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# EXISTENCE OF GLOBAL SOLUTIONS FOR A GIERER-MEINHARDT SYSTEM WITH THREE EQUATIONS

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Abstract. This articles shows the existence of global solutions for a Gierer-Meinhardt model of three substances described by reaction-diffusion equations with fractional reactions. Our technique is based on a suitable Lyapunov functional.

## 1. Introduction

In recent years, systems of Reaction-Diffusion equations have received a great deal of attention, motivated by their widespread occurrence in modeling chemical and biological phenomena. Among these systems, the Gierer-Meinhardt is an important one. Meinhardt, Koch and Bernasconi [7] proposed activator-inhibitor models (an example is given in section 4) to describe a theory of biological pattern formation in plants (*Phyllotaxis*).

We consider a reaction-diffusion system with three components:

$$\begin{aligned} u_t - a_1\Delta u &= f(u,v,w) = \sigma - b_1 u + \frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} \\ v_t - a_2\Delta v &= g(u,v,w) = -b_2 v + \frac{u^{p_2}}{v^{q_2}w^{r_2}} \\ w_t - a_3\Delta w &= h(u,v,w) = -b_3 w + \frac{u^{p_3}}{v^{q_3}w^{r_3}} \end{aligned} \tag{1.1}$$

with $x \in \Omega$, $t > 0$, and with Neummann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times \{t > 0\}, \tag{1.2}$$

and initial data

$$\begin{aligned} u(0,x) &= \varphi_1(x) > 0 \\ v(0,x) &= \varphi_2(x) > 0 \\ w(0,x) &= \varphi_3(x) > 0 \end{aligned} \tag{1.3}$$

on $\Omega$, and $\varphi_i \in C(\overline{\Omega})$ for all $i = 1,2,3$. Here $\Omega$ is an open bounded domain of class $C^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$; $\partial/\partial\eta$ denotes the outward normal derivative on $\partial\Omega$.

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We use the following assumptions: $a_i, b_i, p_i, q_i, r_i$ are nonnegative for $i = 1, 2, 3$, with $\sigma > 0$, $c \geq 0$:

$$0 < p_1 - 1 < \max\left\{p_2 \min(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1),\ p_3 \min(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1)\right\}. \tag{1.4}$$

We set $A_{ij} = \frac{a_i + a_j}{2\sqrt{a_i a_j}}$ for $i, j = 1, 2, 3$. Let $\alpha, \beta$ and $\gamma$ be positive constants such that

$$\alpha > 2\max\left\{1, \frac{b_2 + b_3}{b_1}\right\}, \quad \frac{1}{\beta} > 2A_{12}^2, \tag{1.5}$$

$$(\frac{1}{2\beta} - A_{12}^2)(\frac{1}{2\gamma} - A_{13}^2) > (\frac{\alpha - 1}{\alpha} A_{23} - A_{12}A_{13})^2. \tag{1.6}$$

The main result of the paper reads as follows.

**Theorem 1.1.** *Suppose that the functions $f, g$ and $h$ satisfy condition* (1.4)*. Let $(u(t,\cdot), v(t,\cdot), w(t,\cdot))$ be a solution of* (1.1)-(1.3) *and let*

$$L(t) = \int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x) w^\gamma(t,x)}\, dx. \tag{1.7}$$

*Then the functional $L$ is uniformly bounded on the interval $[0, T^*], T^* < T_{\max}$, where $T_{\max}$ $(\|u_0\|_\infty, \|v_0\|_\infty, \|w_0\|_\infty)$ denotes the eventual blow-up time.*

**Corollary 1.2.** *Under the assumptions of Theorem 1.1, all solutions of* (1.1)-(1.3) *with positive initial data in $C(\overline{\Omega})$ are global. If in addition $b_1$, $b_2$, $b_3$, $\sigma > 0$, then $u, v, w$ are uniformly bounded in $\overline{\Omega} \times [0, \infty)$.*

## 2. Preliminaries

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted respectively by:

$$\begin{gathered} \|u\|_p^p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p dx, \\ \|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|, \\ \|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|. \end{gathered} \tag{2.1}$$

In 1972, following an ingenious idea of Turing [10], Gierer and Meinhardt [6] proposed a mathematical model for pattern formations of spatial tissue structures of hydra in morphogenesis, a biological phenomenon discovered by Trembley in 1744 [9]. It is a system of reaction-diffusion equations of the form

$$\begin{gathered} u_t - a_1 \Delta u = \sigma - \mu u + \frac{u^p}{v^q} \\ v_t - a_2 \Delta v = -\nu v + \frac{u^r}{v^s} \end{gathered} \tag{2.2}$$

for $x \in \Omega$ and $t > 0$, with Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad x \in \partial\Omega, t > 0, \tag{2.3}$$

and initial conditions

$$u(x,0) = \varphi_1(x) > 0, \quad v(x,0) = \varphi_2(x) > 0, \quad x \in \Omega, \tag{2.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $a_1, a_2 > 0$, $\mu, \nu, \sigma > 0$, the indexes $p, q, r$ and $s$ are non negative with $p > 1$.

Existence of solutions in $(0,\infty)$ is proved by Rothe in 1984 [12] in a particular situation when $p = 2$, $q = 1$, $r = 2$, $s = 0$ and $N = 3$. Rothe's method cannot be applied (at least directly) to the general case. Wu and Li [5] obtained the same results for (2.2)-(2.4) so long as $u, v^{-1}$ and $\sigma$ are suitably small. Mingde, Shaohua and Yuchun [8] show that solutions of this problem are bounded all the time for any initial values if

$$\frac{p-1}{r} < \frac{q}{s+1}, \quad \frac{p-1}{r} < 1. \tag{2.5}$$

Masuda and Takahashi [11] considered a more general system for $(u, v)$,

$$\begin{gathered} u_t - a_1\Delta u = \sigma_1(x) - \mu u + \rho_1(x,u)\frac{u^p}{v^q}, \\ v_t - a_2\Delta v = \sigma_2(x) - \nu v + \rho_2(x,u)\frac{u^r}{v^s}, \end{gathered} \tag{2.6}$$

with $\sigma_1, \sigma_2 \in C^1(\overline{\Omega})$, $\sigma_1 \geq 0, \sigma_2 \geq 0$, $\rho_1, \rho_2 \in C^1(\overline{\Omega} \times \overline{\mathbb{R}}^2_+) \cap L^\infty(\overline{\Omega} \times \overline{\mathbb{R}}^2_+)$ satisfying $\rho_1 \geq 0, \rho_2 > 0$ and $p, q, r, s$ are nonnegative constants satisfying (2.5). Obviously, system (2.4) is a special case of system (2.6).

In 1987, Masuda and Takahashi [11] extended the result to $\frac{p-1}{r} < \frac{2}{N+2}$ under the sole condition $\sigma_1 > 0$. In 2006, Jiang [4], under the conditions (2.5), $\varphi_1, \varphi_2 \in W^{2,l}(\Omega)$, $l > \max\{N, 2\}$, $\frac{\partial \varphi_1}{\partial \eta} = \frac{\partial \varphi_2}{\partial \eta} = 0$ on $\partial\Omega$ and $\varphi_1 \geq 0, \varphi_2 > 0$ in $\overline{\Omega}$, showed that (2.6) has a unique nonnegative global solution $(u, v)$ satisfying (2.3)-(2.4).

It is well-known that to prove existence of global solutions to (1.1)–(1.3), it suffices to derive a uniform estimate of $\|f(u,v,w)\|_p$, $\|g(u,v,w)\|_p$ and $\|h(u,v,w)\|_p$ on $[0;T_{\max})$ in the space $L^p(\Omega)$ for some $p > N/2$ (see Henry [3]). Our aim is to construct a polynomial Lyapunov functional allowing us to obtain $L^p-$ bounds on $u, v$ and $w$ that lead to global existence. Since the functions $f, g$ and $h$ are continuously differentiable on $\mathbb{R}^3_+$, then for any initial data in $C(\overline{\Omega})$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\mathcal{A} = -\begin{pmatrix} a_1\Delta & 0 & 0 \\ 0 & a_2\Delta & 0 \\ 0 & 0 & a_3\Delta \end{pmatrix}. \tag{2.7}$$

Under these assumptions, the following local existence result is well known (see Henry [3]).

**Proposition 2.1.** *System* (1.1)-(1.3) *admits a local unique, classical solution* $(u, v, w)$ *on* $(0, T_{\max}) \times \Omega$. *If* $T_{\max} < \infty$ *then*

$$\lim_{t \nearrow T_{\max}} (\|u(t,.)\|_\infty + \|v(t,.)\|_\infty + \|w(t,.)\|_\infty) = \infty. \tag{2.8}$$

## 3. Proofs of main results

For the proof of Theorem 1.1, we need some preparatory Lemmas.

**Lemma 3.1.** *Assume that* $p$, $q$, $r$, $s$, $m$, *and* $n$ *satisfy*

$$\frac{p-1}{r} < \min\Big(\frac{q}{s+1}, \frac{m}{n}, 1\Big).$$

*For all $h$, $l$, $\alpha$, $\beta$, $\gamma > 0$, there exist $C = C(h, l, \alpha, \beta, \gamma) > 0$ and $\theta = \theta(\alpha) \in (0, 1)$, such that*

$$\alpha \frac{x^{p-1+\alpha}}{y^{q+\beta} z^{m+\gamma}} \le \beta \frac{x^{r+\alpha}}{y^{s+1+\beta} z^{n+\gamma}} + C\Big(\frac{x^\alpha}{y^\beta z^\gamma}\Big)^\theta, \quad x \ge 0,\ y \ge h,\ z \ge l. \tag{3.1}$$

*Proof.* For all $x \ge 0, y \ge h, z \ge l$, from the inequality (3.1), we have

$$\alpha \frac{x^{p-1}}{y^q z^m} \le \beta \frac{x^r}{y^{s+1} z^n} + C\Big(\frac{x^\alpha}{y^\beta z^\gamma}\Big)^{\theta - 1}. \tag{3.2}$$

We can write

$$\alpha \frac{x^{p-1}}{y^q z^m} = \alpha \beta^{-(p-1)/r} \Big(\beta \frac{x^r}{y^{s+1} z^n}\Big)^{\frac{p-1}{r}} y^{\frac{(s+1)(p-1)}{r} - q} z^{\frac{n(p-1)}{r} - m}.$$

For each $\epsilon$ such that $0 < \epsilon < \min(\frac{q}{s+1}, \frac{m}{n}, 1) - \frac{p-1}{r}$, we have

$$\alpha \frac{x^{p-1}}{y^q z^m} = \alpha \beta^{-(p-1)/r} \Big(\beta \frac{x^r}{y^{s+1} z^n}\Big)^{\frac{p-1}{r} + \epsilon} \Big(\beta \frac{x^r}{y^{s+1} z^n}\Big)^{-\epsilon} v^{\frac{(s+1)(p-1)}{r} - q} z^{\frac{n(p-1)}{r} - m}.$$

Then

$$\begin{aligned}
\alpha \frac{x^{p-1}}{y^q z^m} &= \alpha(\beta)^{-\frac{p-1}{r} - \epsilon} \Big(\beta \frac{x^r}{y^{s+1} z^n}\Big)^{\frac{p-1}{r} + \epsilon} \Big(\frac{1}{x^\alpha}\Big)^{\frac{r\epsilon}{\alpha}} (y)^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1)} \\
&\quad \times z^{\frac{n(p-1)}{r} - m + \epsilon n} \\
&\le \alpha(\beta)^{-\frac{p-1}{r} - \epsilon} \Big(\beta \frac{x^r}{y^{s+1} z^n}\Big)^{\frac{p-1}{r} + \epsilon} \Big(\frac{1}{x^\alpha}\Big)^{\frac{r\epsilon}{\alpha}} (h)^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1)} \\
&\quad \times l^{\frac{n(p-1)}{r} - m + \epsilon n} \\
&\le \alpha(\beta)^{-\frac{p-1}{r} - \epsilon} \Big(\beta \frac{x^r}{y^{s+1} z^n}\Big)^{\frac{p-1}{r} + \epsilon} \Big(\frac{1}{x^\alpha}\Big)^{\frac{r\epsilon}{\alpha}} (h)^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1)} \\
&\quad \times l^{\frac{n(p-1)}{r} - m + \epsilon n} \Big(\frac{y}{h}\Big)^{\frac{\beta r\epsilon}{\alpha}} \Big(\frac{z}{l}\Big)^{\frac{\gamma r\epsilon}{\alpha}} \\
&\le C_1 \Big(\beta \frac{x^r}{y^{s+1} z^n}\Big)^{\frac{p-1}{r} + \epsilon} \Big(\frac{y^\beta z^\gamma}{x^\alpha}\Big)^{r\epsilon/\alpha},
\end{aligned} \tag{3.3}$$

where

$$C_1 = \alpha(\beta)^{-\frac{p-1}{r} - \epsilon} h^{\frac{(s+1)(p-1)}{r} - q + \epsilon(s+1) - \frac{\beta r\epsilon}{\alpha}} l^{\frac{(n)(p-1)}{r} - m + \epsilon n - \frac{\gamma r\epsilon}{\alpha}}.$$

Using Young's inequality for (3.3) by taking

$$C = C_1^{1 + \frac{p-1+r\epsilon}{r-(p-1)-r\epsilon}}, \quad \theta = 1 - \frac{r\epsilon}{\alpha(1 - \frac{p-1}{r} - \epsilon)},$$

where $\epsilon$ is sufficiently small, we obtain inequality (3.2). ∎

**Lemma 3.2.** *Let $T > 0$ and $f = f(t)$ be a non-negative integrable function on $[0, T)$. Let $0 < \theta < 1$ and $W = W(t)$ be a positive function on $[0, T)$ satisfying the differential inequality*

$$\frac{dW}{dt} \le -W(t) + f(t) W^\theta(t), \quad 0 \le t < T.$$

*Then $W(t) \le \kappa$, where $\kappa$ is the positive root of the algebraic equation*

$$x - \Big(\sup_{0<t<T} \int_0^t e^{-(t-\xi)} f(\xi) d\xi\Big) x^\theta = W(0).$$

**Lemma 3.3.** *Let* $(u(t,\cdot),v(t,\cdot),w(t,\cdot))$ *be a solution of* (1.1)-(1.3). *Then for any* $(t,x)$ *in* $(0,T_{\max})\times\Omega$, *we have*

$$\begin{aligned} u(t,x) &\geq e^{-b_1 t}\min(\varphi_1(x))>0,\\ v(t,x) &\geq e^{-b_2 t}\min(\varphi_2(x))>0,\\ w(t,x) &\geq e^{-b_3 t}\min(\varphi_3(x))>0. \end{aligned}\tag{3.4}$$

The proof of the above lemma follows immediately from the maximum principle, and it is omitted.

*Proof of Theorem 1.1.* Differentiating $L(t)$ with respect to $t$ yields

$$\begin{aligned} L'(t) &= \int_\Omega \frac{d}{dt}\Big(\frac{u^\alpha}{v^\beta w^\gamma}\Big)dx\\ &= \int_\Omega \Big(\alpha\frac{u^{\alpha-1}}{v^\beta w^\gamma}\partial_t u-\beta\frac{u^\alpha}{v^{\beta+1}w^\gamma}\partial_t v-\gamma\frac{u^\alpha}{v^\beta w^{\gamma+1}}\partial_t w\Big)dx. \end{aligned}$$

Replacing $\partial_t u$, $\partial_t v$ and $\partial_t w$ by their values in (1.1), we obtain

$$L'(t)=I+J,$$

where $I$ contains the Laplacian terms and $J$ contains the other terms,

$$I=a_1\alpha\int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma}\,\Delta u dx-a_2\beta\int_\Omega\frac{u^\alpha}{v^{\beta+1}w^\gamma}\,\Delta v dx-a_3\gamma\int_\Omega\frac{u^\alpha}{v^\beta w^{\gamma+1}}\,\Delta w\,dx,$$

and

$$\begin{aligned} J&=(-b_1\alpha+b_2\beta+b_3\gamma)L(t)+\alpha\int_\Omega\frac{u^{p_1+\alpha-1}}{v^{q_1+\beta}w_3^\gamma(w^{r_1}+c)}\,dx\\ &\quad-\beta\int_\Omega\frac{u^{p_2+\alpha}}{v^{q_2+\beta+1}w^{r_2+\gamma}}dx-\gamma\int_\Omega\frac{u^{p_3+\alpha}}{v^{q_3+\beta}w^{r_3+\gamma+1}}\,dx+\sigma\alpha\int_\Omega\frac{u^{\alpha-1}}{v^\beta w^\gamma}\,dx. \end{aligned}$$

**Estimation of $I$.** Using Green's formula, we obtain

$$\begin{aligned} I&=\int_\Omega\Big(-a_1\alpha(\alpha-1)\frac{u^{\alpha-2}}{v^\beta w^\gamma}|\nabla u|^2+a_1\alpha\beta\frac{u^{\alpha-1}}{v^{\beta+1}w^\gamma}\nabla u\nabla v+a_1\alpha\gamma\frac{u^{\alpha-1}}{v^\beta w^{\gamma+1}}\nabla u\nabla w\\ &\quad+a_2\beta\alpha\frac{u^{\alpha-1}}{v^{\beta+1}w^\gamma}\nabla u\nabla v-a_2\beta(\beta+1)\frac{u^\alpha}{v^{\beta+2}w^\gamma}|\nabla v|^2-a_2\beta\gamma\frac{u^\alpha}{v^{\beta+1}w^{\gamma+1}}\nabla v\nabla w\\ &\quad+a_3\gamma\alpha\frac{u^{\alpha-1}}{v^\beta w^{\gamma+1}}\nabla u\nabla w-a_3\gamma\beta\frac{u^\alpha}{v^{\beta+1}w^{\gamma+1}}\nabla v\nabla w-a_3\gamma(\gamma+1)\frac{u^\alpha}{v^\beta w^{\gamma+2}}|\nabla w|^2\Big)dx\\ &=-\int_\Omega\Big[\frac{u^{\alpha-2}}{v^{\beta+2}w^{\gamma+2}}(QT)\cdot T\Big]\,dx, \end{aligned}$$

where

$$Q=\begin{pmatrix} a_1\alpha(\alpha-1) & -\alpha\beta\frac{a_1+a_2}{2} & -\alpha\gamma\frac{a_1+a_3}{2}\\ -\alpha\beta\frac{a_1+a_2}{2} & a_2\beta(\beta+1) & \beta\gamma\frac{a_2+a_3}{2}\\ -\alpha\gamma\frac{a_1+a_3}{2} & \beta\gamma\frac{a_2+a_3}{2} & a_3\gamma(\gamma+1)\end{pmatrix}.$$

The matrix $Q$ is positive definite if, and only if, all its principal successive determinants $\Delta_1,\Delta_2,\Delta_3$ are positive. To see this, we have:

$\Delta_1=a_1\alpha(\alpha-1)>0$ by (1.5). Note that

$$\Delta_2=\begin{vmatrix} a_1\alpha(\alpha-1) & -\alpha\beta\frac{a_1+a_2}{2}\\ -\alpha\beta\frac{a_1+a_2}{2} & a_2\beta(\beta+1)\end{vmatrix}=\alpha^2\beta^2a_1a_2\Big(\frac{\alpha-1}{\alpha}\frac{\beta+1}{\beta}-A_{12}^2\Big)$$

which is positive by (1.5).

Using [1, Theorem 1], we obtain

$$\begin{aligned}(\alpha-1)\Delta_3 &= (\alpha-1)|Q|\\ &= \alpha(\alpha\gamma\beta)^2 a_1a_2a_3\Big(\big(\frac{\alpha-1}{\alpha}\frac{\beta+1}{\beta}-A_{12}^2\big)\big(\frac{\alpha-1}{\alpha}\frac{\gamma+1}{\gamma}-A_{13}^2\big)\\ &\quad -\big(\frac{\alpha-1}{\alpha}A_{23}-A_{12}A_{13}\big)^2\Big).\end{aligned}$$

Then using (1.5)-(1.6), we obtain $\Delta_3 > 0$.

Consequently, $I \le 0$ for all $(t,x) \in [0,T^*]\times\Omega$.

**Estimation of $J$.** According to the maximum principle, there exists $C_0$ depending on $\|\varphi_1\|_\infty$, $\|\varphi_2\|_\infty$, $\|\varphi_3\|_\infty$ such that $v, w \ge C_0 > 0$. We then have

$$\frac{u^{\alpha-1}}{v^\beta w^\gamma} = \big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^{(\alpha-1)/\alpha}\big(\frac{1}{v}\big)^{\beta/\alpha}\big(\frac{1}{w}\big)^{\gamma/\alpha} \le \big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^{\frac{\alpha-1}{\alpha}}\big(\frac{1}{C_0}\big)^{(\beta+\gamma)/\alpha}.$$

So

$$\frac{u^{\alpha-1}}{v^\beta w^\gamma} \le C_2\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^{(\alpha-1)/\alpha} \quad \text{where } C_2 = \big(\frac{1}{C_0}\big)^{(\beta+\gamma)/\alpha}.$$

Using Lemma 3.1, for all $(t,x) \in [0,T^*]\times\Omega$, we obtain

$$\alpha\frac{u^{p_1+\alpha-1}}{v^{q_1+\beta}w^\gamma(w^{r_1}+c)} \le \alpha\frac{u^{p_1+\alpha-1}}{v^{q_1+\beta}w^{\gamma+r_1}} \le \beta\frac{u^{p_2+\alpha}}{v^{q_2+\beta+1}w^{r_2+\gamma}} + C\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^\theta, \tag{3.5}$$

or

$$\alpha\frac{u^{p_1+\alpha-1}}{v^{q_1+\beta}w^{\gamma+r_1}} \le \gamma\frac{u^{p_3+\alpha}}{w^{r_3+1+\gamma}v^{q_3+\beta}}C\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^\theta. \tag{3.6}$$

We have

$$\begin{aligned}J &= (-b_1\alpha+b_2\beta+b_3\gamma)L(t) + \alpha\int_\Omega \frac{u^{p_1+\alpha-1}}{v^{q_1+\beta}w^\gamma(w^{r_1}+c)}dx\\ &\quad -\beta\int_\Omega\frac{u^{p_2+\alpha}}{v^{q_2+\beta+1}w^{r_2+\gamma}}dx - \gamma\int_\Omega\frac{u^{p_3+\alpha}}{v^{q_3+\beta}w^{r_3+\gamma+1}}\,dx + \sigma\alpha\int_\Omega\frac{u^{\alpha-1}}{v^\beta w^\gamma}\,dx.\end{aligned}$$

Using (3.6),

$$J \le (-b_1\alpha+b_2\beta+b_3\gamma)L(t) + \int_\Omega C\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^\theta dx + \alpha\sigma\int_\Omega C_2\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^{(\alpha-1)/\alpha}dx\,.$$

Applying Hölder's inequality, for all $t \in [0,T^*]$, we obtain

$$\int_\Omega C\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^\theta dx \le \Big(\int_\Omega\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)dx\Big)^\theta\Big(\int_\Omega C^{\frac{1}{1-\theta}}\,dx\Big)^{1-\theta}.$$

Then

$$\int_\Omega C\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^\theta dx \le C_3L^\theta(t), \quad \text{where } C_3 = C|\Omega|^{1-\theta}.$$

We have

$$\int_\Omega C_2\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^{(\alpha-1)/\alpha}dx \le \Big(\int_\Omega\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)dx\Big)^{(\alpha-1)/\alpha}\Big(\int_\Omega (C_2)^\alpha dx\Big)^{1/\alpha}.$$

Whereupon

$$\int_\Omega C_2\big(\frac{u^\alpha}{v^\beta w^\gamma}\big)^{(\alpha-1)/\alpha}dx \le C_4L^{(\alpha-1)/\alpha}(t) \quad \text{where } C_4 = C_2|\Omega|^{1/\alpha}.$$

We have

$$J \le (-b_1\alpha+b_2\beta+b_3\gamma)L(t) + C_3L^\theta(t) + \alpha\sigma C_4L^{\frac{\alpha-1}{\alpha}}(t).$$

Whereupon

$$J \le (-b_1\alpha + b_2\beta + b_3\gamma)L(t) + C_5(L^\theta(t) + \alpha\sigma L^{\frac{\alpha-1}{\alpha}}(t)).$$

Thus under conditions (1.5) and (1.6), we obtain the differential inequality

$$L'(t) \le (-b_1\alpha + b_2\beta + b_3\gamma)L(t) + C_5(L^\theta(t) + \alpha\sigma L^{(\alpha-1)/\alpha}(t)).$$

Since $-b_1\alpha + b_2\beta + b_3\gamma < 0$, we obtain

$$L'(t) \le C_5 L^\theta(t) + C_5\alpha\sigma L^{(\alpha-1)/\alpha}(t). \tag{3.7}$$

Using Lemma 3.2, we deduce that $L(t)$ is bounded on $(0, T_{\max})$; i.e, $L(t) \le \gamma_1$, where $\gamma_1$ dependents on the $L^\infty$-norm of $\varphi_1, \varphi_2$ and $\varphi_3$. □

*Proof of Corollary 1.2.* Since $L(t)$ is bounded on $(0, T_{\max})$ and the functions

$$\frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)}, \quad \frac{u^{p_2}}{v^{q_2}w^{r_2}}, \quad \frac{u^{p_3}}{v^{q_3}w^{r_3}}$$

are in $L^\infty((0, T_{\max}), L^m(\Omega))$ for all $m > \frac{N}{2}$, as a consequence of the arguments in Henry [3] and Haraux and Kirane [2], we conclude that the solution of the system (1.1)-(1.6) is global and uniformly bounded on $\Omega \times (0, +\infty)$. □

## 4. Example

In this section we present a particular activator-inhibitor model that illustrates the applicability of Theorem 1.1 and Corollary 1.2. We assume that all reactions take place in bounded region $\Omega$ with smooth boundary $\partial\Omega$.

**Example 4.1.** The model proposed by Meinhrdt, Koch and Bernasconi [7] to describe a theory of biological pattern formation in plants (Phyllotaxis) is

$$\begin{gathered}
\frac{\partial u}{\partial t} - a_1\frac{\partial^2 u}{\partial x^2} = -b_1 u + \frac{a^2}{v(w+k_u)} + \sigma, \\
\frac{\partial v}{\partial t} - a_2\frac{\partial^2 v}{\partial x^2} = -b_2 v + u^2, \\
\frac{\partial w}{\partial t} - a_3\frac{\partial^2 w}{\partial x^2} = -b_3 w + u,
\end{gathered} \tag{4.1}$$

for $x \in \Omega$ and $t > 0$, where $u, v, w$ are the concentrations of the three substances; called activator ($u$) and inhibitors ($v$ and $w$).

We claim that (4.1) with boundary conditions (1.2) and non-negative uniformly bounded initial data (1.3) has a global solution. This claim follows from this model being a special case of (1.1), with $p_1 = 2$, $q_1 = 1$, $r_1 = 1$, $p_2 = 2$, $q_2 = 0$, $r_2 = 0$, $p_3 = 1$, $q_3 = 0$, $r_3 = 0$. Since these indexes satisfy the conditions for global existence: $\frac{p_1-1}{p_2} < \min\left(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\right)$, we have a global solution.

We remark that system (4.1) exhibits all the essential features of phyllotaxis.

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