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FORMAL AND ANALYTIC SOLUTIONS FOR A QUADRIC ITERATIVE FUNCTIONAL EQUATION

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Abstract. In this article, we study a quadric iterative functional equation. We prove the existence of formal solutions, and that every formal solution yields a local analytic solution when the eigenvalue of the linearization for the auxiliary function lying inside the unit circle, lying on the unit circle with a Brjuno number, or a root of 1.

1. Introduction

Solving iterative functional equations is difficult since the unknown arises in the iteration [5, 21]. Using Schauder fixed point theorem, Zhang [22] proved the existence and uniqueness of solutions for a general iterative functional equation, the so-called polynomial-like iterative functional equation,

$$\lambda_1 x(t) + \lambda_2 x^2(t) + \cdots + \lambda_n x^n(t) = F(t), \quad t \in \mathbb{R}.$$

Later various properties of solutions of iterative functional equations, such as continuity, differentiability, monotonicity, convexity, analyticity, stability, have received much more attention; see e.g. [8]–[15], [17]–[20], [23]. Among these studies, the existence of analytic solutions caused more concerns since it is closely related to small divisors problem. In [13], analytic invariant curves for a planar map were obtained by solving the iterative functional equation

$$x(z + x(z)) = x(z) + G(z) + H(z + x(z)),\ z \in \mathbb{C}.$$

We notice that [13] and [11] are all based on eigenvalue of the linearization θ is inside the unit circle or a Diophantine number by using Schröder conversion and majorant series. On the other hand, Reich and his co-authors [8]-[10] have studied the formal solutions of a quadric iterative functional equation, called the generalized Dhombres functional equation,

$$f(zf(z)) = \varphi(f(z)), \quad z \in \mathbb{C},$$

in the ring of formal power series $\mathbb{C}[[z]]$. They described the structure of the set of all formal solutions when the eigenvalue θ of linearization is not a root of 1, and also showed every formal solutions yield a local analytic solutions when θ is not on

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the unit circle or a Diophantine number, as well as represent analytic solutions by infinite products for θ ling in the unit circle. In 2008, Xu and Zhang [18] studied the analytic solutions of a q-difference equation

$$\sum_{j=0}^{k}\sum_{t=1}^{\infty} C_{t,j}(z)(x(q^j z))^t = G(z), \quad z \in \mathbb{C}, \tag{1.1}$$

they obtained local analytic solutions under Brjuno condition, and proved no-existence of analytic solutions when the eigenvalue θ of linearization satisfies Cremer condition. Following that, Si and Li [15] discussed analytic solutions of the (1.1) with a singularity at the origin.

In this article, we study the quadric iterative functional equation

$$x(az + bzx(z)) = H(z) \tag{1.2}$$

in the complex field, where $x(z)$ is unknown function, $H(z)$ is a given holomorphic function, a and b are nonzero complex parameters. It is a more complicated equation than the involutory function $x^2(t) = t$, which is the Babbage equation with $n = 2$. We discuss the existence of formal solutions for (1.2) when a is arbitrary nonzero complex number. Moreover, every formal solution yields a local analytic solution when a is lying inside the unit circle, lying on the unit circle with a Brjuno number or a root of 1. Our idea comes from [15].

Let $y(z) = az + bzx(z)$. Then

$$x(z) = \frac{y(z) - az}{bz}.$$

Therefore,

$$x(y(z)) = \frac{y(y(z)) - ay(z)}{by(z)},$$

Then (1.2) is equivalent to the functional equation

$$y(y(z)) - ay(z) = by(z)H(z). \tag{1.3}$$

Using the conversion $y(z) = g(\theta(g^{-1}(z)))$, Equation (1.3) transforms into the equation without functional iteration

$$g(\theta^2 z) - ag(\theta z) = bg(\theta z)H(g(z)). \tag{1.4}$$

Suppose

$$g(z) = \sum_{n=1}^{\infty} a_n z^n, \quad H(z) = \sum_{n=1}^{\infty} h_n z^n. \tag{1.5}$$

Substituting (1.5) into (1.4), we obtain

$$\begin{aligned}
&(\theta^2 - a\theta)a_1 z + \sum_{n=1}^{\infty}(\theta^{2(n+1)} - a\theta^{n+1})a_{n+1}z^{n+1} \\
&= b\sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \theta^{n+1-j}a_{n+1-j}h_m a_{i_1}a_{i_2}\dots a_{i_m}z^{n+1}.
\end{aligned} \tag{1.6}$$

Comparing coefficients, we obtain

$$(\theta^2 - a\theta)a_1 = 0, \tag{1.7}$$

and

$$(\theta^{2(n+1)} - a\theta^{n+1})a_{n+1} = b\sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \theta^{n+1-j} a_{n+1-j} h_m a_{i_1} a_{i_2} \dots a_{i_m}. \quad (1.8)$$

Under $a_1 \neq 0$, the equality (1.7) implies that $\theta = a$, then (1.8) turns into

$$(\theta^{n} - 1)\theta^{n+2} a_{n+1} = b\sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \theta^{n+1-j} a_{n+1-j} h_m a_{i_1} a_{i_2} \dots a_{i_m}. \quad (1.9)$$

This means the sequence $\{a_n\}_{n=2}^{\infty}$ can be determined successively from (1.9) in a unique manner for any $a_1 \neq 0$; that is, (1.4) has formal solution for arbitrary nonzero complex number a. Noticing that the function $H(z)$ is holomorphic in a neighborhood of the origin, we assume

$$|h_n| \leq 1.$$

The reason for this, is that (1.4) and hypothetic conditions $g(0) = 0$, $g'(0) = a_1$ still hold under the transformations

$$H(z) = \rho^{-1}F(\rho z), \quad g(z) = \rho^{-1}G(\rho z)$$

for $|h_n| \leq \rho^n$. We prove analyticity of solutions to (1.4) under varius hypotheses:

- (A1) (elliptic case) $\theta = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is a Brjuno number; i.e., $B(\alpha) = \sum_{k=0}^{\infty} \frac{\log q_{k+1}}{q_k} < \infty$, where $\{\frac{p_k}{q_k}\}$ denotes the sequence of partial fraction of the continued fraction expansion of α;
- (A2) (parabolic case) $\theta = e^{2\pi i \frac{q}{p}}$ for some integer $p \in \mathbb{N}$ with $p \geq 2, q \in \mathbb{Z}\backslash\{0\}$, and $\theta \neq e^{2\pi i \frac{l}{k}}$ for all $1 \leq k \leq p-1, l \in \mathbb{Z}\backslash\{0\}$.
- (A3) (hyperbolic case) $0 < |\theta| < 1$.

2. Existence of analytic solutions for (1.4)

When (A1) is satisfied, that is, $\theta = e^{2\pi i\alpha}$ with α irrational, small divisors arises inevitably. Since $(\theta^n - 1)$ appears in the denominator and the powers of θ form a dense subset, there will be n such that $\frac{1}{\theta^n - 1}$ is arbitrarily large, see [6]. In 1942, Siegel [16] showed a Diophantine condition that α satisfies

$$|\alpha - \frac{p}{q}| > \frac{\gamma}{q^{\delta}}$$

for some positive γ and δ. In 1965, Brjuno [2] put forward Brjuno number which satisfies

$$B(\alpha) = \sum_{n} \frac{\log q_{n+1}}{q_n} < \infty$$

and improved Diophantine condition, he showed that as long as α is a Brjuno number, small divisors is still dealt with tactfully. In the sequel we discuss the analytic solution of (1.4) with Brjuno number α. For this purpose, the Davie's Lemma is necessary.

Lemma 2.1 (Davie's Lemma [3]). *Assume $K(n) = n\log 2 + \sum_{k=0}^{k(n)} g_k(n)\log(2q_{k+1})$, then the function $K(n)$ satisfies*

(a) *There is a universal constant $\tau > 0$ (independent of n and of α), such that*

$$K(n) \le n\Big(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \tau\Big);$$

(b) *for all n_1 and n_2, we have $K(n_1) + K(n_2) \le K(n_1 + n_2)$;*

(c) $-\log|\theta^n - 1| \le K(n) - K(n-1)$.

Theorem 2.2. *Under assumption* (A1), (1.4) *has an analytic solution of the form*

$$g(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 \ne 0. \tag{2.1}$$

Proof. We prove the formal solution (2.1) is convergent in a neighborhood of the origin. From (1.9), we have

$$\begin{aligned}
|a_{n+1}| &\le |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \Big|\frac{\theta^{n+1-j}}{(\theta^n - 1)\theta^{n+2}}\Big| |a_{n+1-j}||h_m||a_{i_1}||a_{i_2}|\dots|a_{i_m}| \\
&= |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \frac{1}{|\theta^n - 1|} |a_{n+1-j}||h_m||a_{i_1}||a_{i_2}|\dots|a_{i_m}| \\
&\le |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \frac{1}{|\theta^n - 1|} |a_{n+1-j}||a_{i_1}||a_{i_2}|\dots|a_{i_m}|.
\end{aligned} \tag{2.2}$$

To construct a majorant series, we define $\{B_n\}_{n=1}^{\infty}$ by $B_1 = |a_1|$ and

$$B_{n+1} = |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} B_{n+1-j} B_{i_1} B_{i_2} \dots B_{i_m}, \quad n = 1, 2, \dots.$$

We denote

$$G(z) = \sum_{n=1}^{\infty} B_n z^n. \tag{2.3}$$

Then

$$\begin{aligned}
G(z) &= |a_1|z + \sum_{n=1}^{\infty} B_{n+1} z^{n+1} \\
&= |a_1|z + |b| \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} B_{n+1-j} B_{i_1} B_{i_2} \dots B_{i_m} z^{n+1} \\
&= |a_1|z + |b| \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{G(z) - G^{j+1}(z)}{1 - G(z)} \cdot B_{n+1-j} \cdot z^{n+1-j} \\
&= |a_1|z + |b| \frac{G^2(z) - (1-z)G^3(z) - G^4(z)}{(1-z)(1-G(z))(1-G^2(z))}.
\end{aligned}$$

Let

$$R(z, \zeta) = \zeta - |a_1|z - |b| \frac{\zeta^2 - (1-z)\zeta^3 - \zeta^4}{(1-z)(1-\zeta)(1-\zeta^2)} = 0. \tag{2.4}$$

We regard (2.4) as an implicit functional equation, since $R(0,0)=0$, $R'_\zeta(0,0)=1\neq 0$. We know that (2.4) has a unique analytic solution $\zeta(z)$ in a neighborhood of the origin such that $\zeta(0)=0$, $\zeta'(0)=|a_1|$ and $R(z,\zeta(z))=0$, so we have $G(z)=\zeta(z)$. Naturally, there exists constant $T>0$ such that $B_n\leq T^n, n=1,2,\dots$. We now deduce by induction on n that

$$|a_{n+1}|\leq B_{n+1}e^{k(n)},\quad n\geq 0. \tag{2.5}$$

In fact, $|a_1|=B_1$, since $k(0)=0$. We assume that $|a_{i+1}|\leq B_{i+1}$, $i<n$, $n=1,2,\dots$. Then

$$\begin{aligned}
&|a_{n+1}|\\
&\leq |b|\sum_{j=1}^{n}\sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}}\frac{1}{|\theta^n-1|}|a_{n+1-j}||a_{i_1}||a_{i_2}|\dots|a_{i_m}|\\
&\leq |b|\sum_{j=1}^{n}\sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}}\frac{1}{|\theta^n-1|}B_{n+1-j}e^{k(n-j)}B_{i_1}e^{k(i_1-1)}B_{i_2}e^{k(i_2-1)}\dots B_{i_m}e^{k(i_m-1)}\\
&= |b|\sum_{j=1}^{n}\sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}}\frac{1}{|\theta^n-1|}B_{n+1-j}B_{i_1}B_{i_2}\dots B_{i_m}e^{k(n-m)}\\
&= \frac{1}{|\theta^n-1|}B_{n+1}e^{k(n-m)}\\
&\leq \frac{1}{|\theta^n-1|}B_{n+1}e^{k(n-1)}\\
&\leq \frac{1}{|\theta^n-1|}B_{n+1}e^{\log|\theta^n-1|+k(n)}=B_{n+1}e^{k(n)},
\end{aligned}$$

by means of Davie's Lemma, thus (2.5) is proved. Note that

$$K(n)\leq n(B(\alpha)+\tau)$$

for some universal constant $\tau>0$. Then

$$|a_{n+1}|\leq T^{n+1}e^{n(B(\alpha)+\tau)};$$

that is,

$$\lim_{n\to\infty}\sup(|a_{n+1}|)^{1/(n+1)}\leq\lim_{n\to\infty}\sup(T^{n+1}e^{n(B(\alpha)+\tau)})^{1/(n+1)}=Te^{B(\alpha)+\tau}.$$

This implies that th radius of convergence for (2.1) is at least $(Te^{B(\alpha)+\tau})^{-1}$, the proof is complete. $\square$

In what follows, we consider the case that the constant θ is not only on the unit circle, but also a root of unity. Denote the right side of (1.9) as

$$\Lambda(n,\theta)=b\sum_{j=1}^{n}\sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}}\theta^{n+1-j}a_{n+1-j}h_m a_{i_1}a_{i_2}\dots a_{i_m}.$$

Theorem 2.3. *Assume* (A2) *holds and*

$$\Lambda(vp,\theta)\equiv 0,\ \ v=1,2,\dots. \tag{2.6}$$

Then (1.4) *has an analytic solution of the form*

$$g(z) = a_1 z + \sum_{n=vp,\ v\in\mathbb{N}} \zeta_{vp} z^n + \sum_{n\neq vp,\ v\in\mathbb{N}} b_n z^n, \quad a_1 \neq 0,\ \mathbb{N} = \{1, 2, \dots\} \tag{2.7}$$

in a neighborhood of the origin for some ζ_{vp}. *Otherwise,* (1.4) *has no analytic solutions in any neighborhood of the origin.*

Proof. In this parabolic case $\theta = e^{2\pi i \frac{q}{p}}$, the eigenvalue θ is a pth root of unity.

If $\Lambda(vp, \theta) \neq 0$, for some natural number v, then (1.9) does not hold since $\theta^{vp} - 1 = 0$, naturally, (1.4) has no formal solutions.

If $\Lambda(vp, \theta) \equiv 0$, for all natural number v, (1.4) has formal solution (2.1). To prove (2.1) yields a local analytic solution, we define the sequence $\{C_n\}_{n=1}^{\infty}$ satisfies $C_1 = |a_1|$ and

$$C_{n+1} = |b|\Gamma \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} C_{n+1-j} C_{i_1} C_{i_2} \dots C_{i_m},\ n = 1, 2, \dots, \tag{2.8}$$

where $\Gamma = \max\{1, |\theta^i - 1|^{-1} : i = 1, 2, \dots, p-1\}$. Clearly, the convergence of series $\sum_{n=1}^{\infty} C_n z^n$ can be proved similar as in Theorem 2.2.

When (2.6) holds for all natural number v, the coefficients a_{vp} have infinitely many choices in $\mathbb{C}$, choose $a_{vp} = \zeta_{vp}$ arbitrarily such that

$$|a_{vp}| \leq C_{vp}, \quad v = 1, 2, \dots. \tag{2.9}$$

Furthermore, we can prove

$$|a_n| \leq C_n, \quad n \neq vp. \tag{2.10}$$

In fact, $|a_1| = C_1$. If we suppose that $|a_{i+1}| \leq C_{i+1},\ i < n\ (n \neq vp)$, then

$$\begin{aligned} |a_{n+1}| &\leq |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \Big|\frac{\theta^{n+1-j}}{(\theta^n - 1)\theta^{n+2}}\Big| |a_{n+1-j}| |h_m| |a_{i_1}| |a_{i_2}| \dots |a_{i_m}| \\ &\leq |b|\Gamma \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} C_{n+1-j} C_{i_1} C_{i_2} \dots C_{i_m} \\ &= C_{n+1}. \end{aligned}$$

From (2.9), (2.10) and the convergence of series $\sum_{n=1}^{\infty} C_n z^n$, the formal solution (2.1) yields a local analytic solution (2.7) in a neighborhood of the origin. This completes the proof. □

Theorem 2.4. *Suppose* (A3) *holds, then* (1.4) *has an analytic solution of the form*

$$g(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 \neq 0.$$

Proof. We prove the formal solution (2.1) is convergent in a neighborhood of the origin. Since $0 < |\theta| < 1$, so $\lim_{n\to\infty} \frac{1}{\theta^n - 1} = 1$, From (1.9), we have

$$\begin{aligned} |a_{n+1}| &\le |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \Big|\frac{\theta^{n+1-j}}{(\theta^n-1)\theta^{n+2}}\Big| |a_{n+1-j}||h_m||a_{i_1}||a_{i_2}|\dots|a_{i_m}| \\ &\le |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \frac{1}{|\theta^{1+j}|} |a_{n+1-j}||a_{i_1}||a_{i_2}|\dots|a_{i_m}|. \end{aligned} \tag{2.11}$$

Let $\{D_n\}_{n=1}^{\infty}$ be defined by $D_1 = |a_1|$ and

$$D_{n+1} = |b| \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \frac{1}{|\theta^{1+j}|} D_{n+1-j} D_{i_1} D_{i_2} \dots D_{i_m}, \quad n = 1, 2, \dots .$$

Denote

$$F(z) = \sum_{n=1}^{\infty} D_n z^n. \tag{2.12}$$

Then

$$\begin{aligned} F(z) &= |a_1|z + \sum_{n=1}^{\infty} D_{n+1} z^{n+1} \\ &= |a_1|z + |b| \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{\substack{i_1+i_2+\dots+i_m=j;\\ m=1,2,\dots,j}} \frac{1}{|\theta^{1+j}|} D_{n+1-j} D_{i_1} D_{i_2} \dots D_{i_m} z^{n+1} \\ &= |a_1|z + |b| \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{1}{|\theta^{1+j}|} \cdot \frac{F(z) - F^{j+1}(z)}{1 - F(z)} \cdot D_{n+1-j} \cdot z^{n+1-j} \\ &= |a_1|z + |b| \frac{\theta F^2(z)}{(\theta^2 - 1)(\theta - F(z))}. \end{aligned}$$

Let

$$Q(z, \xi) = \xi - |a_1|z - |b| \frac{\theta \xi^2}{(\theta^2 - 1)(\theta - \xi)} = 0. \tag{2.13}$$

Since $Q(0,0) = 0$, $Q'_\xi(0,0) = 1 \ne 0$, then (2.13) has a unique analytic solution $\xi(z)$ in a neighborhood of the origin such that $\xi(0) = 0$, $\xi'(0) = |a_1|$ and $Q(z, \xi(z)) = 0$, so we have $F(z) = \xi(z)$. Similar as in Theorem 2.3, we can prove

$$|a_n| \le D_n, \quad n = 1, 2, \dots, \tag{2.14}$$

by induction. Then the local analytic solution (2.1) is existent in a neighborhood of the origin by means of the convergence of $\sum_{n=1}^{\infty} D_n$ and inequality (2.14). This completes the proof. □

3. Formal solutions and analytic solutions of (1.2)

In this section we prove the existence of formal solutions and analytic solutions of (1.2).

Theorem 3.1. *Equation* (1.3) *has a formal solution* $y(z) = g(\theta\, g^{-1}(z))$ *in a neighborhood of the origin, where* $g(z)$ *is formal solution of* (1.4). *Under one of the conditions in Theorems 2.2–2.4, every formal solution yields an analytic solution of the form* $y(z) = g(\theta\, g^{-1}(z))$, *where* $g(z)$ *is analytic solution of* (1.4).

Proof. Since $g'(0) = a_1 \neq 0$, the inverse $g^{-1}(z)$ exists in a neighborhood of $g(0) = 0$. If we define $y(z) = g(\theta\, g^{-1}(z))$, then

$$\begin{aligned} y(y(z)) - ay(z) &= g(\theta(g^{-1}(g\theta(g^{-1}(z))))) - ag(\theta(g^{-1}(z))) \\ &= g(\theta^2(g^{-1}(z))) - ag(\theta(g^{-1}(z))) \\ &= b(g\theta(g^{-1}(z))H(g^{-1}(z)) \\ &= by(z)H(z). \end{aligned} \tag{3.1}$$

as required, so (1.3) has a formal solution $y(z) = g(\theta\, g^{-1}(z))$ in a neighborhood of the origin.

Under one of the conditions in Theorems 2.2–2.4, the inverse $g^{-1}(z)$ exists and is analytic in a neighborhood of $g(0) = 0$, we obtain analytic solutions of (1.3) in a neighborhood of the origin. The proof is completed. $\square$

Suppose that

$$y(z) = \theta z + b_2 z^2 + b_3 z^3 + \dots,$$

since $a = \theta$ and $x(z) = \frac{y(z)-az}{bz}$, it follows that

$$x(z) = \frac{b_2}{b} z + \frac{b_3}{b} z^2 + \frac{b_4}{b} z^3 + \dots. \tag{3.2}$$

That is, (1.2) has a unique formal solution with the form (3.2) in a neighborhood of the origin. The formal solution also is analytic solution when $y(z)$ is analytic in a neighborhood of the origin.

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