} \subset \operatorname{Im} L.$$

□

Lemma 2.4. *If* (H0) *holds and*

$$\kappa = \frac{\kappa_3}{2}\int_0^1 t(1-t)d\alpha(t) + \frac{\kappa_1}{2}\int_0^1 t(1-t)d\beta(t) \neq 0,$$

then L *is a Fredholm operator of index zero and* $\dim\ker L = \operatorname{codim}\operatorname{Im} L = 1$. *Furthermore, the linear operator* $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *can be defined by*

$$(K_p y)(t) = -\int_0^1 k(t,s)y(s)ds - \frac{t}{\kappa_4}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t).$$

Also

$$\|K_p y\| \leq \triangle \|y\|_1, \quad \text{for all } y \in \operatorname{Im} L,$$

where

$$\triangle = 1 + \frac{\left|\int_0^1 t d|\beta(t)|\right|}{|\kappa_4|}.$$

Proof. Firstly, we construct the mapping $Q : Z \to Z$ defined by

$$Qy = \frac{1}{\kappa}\big(\kappa_3 \int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t)\big).$$

Note that $\int_0^1 k(t,s)ds = \frac{1}{2}t(1-t)$ and

$$Q^2 y = Qy.$$

Thus $Q : Z \to Z$ is a well-defined projector.

Now, it is obvious that $\operatorname{Im} L = \ker Q$. Noting that Q is a linear projector, we have $Z = \operatorname{Im} Q \oplus \ker Q$. Hence $Z = \operatorname{Im} Q \oplus \operatorname{Im} L$ and $\dim\ker L = \operatorname{codim}\operatorname{Im} L$ =1. This means L is a Fredholm mapping of index zero. Taking $P : Y \to Y$ as

$$(Px)(t) = x(0)(1 + (\rho - 1)t),$$

then the generalized inverse $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of L can be rewritten

$$(K_p y)(t) = -\int_0^1 k(t,s)y(s)ds - \frac{t}{\kappa_4}\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t).$$

In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_p)y(t) = ((K_p)y(t))'' = y(t)$$

and for $x \in \operatorname{dom} L \cap \ker P$, we know

$$\begin{aligned}(K_p L)x(t) &= -\int_0^1 k(t,s)x''(s)ds - \frac{t}{\kappa_4}\int_0^1\int_0^1 k(t,s)x''(s)\,ds\,d\beta(t)\\ &= x(t) - x(0)(1-t) - x(1)t\\ &\quad + \frac{t}{\kappa_4}\int_0^1 (x(t) - x(0)(1-t) - x(1)t)d\beta(t).\end{aligned}$$

In view of $x \in \operatorname{dom} L \cap \ker P$, $x(0) = 0, x(1) = \int_0^1 x(t)d\beta(t)$, thus

$$(K_p L)x(t) = x(t).$$

This shows that $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. Since

$$\|K_p y\|_\infty \leq \int_0^1 |y(s)|ds + \frac{1}{|\kappa_4|}\Big|\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t)\Big| \leq \triangle \|y\|_1$$

and

$$\|(K_py)'\| \le \int_0^1 |y(s)|ds + \frac{1}{|\kappa_4|}\Big|\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t)\Big| \le \triangle\|y\|_1,$$

it follows that $\|(K_py)'\|_\infty \le \triangle\|y\|_1$. □

3. Main results

In this section, we will use Theorem 2.2 to prove the existence of solutions to (1.1). For the next theorem we use the assumptions:

(H1) There exist functions $p, q, \gamma \in L^1[0,1]$, such that for all $(x,y)\in\mathbb{R}^2$ and $t\in[0,1]$,

$$|f(t,x,y)| \le p(t)|x| + q(t)|y| + \gamma(t);$$

(H2) There exists a constant $A>0$ such that for $x\in \operatorname{dom} L$, if $|x(t)|>A$ or $|x'(t)|>A$ for all $t\in[0,1]$, then

$$QN(x(t)) \neq 0;$$

(H3) There exists a constant $B>0$ such that for $a\in\mathbb{R}$, if $|a|>B$, then either

$$aQN(a(1+(\rho-1)t))>0, \quad \text{or} \quad aQN(a(1+(\rho-1)t))<0.$$

Theorem 3.1. *Let* (H0)–(H3) *hold and* $\kappa\neq 0$. *Then* (1.1) *has at least one solution in* $C^1[0,1]$, *provided*

$$\|p\|_1 + \|q\|_1 < \frac{1}{1+|\rho-1|+\triangle},$$

where $\triangle$ *is the same as Lemma 2.4.*

Proof. Set

$$\Omega_1 = \{x\in \operatorname{dom} L\setminus \ker L : Lx = \lambda Nx \text{ for some } \lambda\in[0,1]\}.$$

For $x\in\Omega_1$, since $Lx=\lambda Nx$, so $\lambda\neq 0$, $Nx\in \operatorname{Im} L$, hence

$$QN(x(t)) = 0.$$

Thus, from (H2), there exist $t_0,t_1\in[0,1]$ such that $|x(t_0)|\le A$, $|x'(t_1)|\le A$. Since x, x' are absolutely continuous for all $t\in[0,1]$,

$$|x'(t)| = |x'(t_1) - \int_t^{t_1} x''(s)ds \le |x'(t_1)| + \|x''\|_1 \le A + \|Nx\|_1,$$

$$|x(0)| = |x(t_0) - \int_0^{t_0} x'(s)ds| \le |x(t_0)| + t_0(A+\|Nx\|_1) \le 2A + \|Nx\|_1.$$

Thus

$$\|Px\| \le |x(0)|(1+|\rho-1|) \le (1+|\rho-1|)(2A+\|Nx\|_1). \tag{3.1}$$

Also for $x\in\Omega_1$, $x\in\operatorname{dom} L\setminus\ker L$, then $(I-P)x\in \operatorname{dom} L\cap\ker L$, $LPx=0$, thus from Lemma 2.4, we have

$$\|(I-P)x\| = \|K_PL(I-P)x\| \le \triangle\|L(I-P)x\|_1 = \triangle\|Lx\|_1 \le \triangle\|Nx\|_1. \tag{3.2}$$

By using (3.1) and (3.2), we obtain

$$\|x\| = \|Px + (I-P)x\| \le \|Px\| + \|(I-P)x\| \le 2A(1+|\rho-1|) + (1+|\rho-1|+\triangle)\|Nx\|_1$$

By this and (H1), we have

$$\begin{aligned}\|x\| &\le 2A(1+|\rho-1|) + (1+|\rho-1|+\triangle)(\|\alpha\|_1\|x\|_\infty + \|\beta\|_1\|x'\|_\infty + \|\gamma\|_1)\\ &\le 2A(1+|\rho-1|) + (1+|\rho-1|+\triangle)(\|\alpha\|_1\|x\| + \|\beta\|_1\|x\| + \|\gamma\|_1),\end{aligned}$$

and

$$\|x\| \le \frac{2A(1+|\rho-1|)+\|\gamma\|_1(1+|\rho-1|+\triangle)}{1-(1+|\rho-1|+\triangle)(\|\alpha\|_1+\|\beta\|_1)}.$$

Therefore, Ω_1 is bounded. Let

$$\Omega_2=\{x\in\ker L: Nx\in\operatorname{Im} L\}.$$

For $x\in\Omega_2$, $x\in\ker L$ implies that x can be defined by $x=a(1+(\rho-1)t)$, $t\in[0,1]$, $a\in\mathbb{R}$. By (H2), there exist $t_0,t_1\in[0,1]$ such that $|x(t_0)|\le A$, $|x'(t_1)|\le A$, then

$$\|x'\|_\infty=|a(\rho-1)|\le A.$$

Moreover,

$$\|x\|_\infty\le\|x'\|_\infty+A.$$

So $\|x\|\le 2A$. Thus, Ω_2 is bounded.

Next, according to the condition (H3), for any $a\in\mathbb{R}$, if $|a|>B$, then either

$$aQN(a(1+(\rho-1)t)>0, \tag{3.3}$$

or

$$aQN(a(1+(\rho-1)t)<0. \tag{3.4}$$

When (3.3) holds, set

$$\Omega_3=\{x\in\ker L:\lambda Jx+(1-\lambda)QNx=0,\ \lambda\in[0,1]\},$$

where $J:\ker L\to\operatorname{Im} Q$ is the linear isomorphism given by $J(a(1+(\rho-1)t))=a$, for all $a\in\mathbb{R}$. Since for any $x=a(1+(\rho-1)t)$, we have

$$\lambda a=-(1-\lambda)QN(a(1+(\rho-1)t),$$

if $\lambda=1$, then $a=0$. Otherwise, if $|a|>B$, in view of (3.3), we have

$$-(1-\lambda)aQN(a(1+(\rho-1)t)<0,$$

which contradict $\lambda a\ge 0$. Thus Ω_3 is bounded. If (3.4) holds, then let

$$\Omega_3=\{x\in\ker L:-\lambda Jx+(1-\lambda)QNx=0,\ \lambda\in[0,1]\}.$$

By the same method as above, we obtain that Ω_3 is bounded.

In the following, we shall prove that the all conditions of Theorem 2.2 are satisfied. Set Ω be a bound open subset of Y such that $\cup_{i=1}^{3}\overline{\Omega_i}\subset\Omega$. By using the Ascoli-Arzela theorem, we can prove that $K_P(I-Q)N:\Omega\to Y$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by the above argument we have

(i) $Lx\ne\lambda Nx$, for every $(x,\lambda)\in[(\operatorname{dom}L\setminus\ker L)\cap\partial\Omega]\times(0,1)$,
(ii) $Nx\notin\operatorname{Im}L$ for $x\in\ker L\cap\partial\Omega$.

At last we will prove that (iii) of Theorem 2.2 is satisfied. Let $H(t,\lambda)=\pm\lambda Jx+(1-\lambda)QNx$. According to above argument, we know

$$H(t,\lambda)\ne 0 \text{ for } x\in\ker L\cap\partial\Omega,$$

thus, by the homotopy property of degree

$$\begin{aligned}\deg(QN|_{\ker L},\ker L\cap\Omega,0)&=\deg(H(\cdot,0),\ker L\cap\Omega,0)\\&=\deg(H(\cdot,1),\ker L\cap\Omega,0)\\&=\deg(\pm J,\ker L\cap\Omega,0)\ne 0.\end{aligned}$$

Then by Theorem 2.2, $Lx=Nx$ has at least one solution in $\operatorname{dom}L\cap\overline{\Omega}$, so that (1.1) has solution in $C^1[0,1]$. □

To illustrate our main results we present an example. Consider the boundary-value problem

$$
\begin{gathered}
x'' = \cos t - 1 + \frac{1}{7}\sin x + \frac{1}{12}(|x| + |x'|), \quad t \in (0,1), \\
x(0) = -\frac{1}{2}x(\frac{1}{6}) + 2x(\frac{1}{2}),\ x(1) = \frac{5}{8}\int_0^1 x(s)ds.
\end{gathered}
$$

Let

$$
\begin{gathered}
f(t,x,y) = \cos t - 1 + \frac{1}{7}\sin x + \frac{1}{12}(|x| + |y|), \quad \beta(t) = \frac{5}{8}t, \\
\alpha(t) = \begin{cases} 0 & t \in [0, \frac{1}{6}), \\ -\frac{1}{2} & t \in [\frac{1}{6}, \frac{1}{2}), \\ \frac{3}{2} & t \in [\frac{1}{2}, 1]. \end{cases}
\end{gathered}
$$

then

$$
\begin{gathered}
|f(t,x,y)| \le \frac{19}{84}|x| + \frac{1}{12}|y| + 2, \quad \kappa_1 = \frac{5}{12}, \quad \kappa_2 = \frac{11}{12}, \quad \kappa_3 = \frac{5}{16}, \\
\kappa_4 = \frac{11}{16}, \quad \kappa = \frac{205}{2304}, \quad \rho = \frac{5}{11}, \quad \triangle = \frac{16}{11}.
\end{gathered}
$$

Again taking $p = \frac{19}{84}$ and $q = \frac{1}{12}$, we have

$$
\|p\|_1 + \|q\|_1 = \frac{19}{84} + \frac{1}{12} = \frac{13}{42} < \frac{1}{3} = \frac{1}{1 + |\rho - 1| + \triangle}.
$$

Finally taking $A = 36$. So, as $|x(t)| \ge 36$ or $|x'(t)| \ge 36$, we have $f(t, x(t), x'(t)) > 0$. Therefore,

$$
\begin{aligned}
&QN(x(t)) \\
&= \frac{\kappa_3}{\kappa}\int_0^1\int_0^1 k(t,s)f(s,x(s),x'(s))\,ds\,d\alpha(t) \\
&\quad + \frac{\kappa_1}{\kappa}\int_0^1\int_0^1 k(t,s)f(s,x(s),x'(s))\,ds\,d\beta(t) \\
&> \frac{\kappa_3}{\kappa}\int_0^1\int_0^1 k(t,s)f(s,x(s),x'(s))\,ds\,d\alpha(t) \\
&= \frac{\kappa_3}{\kappa}\Big(-\frac{1}{2}\int_0^1 k(\frac{1}{6},s)f(s,x(s),x'(s))ds + 2\int_0^1 k(\frac{1}{2},s)f(s,x(s),x'(s))ds\Big) \\
&\ge \frac{\kappa_3}{\kappa}\Big(2\int_0^1 \frac{1}{2}(1-\frac{1}{2})s(1-s)f(s,x(s),x'(s))ds \\
&\quad - \frac{1}{2}\int_0^1 s(1-s)f(s,x(s),x'(s))ds\Big) = 0.
\end{aligned}
$$

Thus condition (H2) holds. Again taking $B = 50$, for any $a \in \mathbb{R}$, when $|a| > 50$, we have $N(a(1 + (\rho - 1)t)) > 0$. So condition (H3) holds. Hence from Theorem 3.1, BVP (1.1) has at least one solution $x \in C^1[0,1]$.

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Addendum posted on March 30, 2012

In response to the comments from a reader, the author wanted to make several corrections and add references [16]–[22] to the original article. However, due to the large number of corrections, the editors decided to attached a revised version of all the sections at the end of the article, and to keep original sections for historical purposes.

4. Introduction

We consider the nonlinear second-order differential equation with integral boundary conditions

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)), \quad t \in (0,1), \\ x(0) = \int_0^1 x(s)d\alpha(s), \quad x(1) = \int_0^1 x(s)d\beta(s), \end{gathered} \tag{4.1}$$

where $f \in C([0,1]\times\mathbb{R}^2, \mathbb{R})$; α and β are functions of bounded variation; $\int_0^1 u(s)d\alpha(s)$ and $\int_0^1 u(s)d\beta(s)$ denote the Riemann-Stieltjes integrals of u with respect to α and β, respectively.

The boundary-value problem (4.1) is at resonance in the sense that the associated linear homogeneous boundary-value problem

$$\begin{gathered} x''(t) = 0, \quad t \in (0,1), \\ x(0) = \int_0^1 x(s)d\alpha(s), \quad x(1) = \int_0^1 x(s)d\beta(s) \end{gathered} \tag{4.2}$$

has nontrivial solutions. The resonance condition is $\kappa_1\kappa_4 - \kappa_2\kappa_3 = 0$, where

$$\begin{gathered} \kappa_1 = 1 - \int_0^1 (1-t)d\alpha(t), \quad \kappa_2 = \int_0^1 t d\alpha(t), \\ \kappa_3 = \int_0^1 (1-t)d\beta(t), \quad \kappa_4 = 1 - \int_0^1 t d\beta(t). \end{gathered}$$

Boundary value problems with integral boundary conditions for ordinary differential equations arise in different fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. Moreover, boundary-value problems with Riemann-Stieltjes integral conditions constitute a very interesting and important class of problems. They include two, three, multi-point and integral boundary-value problems as special cases, see [2, 3, 8, 9]. The existence and multiplicity of solutions for such problems have received a great deal of attention in the literature.

We refer the reader to [8, 9, 10, 11, 12, 14, 19] for some recent results at nonresonance and to [1, 4, 5, 13, 15, 16, 17, 18, 19, 20, 21, 22] at resonance. Zhang, Feng and Ge [13] obtained some excellent results for certain integral boundary conditions at resonance with dim $\ker L = 2$. Zhao and Liang [15] studied the following second-order functional boundary-value problem

$$x''(t) = f(t, x(t), x'(t)), \quad t \in (0,1),$$

$$\Gamma_1(x) = 0, \quad \Gamma_2(x) = 0,$$

where $\Gamma_1, \Gamma_2 : C^1[0,1] \to \mathbb{R}$ are continuous linear functionals.

We should note that all boundary-value conditions in the work of Zhao and Liang are relied on both x and x'. By using the Mawhin's continuation theorem [6, 7], some existence results were obtained when certain resonance conditions hold. However, the work of Zhao and Liang concentrate on the resonance condition that at least two constants of $\{\kappa_i\}_{i=1}^4$ is equals to 0. In particular, there has been no work done for (4.1) under the resonance condition

$$\kappa_1\kappa_2\kappa_3\kappa_4 \neq 0;\ \kappa_1\kappa_4 - \kappa_2\kappa_3 = 0.$$

Motivated by all the above works, we give some sufficient conditions for the existence of solutions to (4.1) at resonance. Our method is based upon the coincidence degree theory of Mawhin [6, 7].

Throughout this paper, we suppose that $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ satisfy

(H0) $\kappa_1\kappa_2\kappa_3\kappa_4 \neq 0$; $\kappa_1\kappa_4 - \kappa_2\kappa_3 = 0$.

5. Preliminaries

In this section, we provide some definitions and lemmas used for establishing the existence of solutions in $C^1[0,1]$.

Definition 5.1. Let Y, Z be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator. L is said to be the Fredholm operator of index zero provided that

(i) $\operatorname{Im} L$ is a closed subset of Z,
(ii) $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$.

Let Y, Z be real Banach spaces and $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm operator of index zero. $P : Y \to Y$, $Q : Z \to Z$ are continuous projectors such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$, $Y = \ker L \oplus \ker P$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L\cap\ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible. We denote the inverse of the mapping by K_P (generalized inverse operator of L). If Ω is an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the mapping $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

Our main tools are [6, Theorem 2.4] and [7, Theorem IV.13].

Theorem 5.2. *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L\setminus \ker L) \cap \partial\Omega] \times (0,1)$.*
(ii) *$Nx \notin ImL$ for every $x \in \ker L \cap \partial\Omega$.*
(iii) *$\deg(QN|_{\ker L}, \ker L \cap \Omega, 0) \neq 0$, where $Q : Z \to Z$ is a projector as above with $\operatorname{Im} L = \ker Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

We use the classical spaces $C[0,1]$, $C^1[0,1]$ and $L^1[0,1]$. For $x \in C^1[0,1]$, we use the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. And denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We also use the Sobolev space $W^{2,1}(0,1)$ defined by

$$W^{2,1}(0,1) = \{x : [0,1] \to \mathbb{R} \mid x, x' \text{ are absolutely cont.on } [0,1],\ x'' \in L^1[0,1]\}$$

with its usual norm.

Let $Y = C^1[0,1]$, $Z = L^1[0,1]$. Let the linear operator $L : \operatorname{dom} L \subset Y \to Z$ with

$$\operatorname{dom} L = \{x \in W^{2,1}(0,1) : u(0) = \int_0^1 u(s)d\alpha(s),\ u(1) = \int_0^1 u(s)d\beta(s)\}$$

be define by $Lx = x''$. Let the nonlinear operator $N : Y \to Z$ be defined by

$$(Nx)(t) = f(t, x(t), x'(t)).$$

Then (4.1) can be written as

$$Lx = Nx.$$

Lemma 5.3. *Let L be the linear operator defined as above. Then*

$$\ker L = \{x \in \operatorname{dom} L : c(1 + (\rho - 1)t),\ c \in \mathbb{R},\ t \in [0,1]\}$$

and

$$\operatorname{Im} L = \{y \in Z : \kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\},$$

where $\rho = \kappa_3/\kappa_4 = \kappa_1/\kappa_2$, and

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Proof. Let $x(t) = 1 + (\rho - 1)t$. Considering $\rho = \kappa_3/\kappa_4 = \kappa_1/\kappa_2$,

$$\int_0^1 x(t)d\alpha(t) = \int_0^1 ((1-t) + \rho t)d\alpha(t) = 1 - \kappa_1 + \rho\kappa_2 = 1 = x(0)$$

and $\int_0^1 x(t)d\beta(t) = \int_0^1 ((1-t) + \rho t)d\beta(t) = \kappa_3 + \rho(1 - \kappa_4) = \rho = x(1)$. So

$$\{x \in \operatorname{dom} L : c(1 + (\rho - 1)t),\ c \in \mathbb{R},\ t \in [0,1]\} \subset \ker L.$$

If $Lx = x'' = 0$, then $x(t) = a(1-t) + bt$. Considering $x(0) = \int_0^1 u(t)d\alpha(t)$ and $x(1) = \int_0^1 x(t)d\beta(t)$, we can obtain that

$$a = \int_0^1 x(t)d\alpha(t) = \int_0^1 (a(1-t) + bt)d\alpha(t) = a(1 - \kappa_1) + b\kappa_2.$$

It yields $a\kappa_1 = b\kappa_2$ and $\ker L \subset \{x \in \mathrm{domL} : c(1 + (\rho - 1)t),\ c \in \mathbb{R},\ t \in [0,1]\}$.

We now show that

$$\operatorname{Im} L = \{y \in Z : \kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\}.$$

If $y \in \operatorname{Im} L$, then there exists $x \in \operatorname{dom} L$ such that $x''(t) = y(t)$. Hence

$$x(t) = -\int_0^1 k(t,s)y(s)ds + x(0)(1-t) + x(1)t.$$

Integrating with respect to $d\alpha(t)$ and $d\beta(t)$ respectively on $[0,1]$ gives

$$\int_0^1 x(t)d\alpha(t) = -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + x(0)(1 - \kappa_1) + x(1)\kappa_2$$

and

$$\int_0^1 x(t)d\beta(t) = -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) + x(0)\kappa_3 + x(1)(1 - \kappa_4).$$

Therefore,

$$\begin{pmatrix} \kappa_1 & -\kappa_2 \\ -\kappa_3 & \kappa_4 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \end{pmatrix} = \begin{pmatrix} -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) \\ -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) \end{pmatrix}$$

and so

$$-\frac{\kappa_1}{\kappa_3} = -\frac{\kappa_2}{\kappa_4} = \frac{\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t)}{\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t)}.$$

It yields

$$\operatorname{Im} L \subset \{y \in Z : \kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\}.$$

On the other hand, suppose $y \in Z$ satisfies

$$\kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0.$$

Let

$$x(t) = -\int_0^1 k(t,s)y(s)ds + \frac{t}{\kappa_2} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t),$$

then $Lx = x'' = y(t)$, $x(0) = 0$ and $x(1) = \frac{1}{\kappa_2} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t)$. Simple computations yield

$$\int_0^1 x(t)d\alpha(t) = -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) = 0$$

and

$$\begin{aligned} \int_0^1 x(t)d\beta(t) &= -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) + \frac{1-\kappa_4}{\kappa_2} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) \\ &= \frac{\kappa_3}{\kappa_1} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \frac{1-\kappa_4}{\kappa_2} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) \\ &= \frac{\kappa_4}{\kappa_2} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \frac{1-\kappa_4}{\kappa_2} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) \\ &= \frac{1}{\kappa_2} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) = x(1). \end{aligned}$$

Therefore,

$$\{y \in Z : \kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = 0\} \subset \operatorname{Im} L.$$

□

Lemma 5.4. *If*

$$\kappa = \frac{\kappa_3}{2} \int_0^1 t(1-t)d\alpha(t) + \frac{\kappa_1}{2} \int_0^1 t(1-t)d\beta(t) \neq 0,$$

then L is a Fredholm operator of index zero and $\dim\ker L = \operatorname{codim}\operatorname{Im} L = 1$. *Furthermore, the linear operator $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ can be defined by*

$$(K_p y)(t) = -\int_0^1 k(t,s)y(s)ds - \frac{t}{\kappa_4} \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t).$$

Also

$$\|K_p y\| \le \triangle \|y\|_1, \quad \textit{for all } y \in \operatorname{Im} L,$$

where

$$\triangle = 1 + \frac{\big|\int_0^1 t d|\beta(t)|\big|}{|\kappa_4|}.$$

Proof. Firstly, we construct the mapping $Q : Z \to Z$ defined by

$$Qy = \frac{1}{\kappa}\big(\kappa_3 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) + \kappa_1 \int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t)\big).$$

Note that $\int_0^1 k(t,s)ds = \frac{1}{2}t(1-t)$ and

$$Q^2 y = Qy.$$

Thus $Q : Z \to Z$ is a well-defined projector.

Now, it is obvious that $\operatorname{Im} L = \ker Q$. Noting that Q is a linear projector, we have $Z = \operatorname{Im} Q \oplus \ker Q$. Hence $Z = \operatorname{Im} Q \oplus \operatorname{Im} L$ and $\dim\ker L = \operatorname{codim} \operatorname{Im} L$ =1. This means L is a Fredholm mapping of index zero. Taking $P : Y \to Y$ as

$$(Px)(t) = x(0)(1 + (\rho - 1)t),$$

then the generalized inverse $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of L can be rewritten

$$(K_p y)(t) = -\int_0^1 k(t,s)y(s)ds - \frac{t}{\kappa_4}\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t).$$

In fact, for $y \in \operatorname{Im} L$, we have

$$\begin{aligned}
\int_0^1 (K_p y)(t)d\alpha(t) &= -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) - \frac{\kappa_2}{\kappa_4}\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) \\
&= -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\alpha(t) - \frac{\kappa_1}{\kappa_3}\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) \\
&= 0 = (K_p y)(0)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 (K_p y)(t)d\beta(t) &= -\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) - \frac{1-\kappa_4}{\kappa_4}\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) \\
&= -\frac{1}{\kappa_4}\int_0^1 \int_0^1 k(t,s)y(s)\,ds\,d\beta(t) = (K_p y)(1)
\end{aligned}$$

which implies that K_p is well defined on $\operatorname{Im} L$. Moreover, for $y \in \operatorname{Im} L$, we have

$$(LK_p)y(t) = ((K_p y)(t))'' = y(t)$$

and for $x \in \operatorname{dom} L \cap \ker P$, we know

$$\begin{aligned}
(K_p L)x(t) &= -\int_0^1 k(t,s)x''(s)ds - \frac{t}{\kappa_4}\int_0^1 \int_0^1 k(t,s)x''(s)\,ds\,d\beta(t) \\
&= x(t) - x(0)(1-t) - x(1)t \\
&\quad + \frac{t}{\kappa_4}\int_0^1 (x(t) - x(0)(1-t) - x(1)t)d\beta(t).
\end{aligned}$$

In view of $x \in \operatorname{dom} L \cap \ker P$, $x(0) = 0, x(1) = \int_0^1 x(t)d\beta(t)$, thus

$$(K_p L)x(t) = x(t).$$

This shows that $K_P = (L|_{\operatorname{dom} L\cap \ker P})^{-1}$. Since

$$\|K_p y\|_\infty \le \int_0^1 |y(s)|ds + \frac{1}{|\kappa_4|}\Big|\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t)\Big| \le \triangle \|y\|_1$$

and

$$\|(K_p y)'\|_\infty \le \int_0^1 |y(s)|ds + \frac{1}{|\kappa_4|}\Big|\int_0^1\int_0^1 k(t,s)y(s)\,ds\,d\beta(t)\Big| \le \triangle \|y\|_1,$$

it follows that $\|K_p y\| \le \triangle \|y\|_1$. □

6. Main results

In this section, we will use Theorem 5.2 to prove the existence of solutions to (4.1). For the next theorem we use the assumptions:

(H1) There exist functions $p, q, \gamma \in L^1[0,1]$, such that for all $(x, y) \in \mathbb{R}^2$ and $t \in [0,1]$,
$$|f(t,x,y)| \le p(t)|x| + q(t)|y| + \gamma(t);$$
(H2) There exists a constant $A > 0$ such that for $x \in \operatorname{dom} L$, if $|x(t)| > A$ or $|x'(t)| > A$ for all $t \in [0,1]$, then
$$QN(x(t)) \neq 0;$$
(H3) There exists a constant $B > 0$ such that for $a \in \mathbb{R}$, if $|a| > B$, then either
$$aQN(a(1+(\rho-1)t)) > 0, \quad \text{or} \quad aQN(a(1+(\rho-1)t)) < 0.$$

Theorem 6.1. *Let* (H1)–(H3) *hold and* $\kappa \neq 0$. *Then* (4.1) *has at least one solution in* $C^1[0,1]$*, provided*

$$\|p\|_1 + \|q\|_1 < \frac{1}{\delta + \triangle},$$

where $\delta = max\{1, |\rho|, |\rho - 1|\}$ *and* $\triangle$ *is the same as Lemma 5.4.*

Proof. Set

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\}.$$

For $x \in \Omega_1$, since $Lx = \lambda Nx$, so $\lambda \neq 0$, $Nx \in \operatorname{Im} L$, hence

$$QN(x(t)) = 0.$$

Thus, from (H2), there exist $t_0, t_1 \in [0,1]$ such that $|x(t_0)| \le A$, $|x'(t_1)| \le A$. Since x, x' are absolutely continuous for all $t \in [0,1]$,

$$|x'(t)| = |x'(t_1) - \int_t^{t_1} x''(s)ds| \le |x'(t_1)| + \|x''\|_1 \le A + \|Nx\|_1,$$

$$|x(0)| = |x(t_0) - \int_0^{t_0} x'(s)ds| \le |x(t_0)| + t_0(A + \|Nx\|_1) \le 2A + \|Nx\|_1.$$

Thus

$$\|Px\| = \max\{\|Px\|_\infty, \|(Px)'\|_\infty\} \le \delta|x(0)| \le \delta(2A + \|Nx\|_1). \tag{6.1}$$

Also for $x \in \Omega_1$, $x \in \operatorname{dom} L\setminus \ker L$, then $(I-P)x \in \operatorname{dom} L \cap \ker L$, $LPx = 0$, thus from Lemma 5.4, we have

$$\|(I-P)x\| = \|K_P L(I-P)x\| \le \triangle\|L(I-P)x\|_1 = \triangle\|Lx\|_1 \le \triangle\|Nx\|_1. \tag{6.2}$$

By using (6.1) and (6.2), we obtain

$$\|x\| = \|Px + (I-P)x\| \le \|Px\| + \|(I-P)x\| \le 2A\delta + (\delta + \triangle)\|Nx\|_1$$

By this and (H1), we have

$$\begin{aligned}\|x\| &\le 2A\delta + (\delta+\triangle)(\|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|\gamma\|_1)\\ &\le 2A\delta + (\delta+\triangle)(\|p\|_1\|x\| + \|q\|_1\|x\| + \|\gamma\|_1),\end{aligned}$$

and

$$\|x\| \le \frac{2A\delta + \|\gamma\|_1(\delta+\triangle)}{1-(\delta+\triangle)(\|p\|_1+\|q\|_1)}.$$

Therefore, Ω_1 is bounded. Let

$$\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}.$$

For $x \in \Omega_2$, $x \in \ker L$ implies that x can be defined by $x = a(1+(\rho-1)t)$, $t \in [0,1]$, $a \in \mathbb{R}$. By (H2), there exist $t_0, t_1 \in [0,1]$ such that $|x(t_0)| \le A$, $|x'(t_1)| \le A$, then

$$\|x'\|_\infty = |a(\rho-1)| \le A.$$

Moreover,

$$\|x\|_\infty \le \|x'\|_\infty + A.$$

So $\|x\| \le 2A$. Thus, Ω_2 is bounded.

Next, according to the condition (H3), for any $a \in \mathbb{R}$, if $|a| > B$, then either

$$aQN(a(1+(\rho-1)t) > 0, \tag{6.3}$$

or

$$aQN(a(1+(\rho-1)t) < 0. \tag{6.4}$$

When (6.3) holds, set

$$\Omega_3 = \{x \in \ker L : \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by $J(a(1+(\rho-1)t)) = a$, for all $a \in \mathbb{R}$. Since for any $x = a(1+(\rho-1)t)$, we have

$$\lambda a = -(1-\lambda)QN(a(1+(\rho-1)t),$$

if $\lambda = 1$, then $a = 0$. Otherwise, if $|a| > B$, in view of (6.3), we have

$$-(1-\lambda)aQN(a(1+(\rho-1)t) < 0,$$

which contradict $\lambda a^2 \ge 0$. Thus Ω_3 is bounded. If (6.4) holds, then let

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}.$$

By the same method as above, we obtain that Ω_3 is bounded.

In the following, we shall prove that the all conditions of Theorem 5.2 are satisfied. Set Ω be a bound open subset of Y such that $\cup_{i=1}^3 \overline{\Omega_i} \subset \Omega$. By using the Ascoli-Arzela theorem, we can prove that $K_P(I-Q)N : \Omega \to Y$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by the above argument we have

(i) $Lx \ne \lambda Nx$, for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$,
(ii) $Nx \notin \operatorname{Im} L$ for $x \in \ker L \cap \partial\Omega$.

At last we will prove that (iii) of Theorem 5.2 is satisfied. Let $H(t,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$. According to above argument, we know

$$H(t,\lambda) \ne 0 \text{ for } x \in \ker L \cap \partial\Omega,$$

thus, by the homotopy property of degree

$$\begin{aligned}\deg(QN|_{\ker L}, \ker L \cap \Omega, 0) &= \deg(H(\cdot,0), \ker L \cap \Omega, 0)\\ &= \deg(H(\cdot,1), \ker L \cap \Omega, 0)\end{aligned}$$

$$= \deg(J, \ker L \cap \Omega, 0) \neq 0.$$

Then by Theorem 5.2, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, so that (4.1) has solution in $C^1[0,1]$. □

To illustrate our main results we present an example. Consider the boundary-value problem

$$\begin{gathered} x'' = \cos t - 1 + \frac{1}{7}\sin x + \frac{1}{12}(|x| + |x'|), \quad t \in (0,1), \\ x(0) = -\frac{1}{2}x(\frac{1}{6}) + 2x(\frac{1}{2}),\ x(1) = \frac{5}{8}\int_0^1 x(s)ds. \end{gathered}$$

Let

$$\begin{gathered} f(t,x,y) = \cos t - 1 + \frac{1}{7}\sin x + \frac{1}{12}(|x| + |y|), \quad \beta(t) = \frac{5}{8}t, \\ \alpha(t) = \begin{cases} 0 & t \in [0, \frac{1}{6}), \\ -\frac{1}{2} & t \in [\frac{1}{6}, \frac{1}{2}), \\ \frac{3}{2} & t \in [\frac{1}{2}, 1]. \end{cases} \end{gathered}$$

then

$$\begin{gathered} |f(t,x,y)| \leq \frac{19}{84}|x| + \frac{1}{12}|y| + 2, \quad \kappa_1 = \frac{5}{12}, \quad \kappa_2 = \frac{11}{12}, \quad \kappa_3 = \frac{5}{16}, \\ \kappa_4 = \frac{11}{16}, \quad \kappa = \frac{205}{2304}, \quad \rho = \frac{5}{11}, \quad \delta = 1, \quad \triangle = \frac{16}{11}. \end{gathered}$$

Again taking $p = \frac{19}{84}$ and $q = \frac{1}{12}$, we have

$$\|p\|_1 + \|q\|_1 = \frac{19}{84} + \frac{1}{12} = \frac{13}{42} < \frac{11}{27} = \frac{1}{\delta + \triangle}.$$

Finally taking $A = 36$. So, as $|x(t)| \geq 36$ or $|x'(t)| \geq 36$, we have $f(t, x(t), x'(t)) > 0$. Therefore,

$$\begin{aligned} &QN(x(t)) \\ &= \frac{\kappa_3}{\kappa}\int_0^1\int_0^1 k(t,s)f(s,x(s),x'(s))\,ds\,d\alpha(t) \\ &\quad + \frac{\kappa_1}{\kappa}\int_0^1\int_0^1 k(t,s)f(s,x(s),x'(s))\,ds\,d\beta(t) \\ &> \frac{\kappa_3}{\kappa}\int_0^1\int_0^1 k(t,s)f(s,x(s),x'(s))\,ds\,d\alpha(t) \\ &= \frac{\kappa_3}{\kappa}\Big(-\frac{1}{2}\int_0^1 k(\frac{1}{6},s)f(s,x(s),x'(s))ds + 2\int_0^1 k(\frac{1}{2},s)f(s,x(s),x'(s))ds\Big) \\ &\geq \frac{\kappa_3}{\kappa}\Big(2\int_0^1 \frac{1}{2}(1-\frac{1}{2})s(1-s)f(s,x(s),x'(s))ds \\ &\quad - \frac{1}{2}\int_0^1 s(1-s)f(s,x(s),x'(s))ds\Big) = 0. \end{aligned}$$

Thus condition (H2) holds. Again taking $B = 50$, for any $a \in \mathbb{R}$, when $|a| > 50$, we have $N(a(1 + (\rho - 1)t)) > 0$. So condition (H3) holds. Hence from Theorem 6.1, BVP (4.1) has at least one solution $x \in C^1[0,1]$.

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