Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 48, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

DIFFERENTIABILITY, ANALYTICITY AND OPTIMAL RATES OF DECAY FOR DAMPED WAVE EQUATIONS

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Abstract. We give necessary and sufficient conditions on the damping term of a wave equation for the corresponding semigroup to be analytic. We characterize damped operators for which the corresponding semigroup is analytic, differentiable, or exponentially stable. Also when the damping operator is not strong enough to have the above properties, we show that the solution decays polynomially, and that the polynomial rate of decay is optimal.

1. Introduction

This article is concerned with analyticity, differentiability and asymptotic stability of the C_0 semigroups associated with the initial-value problem

$$u_{tt} + Au + Bu_t = 0 \tag{1.1}$$

$$u(0) = u_0, \quad u_t(0) = u_1 \tag{1.2}$$

where A, and B are a self-adjoint positive definite operators with domain $D(A^\alpha) = D(B)$ dense in a Hilbert space H. We use the following hypotheses:

(H1) There exists positive constants C_1 and C_2 such that

$$C_1 A^\alpha \leq B \leq C_2 A^\alpha.$$

which means

$$C_1(A^\alpha u, u) \leq (Bu, u) \leq C_2(A^\alpha u, u)$$

for any $u \in D(A^\alpha)$.

(H2) The bilinear form $b(u, w) = (B^{1/2}u, B^{1/2}w)$ is continuous on $D(A^{\alpha/2}) \times D(A^{\alpha/2})$. By the Riesz representation theorem, assumption (H2) implies that there exists an operator $S \in \mathcal{L}(D(A^{\alpha/2}))$ such that

$$(Bu, w) = (A^{\alpha/2}Su, A^{\alpha/2}w)$$

for any $u, w \in D(A^{\alpha/2})$.

2000 *Mathematics Subject Classification.* 35L10, 47D06.

Key words and phrases. Dissipative systems; decay rate; analytic semigroups; polynomial stability.

Submitted December 22, 2011. Published March 27, 2012.

Luci Fatori was supported by grant 14423/2009 from the Fundação Araucária.

There exists a large body of literature about the above problem dealing with asymptotic behaviour of the solutions to the damped wave equation see for example [10, 8, 4, 21, 22, 7, 13] and the references therein. In contrast to this results, there exists only a few publications dealing with regularity properties of the damped wave equation, like analyticity and differentiability of the corresponding semigroup. Here we mention two references. First, in [5] the authors proved that the semigroup associated to the damped wave equation is analytic if $1/2 \leq \alpha \leq 1$. This result established a fortiori the conjectures by Chen and Russel on structural damping for elastic systems, which referred to the case $\alpha = 1/2$. Second, Liu and Liu [14] proved also the analyticity of the corresponding semigroup when $\alpha \in [1/2, 1]$ and the differentiability of the semigroup provides $\alpha \in]0, 1/2]$. Their proof is simpler than the proof in [5], the method the authors used is based on contradiction arguments.

In the two above cited papers there is no information about the behaviour of the semigroup for $-1 \leq \alpha \leq 1/2$, which frequently appears in applications. We also cite the book by Liu and Zheng [15], for questions related questions to this problem.

In this article we show a class of operators A and B, for which the above equation is analytic, differentiable and exponentially stable. Here we develop a proof simpler than the one in [5, 14], without using contradiction arguments. In addition, we show in case that the semigroup is not exponentially stable, that the solution of (1.1) decays polynomially to zero as time appraoches infinity. We show the our rate decay is optimal. To do so, we show for any contraction semigroup, a necessary condition to get the polynomial rate of decay. That is to say, the main result of this paper is to get a fully characterization of the damping term for $-1 \leq \alpha \leq 1$. We show as in [5, 14] that the semigroup is analytic if and only if $1/2 \leq \alpha \leq 1$, it is differentiable when $\alpha \in]0, 1[$ and that it is exponentially stable if and only if $\alpha \in [0, 1]$. Finally, in case of $\alpha = -\gamma < 0$ we show that the corresponding semigroup decays polynomially to zero as $t^{-1/\gamma}$ and we show that this rate of decay is optimal in $D(A)$ in the sense that is not possible to improve the rate $t^{-1/\gamma}$ with initial data over the domain of the operator A.

This paper is organized as follows. In sections 2 and 3 we show the analyticity and differentiability of the semigroup respectively. In section 4 we show the polynomial rate of decay of the semigroup when $\alpha < 0$ and we prove the optimality of the rates of decay. Finally, in section 5 we give some applications of the above results.

2. Analyticity

Let us denote $\mathcal{H} = \mathcal{D}(A^{1/2}) \times H$. Denoting by $U = (u, v)$ we define the norm in $\mathcal{H}$ as

$$\|U\|^2_{\mathcal{H}} = \|A^{1/2}u\|^2 + \|v\|^2.$$

Putting $v = u_t$, (1.1) can be written as the initial-value problem

$$\begin{gathered} \frac{dU}{dt} = A_B U \\ U(0) = U_0 \end{gathered} \tag{2.1}$$

with $U = (u, v)^t,\ U_0 = (u_0, u_1)^t$. Let us define

$$\mathcal{D}(A_B) = \Big\{(u, v) \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) : Au + Bv \in H\Big\} \tag{2.2}$$

and

$$A_B = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}, \quad A_B U = \begin{pmatrix} v \\ -(Au + Bv) \end{pmatrix}. \tag{2.3}$$

Clearly, for $U \in \mathcal{D}(A_B)$,

$$(A_B U, U) = (A^{1/2}v, A^{1/2}u) - (Au + Bv, v) = -\|B^{1/2}v\| \leq 0.$$

Thus A_B is a dissipative operator. Therefore we have the following result; see Pazy [18].

Theorem 2.1. *Let us assume that A and B are self adjoint operators positive definite and also a bijection operator from $D(A_B)$ to $\mathcal{H}$. Then the operator A_B is the infinitesimal generator of a C_0-semigroup $S_B(t)$ of contraction in $\mathcal{H}$.*

In this section we will show that the semigroup is analytic. Our main tool is the following theorem whose proof is found in [15].

Theorem 2.2. *Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is analytic if and only if*

$$\rho(A) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\limsup_{|\beta| \to \infty} |\beta| \, \|(i\beta I - A)^{-1}\| < \infty,$$

where $\rho(A)$ is the resolvent set of A.

The main result of this section is to show that the semigroup is analytic if and only if $1/2 \leq \alpha \leq 1$.

Theorem 2.3. *The semigroup $S_B(t) = e^{A_B t}$ is analytic if and only if $1/2 \leq \alpha \leq 1$.*

Proof. For $1/2 \leq \alpha \leq 1$, the domain of the operator A_B is

$$\mathcal{D}(A_B) = \{(u, v) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) : Au + Bv \in H\}. \tag{2.4}$$

Note that in general it is not possible to conclude that $u \in D(A)$. Using the spectral equation we obtain

$$i\beta u - v = f \quad \text{in } D(A^{1/2}) \tag{2.5}$$

$$i\beta v + Au + Bv = g \quad \text{in } H. \tag{2.6}$$

As in the above section we obtain

$$\|A^{\alpha/2}v\|^2 \leq C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{2.7}$$

Multiplying (2.6) by $A^\gamma \overline{u}$ and using (2.5) we obtain

$$\|A^{(1+\gamma)/2}u\|^2 + (Bv, A^\gamma u) = \|A^{\gamma/2}v\|^2 + (A^\gamma v, f) + (g, A^\gamma u);$$

that is,

$$\begin{aligned} &\|A^{(1+\gamma)/2}u\|^2 + (A^{\alpha/2}Sv, A^{\gamma+\alpha/2}u) \\ &= \|A^{\gamma/2}v\|^2 + (A^{\gamma-1/2}v, A^{1/2}f) + (g, A^\gamma u). \end{aligned} \tag{2.8}$$

Taking $\gamma = 1 - \alpha$ in the above identity we obtain

$$\|A^{(2-\alpha)/2}u\|^2 \leq \|A^{(1-\alpha)/2}v\|^2 + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

From where we have that there exists a positive constants C such that

$$\|A^{(2-\alpha)/2}u\|^2 \leq C\|A^{\alpha/2}v\|^2 + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \leq C_0\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{2.9}$$

Multiplying (2.5) by $A\overline{u}$ we obtain

$$i\beta\|A^{1/2}u\|^2 = (A^{1/2}f, A^{1/2}u) + (A^{\alpha/2}v, A^{1-\alpha/2}u).$$

Then using (2.7) and (2.9) we obtain

$$\beta\|A^{1/2}u\|^2 \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Let us decompose v as $v = v_1 + v_2$ such that

$$i\beta v_1 + Bv_1 = g \quad \text{in } H \tag{2.10}$$

$$i\beta v_2 + Au + Bv_2 = 0 \quad \text{in } H. \tag{2.11}$$

Multiplying (2.10) by $\overline{v}_1$ and taking imaginary and real part we obtain

$$|\beta|\|v_1\| \le \|F\|_{\mathcal{H}}, \quad \|B^{1/2}v_1\| \le \|F\|_{\mathcal{H}}. \tag{2.12}$$

Note that $\|v_1\| \le \|v\| + \|v_2\|$ and

$$\begin{aligned}
\|A^{\alpha/2}v_2\| &\le \|A^{\alpha/2}v\| + \|A^{-\alpha/2}Bv_1\| \\
&\le c_1\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + c_2\|v_1\|^{1/2}\|F\|_{\mathcal{H}}^{1/2} \\
&\le c\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + c_2\|v_2\|^{1/2}\|F\|_{\mathcal{H}}^{1/2}.
\end{aligned}$$

From (2.11) and the above inequality, we obtain

$$|\beta|\|A^{-\alpha/2}v_2\| \le \|A^{(2-\alpha)/2}u\| + \|A^{\alpha/2}v_2\| \le c\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + c_2\|v_2\|^{1/2}\|F\|_{\mathcal{H}}^{1/2}.$$

Using interpolation we obtain

$$\begin{aligned}
\|v_2\|^2 &\le c\|A^{-\alpha/2}v_2\|\|A^{\alpha/2}v_2\| \\
&\le \frac{c}{\beta}(\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + c_2\|v_2\|^{1/2}\|F\|_{\mathcal{H}}^{1/2})^2 \\
&\le \frac{c}{\beta}(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \|v_2\|\|F\|_{\mathcal{H}}).
\end{aligned}$$

From where we have

$$\beta^2\|v_2\|^2 \le c\beta\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c_0\|F\|_{\mathcal{H}}^2.$$

From the above inequality and (2.12) we obtain

$$\beta^2\|v\|^2 \le 2\beta^2(\|v_1\|^2 + \|v_2\|^2) \le c\beta\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c_0\|F\|_{\mathcal{H}}^2.$$

Using relation (2), we obtain

$$\beta^2(\|v\|^2 + \|A^{1/2}u\|^2) \le c\beta\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c_0\|F\|_{\mathcal{H}}^2$$

which is equivalent to

$$\beta^2\|U\|_{\mathcal{H}}^2 \le c\beta\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c_0\|F\|_{\mathcal{H}}^2$$

which implies

$$\beta^2\|U\|_{\mathcal{H}}^2 \le c_1\|F\|_{\mathcal{H}}^2.$$

From where the analyticity follows.

Now we show that the corresponding semigroup is not analytic for $0 \le \alpha < 1/2$. Here, we consider that the operator A and B have infinite eigenvector in common. Let us construct a sequence F_ν such that the solutions of

$$i\beta_\nu U_\nu - \mathcal{A}U_\nu = F_\nu$$

satisfies $|\beta_\nu|\|U_\nu\|_{\mathcal{H}} \to \infty$, which in particular implies

$$\|\beta_\nu(i\beta_\nu I - A)^{-1}\|_{\mathcal{H}} \to \infty$$

which means that the corresponding semigroup is not analytic. To see this, let us consider the spectral system

$$i\beta u_\nu - v_\nu = 0 \tag{2.13}$$

$$i\beta v_\nu + Au_\nu + Bv_\nu = w_\nu \tag{2.14}$$

where w_ν is an unitary eigenvector of A and B. Let us denote by λ_ν and $\lambda_{B\nu}$ the eigenvalues of A and B respectively. So we have

$$-\beta^2 u_\nu + Au_\nu + i\beta Bu_\nu = w_\nu.$$

Therefore, we can assume that $u_\nu = Kw_\nu$, with $K \in \mathbb{C}$. Substitution of u_ν yields

$$(-\beta^2 + \lambda_\nu + i\beta\lambda_{B\nu})Kw_\nu = w_\nu.$$

Taking $\beta^2 = \lambda_\nu$ we obtain that

$$i\beta\lambda_{B\nu}K = 1 \quad \Rightarrow \quad K := K_\nu = -i\lambda_\nu^{-1/2}\lambda_{B\nu}^{-1},$$

since

$$v_\nu = i\beta u_\nu = i\beta K_\nu w_\nu = -i\lambda_{B\nu}^{-1} w_\nu.$$

Therefore,

$$\|U_\nu\|_{\mathcal{H}}^2 = \|A^{1/2}u_\nu\|^2 + \|v_\nu\|^2 = 2\lambda_{B\nu}^{-2} \quad \Rightarrow \quad \beta_\nu\|U_\nu\|_{\mathcal{H}} = \sqrt{2}\lambda_\nu^{1/2}\lambda_{B\nu}^{-1}. \tag{2.15}$$

From (H1) we conclude that

$$C_0\lambda_\nu^\alpha \le \lambda_{B\nu} \le C_1\lambda_\nu^\alpha. \tag{2.16}$$

Therefore, if $\alpha < 1/2$ we obtain

$$\beta_\nu\|U_\nu\|_{\mathcal{H}} \ge c_0\lambda_\nu^{1/2-\alpha} \quad \Rightarrow \quad \beta_\nu\|U_\nu\|_{\mathcal{H}} \to \infty$$

From where our conclusion follows. □

3. Differentiability

Our main tool to show differentiability is the following theorem, Pazy [18, Theorem 4.9].

Theorem 3.1. *Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is differentiable if $i\mathbb{R} \subset \rho(A)$ and*

$$\limsup_{|\beta|\to\infty}(\ln|\beta|)\|(i\beta I - A)^{-1}\| < \infty.$$

We use the above result to show that the semigroup S_B is differentiable when $0 < \alpha < 1/2$. The differentiability for $1/2 \le \alpha \le 1$ is an immediate consequence of the analyticity.

Theorem 3.2. *Suppose that $0 < \alpha < 1/2$. Then the semigroup $S_B(t)$ is differentiable.*

Proof. To show the above relation, let us consider the spectral equation

$$i\beta U - A_B U = F.$$

In terms of the coefficients we have (2.5)–(2.6). Multiplying (2.6) by $\overline{v}$ we obtain

$$i\beta\|v\|^2 + (A^{1/2}u, A^{1/2}v) + \|B^{1/2}v\|^2 = (g, v).$$

Multiplying (2.5) by $A\overline{u}$ we obtain

$$i\beta\|A^{1/2}u\|^2 - (A^{1/2}v, A^{1/2}u) = (A^{1/2}f, A^{1/2}u).$$

Adding the above equations and taking the real part we obtain

$$\|B^{1/2}v\|^2 \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.1}$$

From (H1) we obtain

$$\|A^{\alpha/2}v\|^2 \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.2}$$

In particular,

$$\|v\|^2 \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.3}$$

Multiplying (2.6) by $\overline{u}$ we obtain

$$(i\beta v, u) + \|A^{1/2}u\|^2 + (Bv, u) = (g, u).$$

Using (2.5), we obtain

$$\|A^{1/2}u\|^2 \le \|v\|^2 - (B^{1/2}v, B^{1/2}u) + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Since $\alpha \le 1/2$, using (3.1), hypothesis (H1) and (3.3), we obtain

$$\|A^{1/2}u\|^2 \le c\|v\|^2 + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.4}$$

From (3.3) and (3.4) we conclude that

$$\|U\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}. \tag{3.5}$$

From (2.5), (3.2) and (3.5) we obtain that

$$|\beta|\|A^{\alpha/2}u\| \le \|A^{\alpha/2}v\| + \|F\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}. \tag{3.6}$$

This because $\alpha \le 1/2$. Multiplying (2.6) by $A^\gamma\overline{u}$ we obtain

$$(i\beta v, A^\gamma u) + \|A^{(\gamma+1)/2}u\|^2 + (Bv, A^\gamma u) = (g, A^\gamma u)$$

or equivalent

$$-(A^\gamma v, i\beta u) + \|A^{(\gamma+1)/2}u\|^2 + (Bv, A^\gamma u) = (g, A^\gamma u).$$

From (H2) we obtain

$$-(A^\gamma v, i\beta u) + \|A^{(\gamma+1)/2}u\|^2 + (A^{\alpha/2}Sv, A^{\gamma+\alpha/2}u) = (g, A^\gamma u).$$

Using (2.5) we obtain

$$\begin{aligned}\|A^{(\gamma+1)/2}u\|^2 &\le \|A^{\gamma/2}v\|^2 + (A^{\alpha/2}Sv, A^{\gamma+\alpha/2}u) + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\\ &\le \|A^{\gamma/2}v\|^2 + (A^{\alpha/2}Sv, A^{\gamma+\alpha/2}u) + C\|F\|^2_{\mathcal{H}}.\end{aligned} \tag{3.7}$$

From the above relation we conclude that our best choice for γ is $\gamma = \alpha$, then we obtain

$$\|A^{(\alpha+1)/2}u\|^2 \le \|A^{\alpha/2}v\|^2 + (A^{\alpha/2}Sv, A^{3\alpha/2}u) + C\|F\|^2_{\mathcal{H}}.$$

Since $\alpha \le 1/2$ we obtain

$$\|A^{3\alpha/2}u\| \le \|A^{(\alpha+1)/2}u\|$$

which implies

$$\|A^{(\alpha+1)/2}u\|^2 \le c\|A^{\alpha/2}v\|^2 + C\|F\|_{\mathcal{H}}^2.$$

From (3.2) and (3.5) we obtain that there exists a positive constant C such that

$$\|A^{(\alpha+1)/2}u\|^2 \le C\|F\|_{\mathcal{H}}^2. \tag{3.8}$$

Now we use interpolation

$$1/2 = \theta(\alpha+1)/2 + (1-\theta)\alpha/2 \quad \Rightarrow \quad \theta = 1-\alpha.$$

Therefore,

$$\|A^{1/2}u\| \le c\|A^{(\alpha+1)/2}u\|^{\theta}\|A^{\alpha/2}u\|^{1-\theta}.$$

Then

$$|\beta|^{1-\theta}\|A^{1/2}u\| \le c\|A^{(\alpha+1)/2}u\|^{\theta}(|\beta|\|A^{\alpha/2}u\|)^{1-\theta}.$$

So from (3.6)–(3.8) we have

$$|\beta|^{\alpha}\|A^{1/2}u\| \le C\|F\|_{\mathcal{H}}^{\theta}\|F\|_{\mathcal{H}}^{1-\theta} \le C\|F\|_{\mathcal{H}}. \tag{3.9}$$

Applying $A^{(\alpha-1)/2}$ in (2.6) we obtain

$$i\beta A^{(\alpha-1)/2}v + A^{(\alpha+1)/2}u + A^{(\alpha-1)/2}Bv = A^{(\alpha-1)/2}g.$$

From hypothesis (H2) the operator B can be written as $Bv = A^{\alpha}Sv$, so we have

$$|\beta|\|A^{(\alpha-1)/2}v\| \le \|A^{(\alpha+1)/2}u\| + \|A^{(3\alpha-1)/2}Sv\| + c\|F\|_{\mathcal{H}}.$$

Since $\alpha-1<0$ and $(3\alpha-1)/2 \le \alpha/2$, provided $0<\alpha<1/2$, we obtain that

$$|\beta|\|A^{(\alpha-1)/2}v\| \le c\|A^{(\alpha+1)/2}u\| + c\|A^{\alpha/2}v\| + c\|F\|_{\mathcal{H}}$$

for $\beta > 1$. From (3.2), (3.5) and (3.8) we obtain

$$|\beta|\|A^{(\alpha-1)/2}v\| \le C\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + c\|F\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}. \tag{3.10}$$

Using interpolation once more,

$$0 = \theta(\alpha-1)/2 + (1-\theta)\alpha/2 \quad \Rightarrow \quad \theta = \alpha,$$

we obtain

$$\|v\| \le c\|A^{(\alpha-1)/2}v\|^{\theta}\|A^{\alpha/2}v\|^{1-\theta}.$$

So we have that

$$|\beta|^{\alpha}\|v\| \le c(|\beta|\|A^{(\alpha-1)/2}v\|)^{\alpha}\|A^{\alpha/2}v\|^{1-\alpha}.$$

From (3.2), (3.5) and (3.10) it follows that

$$|\beta|^{\alpha}\|v\| \le c(\|F\|_{\mathcal{H}})^{\alpha}\|F\|_{\mathcal{H}}^{1-\alpha} = C\|F\|_{\mathcal{H}}. \tag{3.11}$$

From relation (3.9) and (3.11) we obtain,for β large,

$$|\beta|^{2\alpha}\|U\|_{\mathcal{H}}^2 \le C\|F\|_{\mathcal{H}}^2.$$

Therefore our conclusion follows. □

4. Polynomial rate of decay and optimality

In this section we prove that the solution of (1.1) for $\alpha = -\gamma < 0$ decays polynomially to zero as time approaches infinity. We will show that the corresponding energy decays to zero as $t^{-1/\gamma}$. Moreover we show that this rate of decay is optimal. This result improves the rates established in [17]. Our result is based on [3]. See also also [2, 1].

Theorem 4.1. *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator A such that $i\mathbb{R} \subset \varrho(A)$. Then*

$$\frac{1}{|\eta|^\alpha}\|(i\eta I - A)^1\| \le C, \quad \forall \eta \in \mathbb{R} \quad \Leftrightarrow \quad \|S(t)A^{-1}\| \le \frac{c}{t^{1/\alpha}}$$

To prove polynomial rate of decay we should show that there exist positive constant $C > 0$ independent of β, l or f such that

$$\sup_{\|f\|\le 1} \frac{1}{\beta^l}\|U\| = \sup_{\|f\|\le 1} \frac{1}{\beta^l}\|(i\beta I - A)^{-1} f\| \le C.$$

Remark 4.2. Note that we can improve the polynomial rate of decay by improving the regularity of the initial data, that is

$$\|S(t)A^{-k}\| \le \frac{c_k}{t^{k/\alpha}}$$

for the proof see [20]. In that sense it is important to remark what optimality means. The optimality of course will depend on the domain. So fixing the domain taking $k = 1$, we prove that the rate $1/\gamma$ can not be improved.

Under the above conditions we can establish the main result of this section.

Theorem 4.3. *Let $\alpha = -\gamma < 0$ be a negative real number where $0 < \gamma \le 1$. Then the semigroup $S_B(t)$ decays polynomially to zero as*

$$\|S_B(t)U_0\| \le C(\frac{1}{t})^{1/\gamma}\|U_0\|_{D(A)}.$$

Moreover, when B and $A^{-\gamma}$ have infinite common eigenvectors the rate $1/\gamma$ can not be improved over $D(A)$.

Proof. We consider spectral (2.5) and (2.6) when $\alpha \in]-1, 0[$ or equivalently

$$i\beta u - v = f \in \mathcal{D}(A) \tag{4.1}$$

$$i\beta v + Au + Bv = g \in H. \tag{4.2}$$

Multiplying (4.2) by $\overline{v}$ and (4.1) by $A\overline{u}$ summing the product result and taking real part we obtain

$$\|A^{-\gamma/2}v\|^2 \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.3}$$

Multiplying (4.2) by $A^{-\gamma}\overline{u}$ and using (4.1), we obtain

$$\|A^{(1-\gamma)/2}u\|^2 = \|A^{-\gamma/2}v\|^2 - (Bv, A^{-\gamma}u) + (A^{-\gamma}v, f) + (g, A^{-\gamma}u).$$

Since $(1-\gamma)/2 > -\gamma$ and using (4.3) we obtain that

$$\|A^{(1-\gamma)/2}u\|^2 \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{4.4}$$

Applying $A^{-(1+\gamma)/2}$ on (4.2), we obtain

$$i\beta A^{-(1+\gamma)/2}v + A^{(1-\gamma)/2}u + A^{-(\gamma+1)/2}Bv = A^{-(\gamma+1)/2}g.$$

Since $-(3\gamma+1)/2 < -\gamma/2$ and using (4.3)-(4.4) in the above identities, it follows that

$$|\beta|\|A^{-(1+\gamma)/2}v\| \le C\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + c\|F\|_{\mathcal{H}}. \tag{4.5}$$

Multiplying (4.1) by $A^s\overline{u}$ where $s=(1-2\gamma)/2$ we obtain

$$i\beta\|A^{s/2}u\|^2 = (A^{-\gamma/2}v, A^{\gamma/2+s}u) + (f, A^s u).$$

That is,

$$|\beta|\|A^{(1-2\gamma)/4}u\|^2 \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2. \tag{4.6}$$

Applying $A^{\gamma/2}$ on (4.1) and (4.3), we obtain

$$|\beta|\,\|A^{-\gamma/2}u\| \le C\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + c\|F\|_{\mathcal{H}}. \tag{4.7}$$

Using interpolation,

$$0 = \theta(1-2\gamma)/4 - \gamma/2(1-\theta) \quad \Rightarrow \quad \theta = 2\gamma,$$

we have

$$\|u\| \le c\|A^{-\gamma/2}u\|^{1-\theta}\|A^{(1-2\gamma)/4}u\|^{\theta}.$$

Now using (4.6) and (4.7), we obtain

$$\begin{aligned}\|u\| &\le \frac{C}{|\beta|^{1-\theta/2}}(\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}^2)\\ &= \frac{C}{|\beta|^{1-\gamma}}(\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}^2).\end{aligned}$$

From this,

$$\|v\| \le |\beta|\|u\| + \|f\| \le \frac{C}{|\beta|^{-\gamma}}(\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}) + \|F\|_{\mathcal{H}}\,,$$

and so

$$|\beta|^{-\gamma}\|v\| \le C(\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2} + \|F\|_{\mathcal{H}}). \tag{4.8}$$

Multiplying (4.2) by $\overline{u}$ and using (4.1) we obtain

$$\|A^{1/2}u\|^2 = (Bv,u) - (i\beta v,u) + (g,u) = (Bv,u) + \|v\|^2 - (f,u) + (g,u).$$

From (4.3) we conclude that

$$|\beta|^{-2\gamma}\|A^{1/2}u\|^2 \le |\beta|^{-2\gamma}\|v\|^2 + C|\beta|^{-2\gamma}(\|U\|_{\mathcal{H}}^{3/2}\|F\|_{\mathcal{H}}^{1/2} + \|F\|\|U\|). \tag{4.9}$$

Adding (4.8) and (4.9), we have

$$|\beta|^{-2\gamma}\|U\|_{\mathcal{H}}^2 \le \beta^{-2\gamma}\|v\|^2 + C\beta^{-2\gamma}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2.$$

Applying Young's inequality in the last term,

$$\frac{1}{2}|\beta|^{-2\gamma}\|U\|_{\mathcal{H}}^2 \le C|\beta|^{-2\gamma}\|F\|^2 \le C\|F\|^2.$$

Therefore, the semigroup is polinomially stable and decay as $t^{-1/\gamma}$ over $D(A)$. Finally, to show the optimality. We suppose that the operators A and B have an infinite eigenvector in common. As in section 2, we can assume that $u_\nu = Kw_\nu$, with $K\in\mathbb{C}$. Substitution of u_ν yields

$$(-\beta^2 + \lambda_\nu + i\beta\lambda_{B\nu})Kw_\nu = w_\nu.$$

Taking $\beta^2 = \lambda_\nu - \lambda_\nu^{(1-\gamma)/2}$ we obtain that $\beta \approx \lambda_\nu^{1/2}$ and

$$(\lambda_\nu^{(1-\gamma)/2} + i\beta\lambda_{B\nu})K = 1 \quad \Rightarrow \quad u_\nu = K_\nu w_\nu = \frac{1}{1+i\beta\lambda_{B\nu}}w_\nu,$$

since

$$u_\nu = K_\nu w_\nu = \frac{1}{\lambda_\nu^{(1-\gamma)/2} + i\beta\lambda_{B\nu}} w_\nu.$$

Then we have

$$\begin{aligned} \|U_\nu\|_{\mathcal{H}} \geq \|A^{1/2}u_\nu\| &= \frac{\lambda_\nu^{1/2}}{\sqrt{\lambda_\nu^{1-\gamma} + \beta^2\lambda_{B\nu}^2}} \\ &\geq \frac{\lambda_\nu^{1/2}}{\sqrt{\lambda_\nu^{1-\gamma} + c_0\lambda_\nu^{1-2\gamma}}} \\ &\geq \frac{\lambda_\nu^{\gamma/2}}{\sqrt{1 + c_0\lambda_\nu^{-\gamma}}}. \end{aligned}$$

Note that $\beta \approx \lambda_\nu^{1/2}$ as $\nu \to \infty$. From where we obtain

$$\beta^{-\gamma+\epsilon}\|U_\nu\|_{\mathcal{H}} \geq \frac{\lambda_\nu^{\epsilon}}{\sqrt{1 + c_0\lambda_\nu^{-1}}} \to \infty$$

as $\nu \to \infty$. Therefore is not possible to improve the polynomial rate of decay. □

5. Applications

Here we apply our result to several models.

Viscoelastic plates. Let Ω be a bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and consider the model

$$\begin{gathered} \varrho u_{tt} + \kappa\Delta^2 u - \gamma\Delta u_t = 0 \quad \text{in } \Omega \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \partial\Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Here $A = k/\rho\Delta^2$ and $B = \gamma\rho/k(-\Delta)$.

From Theorem 2.3 we conclude that the semigroup that defines the solution of the above system is analytic. So, in particular we have that the solution decays exponentially to zero and there exists smoothing effect on the initial data, that is no matter where the initial data u_0 and u_1 is, the solution satisfies $u \in C^\infty(]0,T[;C^\infty(\Omega))$.

On the other hand, if we consider the inertial term on the plate we obtain the model

$$\begin{gathered} \varrho u_{tt} - h\Delta u_{tt} + \kappa\Delta^2 u - \gamma\Delta u_t = 0 \quad \text{in } \Omega \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \partial\Omega \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Here $A = k(\rho I - h\Delta)^{-1}\Delta^2$, $B = -\gamma(\rho I - h\Delta)^{-1}\Delta$. So there exists positive constants c_1 and c_0 such that

$$c_1(A^0 w, w) \leq (Bw, w) \leq c_0(A^0 w, w).$$

We conclude that the model is neither analytic nor differentiable. But is exponentially stable.

Mixtures. We consider a beam composed by a mixture of two viscoelastic interacting continually that occupies the interval $(0, L)$. The displacement of typical particles at time t are u and w, where $u = u(x,t)$ and $w = w(y,t)$, $x, y \in (0, L)$. We assume that the particles under consideration occupy the same position at time $t = 0$, so that $x = y$. We denote by ρ_i the mass density of each constituent at time $t = 0$, T, S the partial stresses associated with the constituents, P the internal diffusive force. In the absent of body forces the system of equations consists of the equations of motion

$$\rho_1 u_{tt} = T_x - P, \quad \rho_2 w_{tt} = S_x + P, \tag{5.1}$$

and the constitutive equations

$$\begin{gathered} T = a_{11}u_x + a_{12}w_x + b_{11}u_{xt} + b_{12}w_{xt} \\ S = a_{12}u_x + a_{22}w_x + b_{21}u_{xt} + b_{22}w_{xt} \\ P = \alpha(u - w). \end{gathered} \tag{5.2}$$

If we substitute the constitutive equations into the motion equations and the energy equation, we obtain the system of field equations

$$\rho_1 u_{tt} - a_{11}u_{xx} - a_{12}w_{xx} + \alpha(u - w) - b_{11}u_{xxt} - b_{12}w_{xxt} = 0 \quad \text{in } (0, \infty) \times (0, L),$$
$$\rho_2 w_{tt} - a_{12}u_{xx} - a_{22}w_{xx} - \alpha(u - w) - b_{21}u_{xxt} - b_{22}w_{xxt} = 0 \quad \text{in } (0, \infty) \times (0, L),$$

We assume that the constants ρ_1, ρ_2 , c, and α are positive, and that the matrix $A = (a_{ij})$, $B = (b_{ij})$ are symmetric and positive definite.

$$u(0) = u_0, \quad u_t(0) = u_1, \quad w(0) = w_0, \quad w_t(0) = w_1,$$
$$u(t, 0) = u(t, L) = w(t, 0) = w(t, L) = 0.$$

In vectorial notation, the above system can be written as

$$U_{tt} + AU_{xx} + BU_{xxt} = 0,$$

where $U = (u, w)^t$. Note that

$$c_1(Aw, w) \le (Bw, w) \le c_0(Aw, w).$$

Therefore, by Theorem 2.3 we conclude that the solution of the mixture model is defined by an analytic semigroup.

Elasticity. Let us denote by $\Omega \subset \mathbb{R}^2$ an open bounded set with smooth boundary. Let us consider the plate equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + \gamma u_t = 0, \quad \text{in } \Omega \times]0, \infty[$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega$$
$$u(0) = u_0, u_t(0) = u_1 \quad \text{in } \partial\Omega.$$

Letting $A = [I - \Delta]^{-1}\Delta^2$ and $B = [I - \Delta]^{-1}$, with H and $D(\Delta)$ being $L^2(\Omega)$ and $H_0^1(\Omega) \cap H^2(\Omega)$ respectively, the above model may be written as (1.1). Note that

$$c_1(A^{-1}w, w) \le (Bw, w) \le c_0(A^{-1}w, w).$$

Using Theorem 4.3 we conclude that the corresponding semigroup decays polynomial as

$$\|S_B(t)U_0\|_{\mathcal{H}} \le \frac{C}{t} \|U_0\|_{D(A)}.$$

Where the rate $1/t$ can not be improved over domain of $D(A)$. This result improves the rate of decay given in [17].

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