Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 64, pp. 1–69.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GLOBAL TOPOLOGICAL CLASSIFICATION OF LOTKA-VOLTERRA QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract. The Lotka-Volterra planar quadratic differential systems have numerous applications but the global study of this class proved to be a challenge difficult to handle. Indeed, the four attempts to classify them (Reyn (1987), Wörz-Buserkros (1993), Georgescu (2007) and Cao and Jiang (2008)) produced results which are not in agreement. The lack of adequate global classification tools for the large number of phase portraits encountered, explains this situation. All Lotka-Volterra systems possess invariant straight lines, each with its own multiplicity. In this article we use as a global classification tool for Lotka-Volterra systems the concept of configuration of invariant lines (including the line at infinity). The class splits according to the types of configurations in smaller subclasses which makes it easier to have a good control over the phase portraits in each subclass. At the same time the classification becomes more transparent and easier to grasp. We obtain a total of 112 topologically distinct phase portraits: 60 of them with exactly three invariant lines, all simple; 27 portraits with invariant lines with total multiplicity at least four; 5 with the line at infinity filled up with singularities; 20 phase portraits of degenerate systems. We also make a thorough analysis of the results in the paper of Cao and Jiang [13]. In contrast to the results on the classification in [13], done in terms of inequalities on the coefficients of normal forms, we construct invariant criteria for distinguishing these portraits in the whole parameter space $\mathbb{R}^{12}$ of coefficients.

Contents

2000 *Mathematics Subject Classification.* 58K30, 34A26,34C05, 34C40.
Key words and phrases. Quadratic vector fields; Lotka-Volterra differential systems; phase portraits; affine invariant polynomials; topological invariants.

Submitted January 19, 2012. Published April 25, 2012.
Research supported by NSERC .

1. Introduction

In this article we consider real autonomous differential systems

$$\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y), \tag{1.1}$$

where $p, q \in \mathbb{R}[x, y]$; i.e., p, q are polynomials in x, y over $\mathbb{R}$ and their associated vector fields

$$\tilde{D} = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}. \tag{1.2}$$

We call *degree* of a system (1.1) (or of a vector field (1.2)) the integer

$$n = \max(\deg p, \deg q).$$

In particular we call *quadratic* a differential system (1.1) with $n = 2$ and we denote by QS the class of all such systems.

The study of quadratic differential systems is motivated in part by their many applications. On the other hand they are also interesting for theoretical reasons. Indeed, hard problems on polynomial differential systems, among them Hilbert's 16th problem, have been open for more than a century even for the quadratic case. These problems are of a global nature and while the global study of the whole quadratic class is not within reach at this time, a handful of specific subfamilies of this class have been successfully studied globally.

The goal of this article is to give a complete global topological classification of the subfamily of quadratic differential systems which can be brought by affine transformation of the form

$$\begin{aligned} \dot{x} &= x(a_0 + a_1 x + a_2 y) \equiv p(x, y), \\ \dot{y} &= y(b_0 + b_1 x + b_2 y) \equiv q(x, y), \end{aligned} \tag{1.3}$$

where p, q are polynomials in x, y with real coefficients and $\max(\deg(p), \deg(q)) = 2$.

Usually we say that two systems (S_1) and (S_2) are topological equivalent if and only if there exists a homeomorphism of the plane carrying orbits to orbits and preserving their orientations. In this paper we say that (S_1) and (S_2) are *topological equivalent* if and only if there exists a homeomorphism of the plane carrying orbits to orbits and preserving or reversing their orientations. We use this definition in order to halve the number of phase portraits.

Systems (1.3) are called Lotka-Volterra as they were proposed independently by Alfred J. Lotka in 1925 [22] and Vito Volterra in 1926 [39]. Actually Lotka and Volterra considered initially the systems (1.3) with $a_1 = 0 = b_2$ but in the current literature are called Lotka-Volterra, systems of the more general form (1.3). The scientific literature on this family has been steadily growing due to their many applications (see [37] for references on applications).

It is estimated that the class of quadratic differential systems will yield more than two thousand topologically distinct phase portraits. The study of subfamilies of the quadratic class and in particular the Lotka-Volterra family, forms a good testing ground for the analogous but much more difficult task of classifying the whole quadratic class.

The global study of several families of quadratic vector fields was completely done. Examples of such families are:

- the quadratic vector fields possessing a center [40], [29] [43], [26];
- the quadratic Hamiltonian vector fields [1], [5];
- the quadratic vector fields with invariant straight lines of total multiplicity at least four [33], [35];
- the planar quadratic differential systems possessing a line of singularities at infinity [36];
- the quadratic vector fields possessing an integrable saddle [4].

All the systems in the above mentioned classes are integrable. Indeed, every quadratic Hamiltonian system has a cubic polynomial as first integral. All the systems occurring in the other families above are proven to be integrable on the complement of an algebraic curve in the papers mention above, via the algebro-geometric method of Darboux. The global study of each of the above classes was done using only algebraic methods. This is essentially due to the existence of Darboux *inverse integrating factors* (i.e. $1/R(x, y)$, where $R(x, y)$ is an integrating factor).

While the global study of QS is a distant goal at this time, the global study of the infinite singular points of systems in QS was done. Furthermore this was achieved by using only algebraic and geometric methods in [24], [32]. The global study of the finite singularities was also done by using only algebraic and algebro-geometrical methods in [3].

In [14], Coppel wrote:

> Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients. However, attempts in this direction have met with very limited success ...

This task proved to be impossible. Indeed, Dumortier and Fiddelaers [16] and Roussarie [28] exhibited examples of families of quadratic vector fields which have non-algebraic bifurcation sets. The following two classes of quadratic vector fields were studied globally by algebraic methods coupled with analytic and numerical methods:

- the family of quadratic vector fields with a weak focus of third order [21];
- the family of quadratic vector fields with a weak focus of second order [2].

We point out that in both families, besides algebraic hyper-surfaces of bifurcation points, there are non-algebraic hyper-surfaces of bifurcation points in the parameter space.

It is natural to ask the following questions:

> How much of the behavior of quadratic (or more generally polynomial) vector fields or how far can we go in their global theory by using mainly algebraic means?

Modulo the action of the group of affine transformations and time homotheties, the planar Lotka-Volterra class is 3-dimensional while the class of quadratic differential systems modulo the same group action is 5-dimensional. Due to the global result saying that any system in the Lotka-Volterra class has no limit cycles (see Theorem 2.4 or [8], [14]), it is possible to draw the bifurcation diagram of this class.

The literature on the Lotka-Volterra equations has become quite ample. In particular there were several attempts to give complete classifications of this family [27](1987), [42] (1993), [18] (2007), [13] (2008). A quick check of the references given in the last three papers indicates that none of these authors mentions anyone of the previously published papers suggesting that they were not aware of them. But did they obtain results which are in agreement? In fact they are not. This was shown for the first three articles above in [30]. In this work we shall also discuss the results in [13] and we show that our results are not everywhere in agreement with the results in [13]. Indeed, we prove here that several phase portraits claimed to be topologically distinct in [13] are in fact topologically equivalent, and we indicate 8 phase portraits which are incorrect. On the other hand we point out the following observation:

Observation 1.1. While in this article a system (1.3) is taken to have $\max(\deg(p), \deg(q)) = 2$, in [13] the authors only consider systems (1.3) with $\deg(p) = \deg(q) = 2$. Naturally they got fewer phase portraits than we get here. More precisely 8 phase portraits in our classification do not appear in [13] and we indicate them by writing the word "*omitted*" on the lines where they appear in the diagrams (see Diagram 3 – Diagram 6).

To obtain a clear, transparent classification, one needs to have powerful global classifying tools and they are missing in [13]. In our view, the main shortcoming is neither the fact that some portraits are incorrect or missing, nor that in the list we have topologically equivalent phase portraits which are claimed to be distinct. The main shortcoming of this classification is that it is not helpful for understanding globally this class. Indeed, the classification is done in terms of inequalities on the coefficients of several normal forms for the systems. Since there are so many phase portraits we end up with pages of inequalities giving us no insights into the global phenomena present for this class.

In contrast in this article we use powerful classification tools such as *affine and topological invariants*. The Lotka-Volterra equations have an inherent algebro-geometric structure. We spelled out this algebro-geometric structure in [37]. We used the number of distinct invariant straight lines, as well as their multiplicities, as a basic global geometric classifying tool. We combine this algebro-geometric information with information involving the real singularities of the systems located on these invariant lines in the concept of *configuration of invariant straight lines of a system*, introduced in [31] (see Section 2 further below). This is an example of an affine invariant.

The topological equivalence relation is distinct from the affine equivalence relation and it is in fact coarser than the affine one. However the affine equivalence relation is a very powerful tool for computation and it is of great use in the topological classification due to the possibility of using an arsenal of specific affine invariant polynomials. We base the topological classification on the affine classification of the configurations of invariant lines of Lotka-Volterra systems obtained by us in [37]. Our results split this class into subclasses according to the possibilities we have for the types of configurations occurring for this class. We then focus our attention on these subclasses, each of which has much fewer phase portraits and thus it is much easier to keep track of them. By using this approach we are also able to give necessary and sufficient conditions in terms of polynomial invariants for the realization of each one of the phase portraits.

Clearly, any system which could be brought by affine transformation and time homotheties to a system (1.3) has the same geometric properties as (1.3).

Definition 1.2. We denote by LV the class of all planar differential systems which could be brought by affine transformations and time rescaling to the form (1.3) above and the systems in this class will be called LV-systems.

In [37], it was shown that the LV quadratic systems form a subset of an algebraic set in the parameter space $\mathbb{R}^{12}$ of coefficients. The goal of our work here is to give a rigorous and complete *topological* classification of the phase portraits of this class and to construct invariant criteria for distinguishing these portraits.

Our Main Theorem is the following:

Theorem 1.3. *The class of all Lotka-Volterra quadratic differential systems has a total of 112 topologically distinct phase portraits. Among these, 60 portraits are for systems with three simple invariant lines; 27 are portraits of systems with invariant lines of total multiplicity at least four; 5 phase portraits are for Lotka-Volterra systems which have the line at infinity filled up with singularities; 20 phase portraits are for the degenerate systems.*

(i) Consider the 13 configurations Config. 3.j, $j \in \{1, \dots, 13\}$ (see Definition 2.2) with three simple invariant lines given in Fig. 4. For each configuration Config. 3.j we have a number n_j of topologically distinct phase portraits. Then $\sum_{j=1}^{13} n_j = 65$ and the 65 phase portraits (not necessarily topologically distinct) are given in Fig. 5. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 5.

(ii) Consider the 34 configurations of Lotka-Volterra systems Config. 4.1,..., Config. 6.8 with invariant lines of total multiplicity at least four given in Fig. 4. For each one of these 34 configurations we have a number m_i, $i \in \{1, \dots, 34\}$ of topologically distinct phase portraits. Then $\sum_{i=1}^{34} m_i = 59$ and the 59 phase portraits (not necessarily topologically distinct) are given in Fig. 3. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 3.

(iii) Consider the 4 configurations of Lotka-Volterra systems Config. C_2.j, $j \in \{1, 2, 5, 7\}$ with the line at infinity filled up with singularities given in Fig. 2. For each one of these 4 configurations we have a unique phase portrait, except for the configuration Config. C_2.j for which we have two phase portraits. The 5 phase portraits are topologically distinct and they are given in Fig. 1. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 3.

(iv) Consider the 14 configurations Config. LV_d.j, $j \in \{1, \dots, 14\}$ given in Fig. 2, of the degenerate quadratic Lotka-Volterra systems. For each configuration Config. LV_d.j we have a number s_j of topologically distinct phase portraits. Then $\sum_{j=1}^{14} s_j = 20$ and the 20 phase portraits given in Fig. 6 are topologically distinct. The necessary and sufficient affine invariant conditions for the realization of each one of these portraits are given in Table 6.

(v) Of the 149 phase portraits obtained by listing those occurring in the classes (i)–(iv), only 112 are topologically distinct (see Diagrams 1–6).

To characterize each phase portrait we use affine invariant polynomials which we define in Subsection 2.3; to prove that two arbitrarily chosen phase portraits in the complete list are topologically distinct we use topological invariants which we define in Subsection 3.3.

Observation 1.4. In Fig. 5 we see the phase portraits of the family of LV-systems with exactly three invariant lines. In this figure we have 23 couples of phase portraits such that in any such couple the phase portraits are topologically equivalent and they are distinguished in the picture by the presence of a focus instead of a node (see for example, Picture 3.1(a2) and Picture 3.1($\overset{*}{a}$2)). Our algebraic apparatus allows us to distinguish within each couple the two phase portraits by algebraic means.

The article is organized as follows: In Section 2 we give the definitions of the global concepts used in this article, such as for example the notion of *configuration of invariant lines* and we state the theorem proved in [37] classifying the Lotka-Volterra differential systems according to their configurations of invariant lines. We also state results which we need and which were obtained in [31], [33], [34], [35], [36],[37]. In Section 3 we prove the Main Theorem and we give some concluding comments.

2. Global geometric concepts and preliminary results

Our classification is based on the concept of *configuration of invariant lines* of a differential system and on results obtained in [37].

The concept of invariant algebraic curve of a differential system is due to Darboux [15]. Roughly speaking these are algebraic curves which are unions of phase curves. The presence of such algebraic invariant curves is an important information about a system. For example if we have sufficiently many such curves, the system is integrable, i.e. it has a non-constant analytic first integral on the complement of some algebraic curve ([15]). The following is the formal definition due to Darboux.

Definition 2.1. An *affine algebraic invariant curve* (or an *algebraic particular integral*) of a polynomial system (1.1) or of a vector field (1.2) is a curve $f(x,y)=0$ where $f\in\mathbb{C}[x,y]$, $\deg(f)\geq 1$, such that there exists $k(x,y)\in\mathbb{C}[x,y]$ satisfying $\tilde{D}f=fk$ in $\mathbb{C}[x,y]$. We call k the *cofactor* of $f(x,y)$ with respect to the system.

We stress the fact that we have $f(x,y)\in\mathbb{C}[x,y]$. This is important because even in the case when we are only interested in integrability of real systems, the complex invariant curves are helpful in the search for a real first integral of the systems.

If a planar polynomial differential system has invariant algebraic curves then these curves could have *multiplicities*. Just as a singularity of a system could be a multiple singularity, meaning that in perturbations this singularity splits into two or more singularities, so also algebraic invariant curves could have multiplicities, meaning that in neighboring systems this curve splits into two or more invariant algebraic curves. In [12] the authors define several notions of multiplicity of invariant curves and show that they coincide for irreducible invariant curves under some "generic" conditions.

In this work we shall only need invariant straight lines and their multiplicities (we work with the definitions given in [31]). All planar Lotka-Volterra systems possess at least two distinct affine invariant lines ($x = 0$ and $y = 0$) and the line at infinity is also invariant. We could also have other invariant lines and each invariant line could have multiplicity other than one.

Definition 2.2. Consider a planar quadratic differential system. We call *configuration of invariant lines* of this system and we denote it by $\mathcal{C}$, the set of invariant lines (which may, but not necessarily, have real coefficients) of the system, each one which is not filled up with singularities, endowed with its own multiplicity and together with all the real singular points of this system, located on these invariant lines, each isolated singularity endowed with its own multiplicity. We denote by $\mathcal{C}^*$ the set of all isolated invariant lines which are not filled up with singularities.

This is a more powerful global classifying concept than anyone used in [27], [18], [13].

If a system has a finite number of invariant lines and each one of them has finite multiplicity, we encode globally the information regarding the multiplicities of the invariant lines of its configuration in the notion of *multiplicity divisor* of invariant lines. Moreover we encode globally the information regarding the multiplicities of the real singularities located on the invariant lines in the configuration in the concept of *zero-cycle of multiplicities of singularities* of its configuration. We have the following formal definitions:

Definition 2.3. We consider an LV-system possessing a configuration $\mathcal{C}$ having a finite number of invariant lines not filled up with singularities, each with its multiplicity.

(i) We attach to this system the *multiplicity divisor on the projective plane* corresponding to the configuration $\mathcal{C}$. This is defined as the formal sum:

$$D_{\mathbb{C}}(\mathcal{C}) = \sum_{L\in\mathcal{C}^*} M(L)L,$$

where L is a projective invariant line of $\mathcal{C}$, and $M(L)$ is the multiplicity of this line.

(ii) We attach to a configuration $\mathcal{C}$ the *multiplicity zero-cycle on the projective plane* which counts the multiplicities of the real isolated singularities of the system which are located on the configuration $\mathcal{C}$. This is the formal sum:

$$D_{\mathbb{R}}(\mathrm{Sing},\mathcal{C}) = \sum_{r\in\mathcal{C}} m(r)r,$$

where $m(r)$ is the multiplicity of the isolated singular point r.

(iii) For a system *(S)* with the line at infinity not filled up with singularities we encode the multiplicities of isolated singularities at infinity in the *multiplicity divisor on the line at infinity* which is the formal sum

$$D_{\mathbb{C}}(S,Z) = \sum_{r\in\{Z=0\}} m(r)r,$$

where r is an isolated singular point at infinity and $m(r)$ denotes its multiplicity.

We use the result which affirms that a quadratic Lotka-Volterra differential system cannot have limit cycles. This theorem was proved by Bautin in [8]. Since this is an important ingredient in determining all phase portraits of the Lotka-Volterra systems we give here below its proof. Our proof is a modification of Coppel's proof in [14] in order to make the arguments more transparent by using a bit of Darboux theory which enables us to effectively see the calculations.

Theorem 2.4 (Bautin [8]). *The unique singular point inside a periodic orbit of a Lotka-Volterra quadratic differential systems is a center. Due to this such a system is integrable via the method of Darboux and so it has no limit cycle.*

Proof. Let γ be a periodic orbit of a Lotka-Volterra system. Since the two axes are affinely invariant we may assume that γ is included in the interior of the first quadrant. Let p be the unique singular point (see [14]) inside γ. The two axes $x=0$ and $y=0$ are invariant lines and hence for any α,β in $\mathbb{C}$, $R(x,y)=x^\alpha y^\beta=0$ is an invariant curve so we have $\tilde{D}R=RK$ for

$$K(x,y)=\alpha(a_0+a_1x+a_2y)+\beta(b_0+b_1x+b_2y)\in\mathbb{C}[x,y].$$

To show that p is a center it suffices to show that we can find $\alpha,\beta\in\mathbb{C}$ such that R is an integrating factor of the system, i.e. $\frac{\partial(Rp)}{\partial x}+\frac{\partial(Rq)}{\partial y}=0$. This means

$$\frac{\partial(Rp)}{\partial x}+\frac{\partial(Rq)}{\partial y}=\tilde{D}R+R\,\mathrm{div}(p,q)=R(K+\mathrm{div}(p,q))=0.$$

Hence we search for α,β such that $K+\mathrm{div}(p,q)=0$. This equation yields the system of equations:

$$\begin{aligned}\alpha a_0+\beta b_0&=-a_0-b_0,\\ \alpha a_1+\beta b_1&=-2a_1-b_1,\\ \alpha a_2+\beta b_2&=-a_2-2b_2.\end{aligned}\tag{2.1}$$

Since the singular point p is isolated and it is not on the axes, p is the unique solution of the equations:

$$a_0+a_1x+a_2y=0,\quad b_0+b_1x+b_2y=0$$

and hence $\widehat{D}=a_1b_2-a_2b_1\neq 0$. Therefore we can solve the second and third equations in (2.1) in α and β and obtain

$$\alpha=-1+b_2(b_1-a_1)/\widehat{D},\quad \beta=-1+a_1(a_2-b_2)/\widehat{D}\tag{2.2}$$

Replacing this in $K+\mathrm{div}(p,q)$ we obtain

$$\begin{aligned}&K+\mathrm{div}(p,q)\\ &=\alpha a_0+\beta b_0+a_0+b_0\\ &=(-1+b_2(b_1-a_1)/D)a_0+(-1+(a_1(a_2-b_2))/\widehat{D}b_0+a_0+b_0=g/\widehat{D}\end{aligned}$$

where

$$g=a_0b_2(b_1-a_1)+a_1b_0(a_2-b_2).$$

Hence $\mathrm{div}(Rp,Rq)=R(K+\mathrm{div}(p,q))=Rg/\widehat{D}$. To show that $\mathrm{div}(Rp,Rq)=0$ it suffices to show that $g=0$. Since γ is a periodic orbit we have:

$$\int_\gamma(Rqdx-Rpdy)=\int_0^T(Rq\dot{x}-Rp\dot{y})dt=\int_0^T(Rqp-Rpq)dt=0$$

where T is the period of γ. We now use the formula of Green

$$\int_{\mathring{\gamma}}\mathrm{div}(Rp,Rq)dxdy=\int_\gamma Rqdx-Rpdy=0,$$

where $\overset{o}{\gamma}$ is the interior set of γ. But calculations give

$$\int_{\overset{o}{\gamma}} \operatorname{div}(Rp, Rq) dx dy = (g/\widehat{D}) \int_{\overset{o}{\gamma}} R dx dy = 0.$$

Since $x > 0$ and $y > 0$ we must have $R > 0$ and hence $g = 0$. But this gives $\operatorname{div}(Rp, Rq) = Rg/\widehat{D} = 0$ so R is an integrating factor and therefore p is a center. Furthermore since the system is integrable on the complement in $\mathbb{R}^2$ of the union of the two axes, it has no limit cycle. □

The study of quadratic systems possessing invariant straight lines began in [31] and was continued in [33], [34], and [35]. The four works jointly taken cover the full study of quadratic differential systems possessing invariant lines of at least four total multiplicity. Among these systems some but not all, belong to the class LV and for these systems we therefore already have their topological classification. We also have the topological classification of all LV- systems with the line at infinity filled up with singularities in [36].

To complete the topological classification of all LV-systems it thus suffices to give a topological classification of: *a)* the class of LV-systems possessing exactly three invariant lines all simple; *b)* the class of all degenerate LV-systems.

In [37] all possible 65 distinct configurations of invariant lines of the LV-systems were listed and necessary and sufficient conditions for the realization of each one of them were given. As we need these results we state them in the Subsection 2.4 below. The systems split into six distinct classes according to the multiplicities of their invariant lines (including the line at infinity). The necessary and sufficient conditions for the realization of each one of the configurations are expressed in [37] in terms of invariant polynomials, with respect to the action of the affine group and time homotheties.

2.1. **Group actions on polynomial systems.** Consider real planar polynomial differential systems (1.1). We denote by PS the set of all planar polynomial systems (1.1) of a fixed degree n. On the set PS acts (left action) the group $\operatorname{Aff}(2,\mathbb{R})$ of affine transformations on the plane:

$$\begin{array}{rcl} \operatorname{Aff}(2,\mathbb{R}) \times PS & \rightarrow & PS \\ (g,\, S) & \rightarrow & \tilde{S} = gS \end{array} \tag{2.3}$$

This action is defined as follows:

Consider an affine transformation $g \in \operatorname{Aff}(2,\mathbb{R})$, $g:\ \mathbb{R}^2 \to \mathbb{R}^2$. For this transformation we have:

$$g: \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + B; \quad g^{-1}: \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} - M^{-1}B.$$

where $M = \|M_{ij}\|$ is a 2×2 nonsingular matrix and B is a 2×1 matrix over $\mathbb{R}$. For every $S \in PS$ we can form its induced transformed system $\tilde{S} = gS$:

$$\frac{d\tilde{x}}{dt} = \tilde{p}(\tilde{x}, \tilde{y}), \quad \frac{d\tilde{y}}{dt} = \tilde{q}(\tilde{x}, \tilde{y}), \tag{2.4}$$

where

$$\begin{pmatrix} \tilde{p}(\tilde{x}, \tilde{y}) \\ \tilde{q}(\tilde{x}, \tilde{y}) \end{pmatrix} = M \begin{pmatrix} (p \circ g^{-1})(\tilde{x}, \tilde{y}) \\ (q \circ g^{-1})(\tilde{x}, \tilde{y}) \end{pmatrix}.$$

The map (2.3) verifies the axioms for a left group action. For every subgroup $G \subseteq \operatorname{Aff}(2,\mathbb{R})$ we have an induced action of G on PS.

Definition 2.5. Consider a subset $\mathcal{A}$ of PS and a subgroup G of $\mathrm{Aff}(2,\mathbb{R})$. We say that the subset $\mathcal{A}$ is invariant with respect to the group G if for every g in G and for every system S in $\mathcal{A}$ the transformed system gS is also in $\mathcal{A}$.

We can identify the set of systems in PS with a subset of $\mathbb{R}^m$ via the embedding $PS \hookrightarrow \mathbb{R}^m$ which associates to each system (S) in PS the m-tuple $(a_{00},\dots,b_{0n})$ of its coefficients. We denote by $\mathbb{R}^m_{\mathcal{A}}$ the image of the subset $\mathcal{A}$ of PS under the embedding $PS \hookrightarrow \mathbb{R}^m$.

For every $g \in \mathrm{Aff}(2,\mathbb{R})$ let $r_g: \ \mathbb{R}^m \to \mathbb{R}^m$ be the map which corresponds to g via this action. We know (cf. [38]) that r_g is linear and that the map $r: \mathrm{Aff}(2,\mathbb{R}) \to GL(m,\mathbb{R})$ thus obtained is a group homomorphism. For every subgroup G of $\mathrm{Aff}(2,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(m,\mathbb{R})$.

The group $\mathrm{Aff}(2,\mathbb{R})$ acts on QS and this yields an action of this group on $\mathbb{R}^{12}$. For every subgroup G of $\mathrm{Aff}(2,\mathbb{R})$, r induces a representation of G onto a subgroup $\mathcal{G}$ of $GL(12,\mathbb{R})$.

2.2. Definitions of invariant polynomials.

Definition 2.6. A polynomial $U(a\,,x,y) \in \mathbb{R}[a,x,y]$ is called a comitant with respect to $(\mathcal{A},G)$, where $\mathcal{A}$ is an affine invariant subset of PS and G is a subgroup of $\mathrm{Aff}(2,\mathbb{R})$, if there exists $\chi \in \mathbb{Z}$ such that for every $(g,a) \in G \times \mathbb{R}^m_{\mathcal{A}}$ the following identity holds in $\mathbb{R}[x,y]$:

$$U(r_g(a), g(x,y)\,) \equiv \ (\det\, g)^{-\chi}\, U(a,x,y),$$

where $\det g = \det M$. If the polynomial U does not explicitly depend on x and y then it is called invariant. The number $\chi \in \mathbb{Z}$ is called the *weight* of the comitant $U(a,x,y)$. If $G = GL(2,\mathbb{R})$ (or $G = Aff(2,\mathbb{R})$) and $\mathcal{A} = PS$ then the comitant $U(a,x,y)$ is called GL-comitant (respectively, affine comitant).

Definition 2.7. A subset $X \subset \mathbb{R}^m$ will be called G-invariant, if for every $g \in G$ we have $r_g(X) \subseteq X$.

Let $T(2,\mathbb{R})$ be the subgroup of $\mathrm{Aff}(2,\mathbb{R})$ formed by translations. Consider the linear representation of $T(2,\mathbb{R})$ into its corresponding subgroup $\mathcal{T} \subset GL(m,\mathbb{R})$, i.e. for every $\tau \in T(2,\mathbb{R})$, $\tau: \ x = \tilde{x}+\alpha, y = \tilde{y}+\beta$ we consider as above $r_\tau: \ \mathbb{R}^m \to \mathbb{R}^m$.

Definition 2.8. A comitant $U(a,x,y)$ with respect to $(\mathcal{A},G)$ is called a T-comitant if for every $(\tau,\boldsymbol{a}) \in T(2,\mathbb{R}) \times \mathbb{R}^m_{\mathcal{A}}$ the identity $U(r_\tau \cdot \boldsymbol{a}, \tilde{x}, \tilde{y}) = U(\boldsymbol{a}, \tilde{x}, \tilde{y})$ holds in $\mathbb{R}[\tilde{x},\tilde{y}]$.

Definition 2.9. The polynomial $U(a,x,y) \in \mathbb{R}[a,x,y]$ has well determined sign on $V \subset \mathbb{R}^m$ with respect to x, y if for every fixed $a \in V$, the polynomial function $U(a,x,y)$ is not identically zero on V and has constant sign outside its set of zeroes on V.

Observation 2.10. We draw attention to the fact, that if a T-comitant $U(a,x,y)$ with respect to $(\mathcal{A},G)$ of even weight is a binary form in x, y, of even degree in the coefficients of (1.1) and has well determined sign on the affine invariant algebraic subset $\mathbb{R}^m_{\mathcal{A}}$ then this property is conserved by any affine transformation and the sign is conserved.

2.3. **Main invariant polynomials associated with LV-systems.** Consider real quadratic systems; i.e., systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(a,x,y) + p_2(a,x,y) \equiv p(a,x,y), \\ \dot{y} &= q_0 + q_1(a,x,y) + q_2(a,x,y) \equiv q(a,x,y) \end{aligned} \tag{2.5}$$

with $\max(\deg(p), \deg(q)) = 2$ and

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(a,x,y) = a_{10}x + a_{01}y, \quad p_2(a,x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(a,x,y) = b_{10}x + b_{01}y, \quad q_2(a,x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2, \end{aligned}$$

where $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ is the 12-tuple of the coefficients of an arbitrary system (2.5) and denote

$$\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y].$$

Notation 2.11. We denote by $a = (a_{00}, a_{10} \dots, b_{02})$ a specific point in $\mathbb{R}^{12}$ and we keep a_{ij} and b_{ij} as parameters. Each particular system (2.5) yields an ordered 12-tuple a of its coefficients.

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y], i = 0,1,2, \\ D_i(a,x,y) &= \frac{\partial}{\partial x} p_i(a,x,y) + \frac{\partial}{\partial y} q_i(a,x,y) \in \mathbb{R}[a,x,y], i = 1,2. \end{aligned} \tag{2.6}$$

As it was shown in [38] the polynomials

$$\{\ C_0(a,x,y), \quad C_1(a,x,y), \quad C_2(a,x,y), \quad D_1(a), \quad D_2(a,x,y)\ \} \tag{2.7}$$

of degree one in the coefficients of systems (2.5) are GL-comitants of these systems.

Notation 2.12. Let $f, g \in \mathbb{R}[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{2.8}$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [19], [25])

Theorem 2.13 (see [41]). *Any GL-comitant of systems* (2.5) *can be constructed from the elements of the set* (2.7) *by using the operations:* $+, -, \times$, *and by applying the differential operation* $(*,*)^{(k)}$.

Remark 2.14. We point out that the elements of the set (2.7) generate the whole set of GL-comitants and hence also the set of affine comitants as well as of set of the T-comitants.

Notation 2.15. Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbb{R}[a, X, Y, Z, \alpha, \beta]$ where $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z, Y/Z)$, p, $q \in \mathbb{R}[a,x,y]$ and $\max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2$. Then

$$\begin{aligned} \Phi_{\alpha,\beta} = c_{11}(a,\alpha,\beta)X^2 &+ 2c_{12}(a,\alpha,\beta)XY + c_{22}(a,\alpha,\beta)Y^2 + 2c_{13}(a,\alpha,\beta)XZ \\ &+ 2c_{23}(a,\alpha,\beta)YZ + c_{33}(a,\alpha,\beta)Z^2 \end{aligned}$$

and we denote

$$\begin{aligned} D(a,x,y) &= 4\big[\det ||c_{ij}(a,y,-x)||_{i,j\in\{1,2,3\}}\big], \\ H(a,x,y) &= 4\big[\det ||c_{ij}(a,y,-x)||_{i,j\in\{1,2\}}\big]. \end{aligned} \tag{2.9}$$

We construct the following T-comitants:

Notation 2.16.

$$\begin{gathered} B_3(a,x,y) = (C_2,D)^{(1)} = \det\left[\text{Jacobian}\,(C_2,D)\right], \\ B_2(a,x,y) = (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)}, \\ B_1(a) = \text{Res}_x\,(C_2,D)\,/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,. \end{gathered} \tag{2.10}$$

Lemma 2.17 (see [31]). *For the existence of invariant affine straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).*

Let us consider the following GL-comitants of systems (2.5):

Notation 2.18.

$$\begin{gathered} M(a,x,y) = (C_2,C_2)^{(2)} = 2\ \text{Hessian}\big(C_2(x,y)\big), \quad \eta(a) = \text{Discrim}\big(C_2(x,y)\big), \\ K(a,x,y) = \det\big[\ \text{Jacobian}\big(p_2(x,y),q_2(x,y)\big)\big], \quad \mu_0(a) = \text{Discrim}\big(K(a,x,y)\big)/16, \\ N(a,x,y) = K(a,x,y) + H(a,x,y), \quad \theta(a) = \text{Discrim}\big(N(a,x,y)\big). \end{gathered} \tag{2.11}$$

Remark 2.19. We note that by the discriminant of the cubic form $C_2(a,x,y)$ we mean the expression given in Maple via the function "discrim$(C_2,x)/y^6$".

The geometrical meaning of these invariant polynomials is revealed by the next 3 lemmas (see [31]).

Lemma 2.20. *Let $(S) \in QS$ and let $a \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. The common points of $P = 0$ and $Q = 0$ (P, Q are the homogenizations of p, q) on the line $Z = 0$ are given by the common linear factors over $\mathbb{C}$ of p_2 and q_2. This yields the geometrical meaning of the comitants μ_0, K and H:*

$$\begin{aligned} &\gcd(p_2(x,y),q_2(x,y)) \\ &= \begin{cases} \text{constant} & \text{if } \mu_0(a) \neq 0; \\ bx+cy & \text{if } \mu_0(a) = 0 \text{ and } K(a,x,y) \neq 0; \\ (bx+cy)(dx+ey) & \text{if } \mu_0(a) = 0, K(a,x,y) = 0 \text{ and } H(a,x,y) \neq 0; \\ (bx+cy)^2 & \text{if } \mu_0(a) = 0, K(a,x,y) = 0 \text{ and } H(a,x,y) = 0, \end{cases} \end{aligned}$$

where $bx+cy, dx+ey \in \mathbb{C}[x,y]$ are some linear forms and $be - cd \neq 0$.

Lemma 2.21. *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a system* (2.5) *corresponding to $a \in \mathbb{R}^{12}$ is the condition $\theta(a) = 0$ (respectively, $N(a,x,y) = 0$).*

Lemma 2.22. *The form of the divisor $D_{\mathbb{C}}(\mathrm{S,Z})$ for systems* (2.5) *is determined by the corresponding conditions indicated in Table 1, where we write $\omega_1^c + \omega_2^c + \omega_3$ if two of the points, i.e. ω_1^c, ω_2^c, are complex but not real.*

To construct other necessary invariant polynomials let us consider the differential operator $\mathcal{L} = x \cdot L_2 - y \cdot L_1$ acting on $\mathbb{R}[a,x,y]$ constructed in [6] (see also [7]), where

$$L_1 = 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}};$$

Table 1

Case	Form of $D_{\mathbb{C}}(S,Z)$	Necessary and sufficient conditions on the comitants
1	$\omega_1+\omega_2+\omega_3$	$\eta>0$
2	$\omega_1^c+\omega_2^c+\omega_3$	$\eta<0$
3	$2\omega_1+\omega_2$	$\eta=0, \quad M\neq 0$
4	3ω	$M=0, \quad C_2\neq 0$
5	$D_{\mathbb{C}}(S,Z)$ undefined	$C_2=0$

$$L_2 = 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.$$

In [6] it is shown that if a polynomial $U \in \mathbb{R}[a,x,y]$ is a GL-comitant of system (2.5) then $\mathcal{L}(U)$ is also a GL-comitant.

By using this operator and the GL-comitant $\mu_0(a)=\mathrm{Res}_x(p_2(x,y),q_2(x,y))/y^4$ we construct the following polynomials:

$$\mu_i(a,x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,4, \quad \text{where} \quad \mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)). \tag{2.12}$$

These polynomials are in fact GL-comitants of systems (2.5). The geometrical meaning of the GL-comitants $\mu_i(a,x,y), i=0,1,\dots,4$ is revealed by the next 2 lemmas (see [32]).

Lemma 2.23. *The system $P(X,Y,Z)=Q(X,Y,Z)=0$ possesses exactly four solutions counted with the multiplicities. Then m $(1\le m\le 4)$ of these solutions lie on $Z=0$ if and only if for every $i\in\{0,1,\dots,m-1\}$ we have $\mu_i(a,x,y)=0$ and $\mu_m(a,x,y)\neq 0$ as polynomials in $\mathbb{R}[x,y]$.*

Lemma 2.24. *A quadratic system (2.5) is degenerate (i.e. $\gcd(p,q)\neq \text{constant}$) if and only if $\mu_i(a,x,y)=0$ as polynomials in $\mathbb{R}[x,y]$ for every $i=0,1,2,3,4$.*

Using the transvectant differential operator (2.8) and the invariant polynomials (2.6), (2.9) and (2.11) constructed earlier, we define the following invariant polynomials which will be needed later (see also [34], [35]):

$$H_1(a)=-\big((C_2,C_2)^{(2)},C_2)^{(1)},D\big)^{(3)};$$
$$H_2(a,x,y)=(C_1,2H-N)^{(1)}-2D_1N;$$
$$H_3(a,x,y)=(C_2,D)^{(2)};$$
$$H_4(a)=\big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)};$$
$$H_5(a)=\big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)}+8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)};$$
$$H_6(a,x,y)=16N^2(C_2,D)^{(2)}+H_2^2(C_2,C_2)^{(2)};$$
$$H_7(a)=(N,C_1)^{(2)};$$
$$H_8(a)=9\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)}+2\Big[(C_2,D)^{(3)}\Big]^2;$$

$$H_9(a) = -\Big(\big((D,D)^{(2)},D\big)^{(1)},D\Big)^{(3)};$$
$$H_{10}(a) = \big((N,D)^{(2)},D_2\big)^{(1)};$$
$$H_{11}(a,x,y) = 8H\big[(C_2,D)^{(2)} + 8(D,D_2)^{(1)}\big] + 3H_2^2;$$
$$H_{12}(a,x,y) = (D,D)^{(2)} \equiv \text{Hessian}(D);$$
$$H_{13}(a,x,y) = 2(\tilde{B},C_2)^{(3)} + \big((C_2,D)^{(2)} + (D_2,D)^{(1)},\tilde{E}\big)^{(2)};$$
$$\begin{aligned}H_{14}(a,x,y) = {}& 96(D,C_2)^{(3)}(9\mu_0+\eta)\\ & - 4\Big(\Big(\big((B_3,D_2)^{(1)},D_2\big)^{(1)},D_2\Big)^{(1)},D_2\Big)^{(1)} - 54\big((H,\tilde{F})^{(1)},K\big)^{(2)}\\ & - 9\Big[\Big(\big(2(C_2,D)^{(2)} + 11(D_2,D)^{(1)},H\big)^{(1)},K\Big)^{(2)}\Big];\end{aligned}$$
$$N_1(a,x,y) = C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)};$$
$$N_2(a,x,y) = D_1(C_1,C_2)^{(2)} - \big((C_2,C_2)^{(2)},C_0\big)^{(1)};$$
$$N_5(a,x,y) = \big[(D_2,C_1)^{(1)} + D_1D_2\big]^2 - 4\big(C_2,C_2\big)^{(2)}\big(C_0,D_2\big)^{(1)},$$
$$\mathcal{G}_2(a) = 8H_8 - 9H_5,$$
$$\mathcal{G}_3(a) = (\mu_0-\eta)H_1 - 6\eta(H_4 + 12H_{10}),$$

where $\tilde{B}(a,x,y)$, $\tilde{E}(a,x,y)$ and $\tilde{F}(a,x,y)$ are defined on the page 14 below.

Apart from the invariant polynomials constructed above, which in fact are responsible for the configurations of the invariant lines for the family LV-systems, we also need polynomials for distinguishing phase portraits.

First we construct the following GL—comitants of the second degree with respect to the coefficients of the initial system

$$\begin{aligned}&T_1 = (C_0,C_1)^{(1)}, \quad T_2 = (C_0,C_2)^{(1)}, \quad T_3 = (C_0,D_2)^{(1)},\\ &T_4 = (C_1,C_1)^{(2)}, \quad T_5 = (C_1,C_2)^{(1)}, \quad T_6 = (C_1,C_2)^{(2)},\\ &T_7 = (C_1,D_2)^{(1)}, \quad T_8 = (C_2,C_2)^{(2)}, \quad T_9 = (C_2,D_2)^{(1)}.\end{aligned} \tag{2.13}$$

Then we define a family of T-comitants expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$ (see [10]):

$$\tilde{A} = \big(C_1, T_8 - 2T_9 + D_2^2\big)^{(2)}/144,$$

$$\begin{aligned}\widetilde{B} = \Big\{&16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2)\\ &+ 32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7)\\ &+ 2\,(D_2,T_9)^{(1)}\,\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0,T_5)^{(1)}\,\big)\\ &+ 6\,(D_2,T_7)^{(1)}\,\big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5)\\ &+ C_2(9T_4 + 96T_3)\big]\\ &+ 6\,(D_2,T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2]\\ &+ 48D_2\,(D_2,T_1)^{(1)}\,\big(2D_2^2 - T_8\big)\\ &- 32D_1T_8\,(D_2,T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2,T_8)^{(1)}\,\big(D_1^2 + 4T_3\big)\end{aligned}$$

$$
\begin{aligned}
&+ 12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2-2C_2D_1)+6D_1D_2T_4\,\big(T_8-7D_2^2-42T_9\big)\\
&+ 12D_1\,(C_1,T_8)^{(1)}\,(T_7+2D_1D_2)+96D_2^2\Big[D_1\,(C_1,T_6)^{(1)}+D_2\,(C_0,T_6)^{(1)}\Big]\\
&- 16D_1D_2T_3\,\big(2D_2^2+3T_8\big)-4D_1^3D_2\,\big(D_2^2+3T_8+6T_9\big)+6D_1^2D_2^2\,(7T_6+2T_7)\\
&- 252D_1D_2T_4T_9\Big\}/(2^83^3),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{D} =&\big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}\\
&+6D_1(C_1D_2-T_5)-9D_1^2C_2\big]/36,\\
\widetilde{E} =& \Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,\\
\widetilde{F} =&\Big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widetilde{E}\\
&-24\Big(C_2,\widetilde{D}\Big)^{(2)}+120\Big(D_2,\widetilde{D}\Big)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widetilde{K} =& (T_8+4T_9+4D_2^2)/72\equiv\big(p_2(x,y),q_2(x,y)\big)^{(1)}/4,\\
\widetilde{H} =& (-T_8+8T_9+2D_2^2)/72.
\end{aligned}
$$

These polynomials in addition with (2.6) and (2.13) will serve as bricks in constructing algebraic affine invariants for systems (2.5). Using these bricks, the minimal polynomial basis of affine invariants up to degree 12, containing 42 elements $A_1,\dots,A_{42}$, was constructed in [10]. The following are the elements of this polynomial basis:

$A_1=\tilde{A}$,

$A_2=(C_2,\widetilde{D})^{(3)}/12$,

$A_3=\big[C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48$,

$A_4=(\widetilde{H},\widetilde{H})^{(2)}$,

$A_5=(\widetilde{H},\widetilde{K})^{(2)}/2$,

$A_6=(\widetilde{E},\widetilde{H})^{(2)}/2$,

$A_7=\big[C_2,\widetilde{E})^{(2)},D_2\big)^{(1)}/8$,

$A_8=\big[\widetilde{D},\widetilde{H})^{(2)},D_2\big)^{(1)}/8$,

$A_9=\big[\widetilde{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48$,

$A_{10}=\big[\widetilde{D},\widetilde{K})^{(2)},D_2\big)^{(1)}/8$,

$A_{11}=(\widetilde{F},\widetilde{K})^{(2)}/4$,

$A_{12}=(\widetilde{F},\widetilde{H})^{(2)}/4$,

$A_{13}=\big[C_2,\widetilde{H})^{(1)},\widetilde{H}\big)^{(2)},D_2\big)^{(1)}/24$,

$A_{14}=(\widetilde{B},C_2)^{(3)}/36$,

$A_{22}=\big[C_2,\widetilde{D})^{(1)},D_2\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}D_2\big)^{(1)}/1152$,

$A_{23}=\big[\widetilde{F},\widetilde{H})^{(1)},\widetilde{K}\big)^{(2)}/8$,

$A_{24}=\big[C_2,\widetilde{D})^{(2)},\widetilde{K}\big)^{(1)},\widetilde{H}\big)^{(2)}/32$,

$A_{25}=\big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{E}\big)^{(2)}/16$,

$A_{26}=(\widetilde{B},\widetilde{D})^{(3)}/36$,

$A_{27}=\big[\widetilde{B},D_2)^{(1)},\widetilde{H}\big)^{(2)}/24$,

$A_{28}=\big[C_2,\widetilde{K})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{E}\big)^{(2)}/16$,

$A_{29}=\big[\widetilde{D},\widetilde{F})^{(1)},\widetilde{D}\big)^{(3)}/96$,

$A_{30}=\big[C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{D}\big)^{(3)}/288$,

$A_{31}=\big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{K}\big)^{(1)},\widetilde{H}\big)^{(2)}/64$,

$A_{32}=\big[\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},\widetilde{H}\big)^{(1)},D_2\big)^{(1)}/64$,

$A_{33}=\big[\widetilde{D},D_2)^{(1)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128$,

$A_{34}=\big[\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},\widetilde{K}\big)^{(1)},D_2\big)^{(1)}/64$,

$A_{35}=\big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128$,

$$
\begin{aligned}
&A_{15} = (\widetilde{E},\widetilde{F})^{(2)}/4, && A_{36} = \big[\widetilde{D},\widetilde{E})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{H}\big)^{(2)}/16,\\
&A_{16} = \big[\widetilde{E},D_2)^{(1)},C_2\big)^{(1)},\widetilde{K}\big)^{(2)}/16, && A_{37} = \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{D}\big)^{(1)},\widetilde{D}\big)^{(3)}/576,\\
&A_{17} = \big[\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64, && A_{38} = \big[C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(2)},\widetilde{D}\big)^{(1)},\widetilde{H}\big)^{(2)}/64,\\
&A_{18} = \big[\widetilde{D},\widetilde{F})^{(2)},D_2\big)^{(1)}/16, && A_{39} = \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{F}\big)^{(1)},\widetilde{H}\big)^{(2)}/64,\\
&A_{19} = \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{H}\big)^{(2)}/16, && A_{40} = \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{F}\big)^{(1)},\widetilde{K}\big)^{(2)}/64,\\
&A_{20} = \big[C_2,\widetilde{D})^{(2)},\widetilde{F}\big)^{(2)}/16, && A_{41} = \big[C_2,\widetilde{D})^{(2)},\widetilde{D}\big)^{(2)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)}/64,\\
&A_{21} = \big[\widetilde{D},\widetilde{D})^{(2)},\widetilde{K}\big)^{(2)}/16, && A_{42} = \big[\widetilde{D},\widetilde{F})^{(2)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is a shorthand to avoid placing up to five parenthesizes "(" which otherwise would be necessary.

Finally we construct the affine invariants which we need (see also [3]):

$$
\begin{gathered}
U_1(a) = A_1(A_1A_2 - A_{14} - A_{15}),\\
U_2(a) = -2A_2^2 - 2A_{17} - 3A_{19} + 6A_{21},\\
U_3(a) = 6A_1^2 - 3A_8 + A_{10} + A_{11} - 3A_{12},\\
U_4(a) = A_{30}, \quad G_9(a) = (A_4 + 2A_5)/4,
\end{gathered}
$$

$$
\begin{aligned}
&W_3(a)\\
&= \big[9A_1^2(36A_{18} - 19A_2^2 + 134A_{17} + 165A_{19}) + 3A_{11}(42A_{18} - 102A_{17} + 195A_{19})\\
&\quad + 2A_2^2(A_{10} + 3A_{11}) + 102A_3(3A_{30} - 14A_{29}) - 63A_6(17A_{25} + 30A_{26})\\
&\quad + 3A_{10}(14A_{18} - 118A_{17} + 153A_{19} + 120A_{21}) + 6A_7(329A_{25} - 108A_{26})\\
&\quad + 3A_8(164A_{18} + 153A_{19} - 442A_{17}) + 9A_{12}(2A_{20} - 160A_{17} - 2A_{18} - 59A_{19})\\
&\quad + 3A_1(77A_2A_{14} + 235A_2A_{15} - 54A_{36}) + 18A_{21}(21A_9 - 5A_{11}) + 302A_2A_{34}\\
&\quad - 366A_{14}^2 - 12A_{15}(71A_{14} + 80A_{15})\big]/9,\\
&W_4(a)\\
&= \big[1512A_1^2(A_{30} - 2A_{29}) - 648A_{15}A_{26} + 72A_1A_2(49A_{25} + 39A_{26})\\
&\quad + 6A_2^2(23A_{21} - 1093A_{19}) - 87A_2^4 + 4A_2^2(61A_{17} + 52A_{18} + 11A_{20})\\
&\quad - 6A_{37}(352A_3 + 939A_4 - 1578A_5) - 36A_8(396A_{29} + 265A_{30})\\
&\quad + 72A_{29}(17A_{12} - 38A_9 - 109A_{11}) + 12A_{30}(76A_9 - 189A_{10} - 273A_{11} - 651A_{12})\\
&\quad - 648A_{14}(23A_{25} + 5A_{26}) - 24A_{18}(3A_{20} + 31A_{17}) + 36A_{19}(63A_{20} + 478A_{21})\\
&\quad + 18A_{21}(2A_{20} + 137A_{21}) - 4A_{17}(158A_{17} + 30A_{20} + 87A_{21})\\
&\quad - 18A_{19}(238A_{17} + 669A_{19})\big]/81.
\end{aligned}
$$

2.4. **Preliminary results involving the use of polynomial invariants.** We consider the family of real quadratic systems (2.5). We shall use the following lemma, which gives the conditions on the coefficients of the systems (2.5) so that the origin of coordinates be a center. To do this we present the systems (2.5) with

$a_{00} = b_{00} = 0$ in the following tensorial form (see [38]):

$$
\begin{gathered}
\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \quad (j, \alpha, \beta = 1, 2);\\
a^1_1 = a_{10}, \quad a^1_2 = a_{01}, \quad a^1_{11} = a_{20}, \quad a^1_{22} = a_{02},\\
a^2_1 = b_{10}, \quad a^2_2 = b_{01}, \quad a^2_{11} = b_{20}, \quad a^2_{22} = b_{02},\\
a^1_{12} = a^1_{21} = a_{11}, \quad a^2_{12} = a^2_{21} = b_{11}.
\end{gathered} \tag{2.14}
$$

Lemma 2.25 ([38]). *The singular point* $(0,0)$ *of a quadratic system* (2.14) *is a center if and only if* $I_2 < 0$, $I_1 = I_6 = 0$ *and one of the following sets of conditions holds:*

$$(1)\ I_3 = 0; \quad (2)\ I_{13} = 0; \quad (3)\ 5I_3 - 2I_4 = 13I_3 - 10I_5 = 0,$$

where

$$
\begin{gathered}
I_1 = a^\alpha_\alpha, \quad I_2 = a^\alpha_\beta a^\beta_\alpha, \quad I_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma} \varepsilon^{pq}, \quad I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma} \varepsilon^{pq},\\
I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq}, \quad I_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta} \varepsilon^{pq}, \quad I_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu} \varepsilon^{pq} \varepsilon^{rs}.
\end{gathered}
$$

and the unit bi-vector ε^{pq} *has the coordinates:* $\varepsilon^{12} = -\varepsilon^{21} = 1$, $\varepsilon^{11} = \varepsilon^{22} = 0$.

Following [37] we denote by QSL_i the family of all non-degenerate quadratic differential systems possessing invariant straight lines (including the line at infinity not filled up with singularities) of total multiplicity i with $i \in \{3, 4, 5, 6\}$

The following is a corollary of Lemma 2.17.

Corollary 2.26. *A necessary condition for a quadratic system* (2.5) *to be in the class LV (i.e. to possess two intersecting real invariant affine lines) is that the condition* $B_2(a, x, y) = 0$ *be verified in* $\mathbb{R}[x, y]$.

According to [31] and [34] we have:

Lemma 2.27. *If a quadratic system* (S) *corresponding to a point* $a \in \mathbb{R}^{12}$ *belongs to the class* $\mathrm{QSL}_4 \cup \mathrm{QSL}_5 \cup \mathrm{QSL}_6$, *then for this system one of the following sets of conditions are satisfied in* $\mathbb{R}[x, y]$, *respectively:*
$(S) \in \mathrm{QSL}_4 \Rightarrow$ *either* $\theta(a) \neq 0$ *and* $B_3(a, x, y) = 0$, *or* $\theta(a) = 0 = B_2(a, x, y)$;
$(S) \in \mathrm{QSL}_5 \Rightarrow$ *either* $\theta(a) = 0 = B_3(a, x, y)$, *or* $N(a, x, y) = 0 = B_2(a, x, y)$;
$(S) \in \mathrm{QSL}_6 \Rightarrow N(a, x, y) = 0 = B_3(a, x, y)$.

The next theorem sums up several results in [37, 33, 35, 36].

Theorem 2.28. *There are 65 distinct configurations of planar quadratic differential LV-systems, given in Fig. 2 and Fig. 4. The systems split into six distinct classes according to the multiplicities of their invariant lines (including the line at infinity) and to the presence of lines filled up with singularities, as follows:*

I. The LV-systems with exactly three invariant straight lines which are all simple. These have 13 configurations Config. 3.j, $j = 1, 2, \dots, 13$. *The affine invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 2.*

II. The LV-systems with four invariant straight lines counted with multiplicity. These have 19 configurations Config. 4.j with $j \in \{1, 3, 4, 5, 9, 10, 11, 12, 16, \dots, 26\}$. *The affine invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective phase portraits given in Fig. 3 are indicated in Table 3.*

III. The LV-systems with five invariant straight lines counted with multiplicity. These have 11 configurations Config. 5.j with $j \in \{1, 3, 7, 8, 11, 12, 13, 14, 17, 18, 19\}$*. The affine invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective phase portraits given in Fig. 3 are indicated in Table 3.*

IV. The LV-systems with six invariant straight lines counted with multiplicity. These have four configurations Config. 6.j with $j \in \{1, 5, 7, 8\}$*. The affine invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective phase portraits given in Fig. 3 are indicated in Table 3.*

V. The non-degenerate LV-systems with a line of singularities at infinity. For these systems the condition $C_2 = 0$ *holds and they have four configurations Config.* $C_2.j$ *with* $j \in \{1, 3, 5, 7\}$*. The affine invariant necessary and sufficient conditions for the realization of each one of these configurations as well as the additional conditions for the respective five phase portraits given in Fig. 1 are indicated in Table 3.*

VI. The degenerate LV-systems defined by the conditions $\mu_i = 0$*,* $i = 0, 1, \dots, 4$ *possessing at least one affine line filled with singularities. These have 14 configurations Config.* $LV_d.j$ *with* $j \in \{1, \dots, 14\}$*. The affine invariant necessary and sufficient conditions for the realization of each one of these configurations as well as its respective representative are indicated in Table 4.*

3. Proof of the main theorem

We first prove two Lemmas (3.3 and 3.4) which will be needed later.

We shall denote by $\tilde{s}$ (respectively $\tilde{n}$; $\tilde{f}$; $\tilde{c}$; $\tilde{s}\tilde{n}$) a singular point of *saddle* (respectively *node*; *focus*; *center*; *saddle-node*) type.

Assume that a quadratic system is an LV-system. Then due to an affine transformation we can assume that this system belongs to the family of the systems

$$\dot{x} = x(c + gx + hy), \quad \dot{y} = y(f + mx + ny), \tag{3.1}$$

which in the generic case possess the following finite singularities: the origin which we denote by M_1, a singularity on the x-axis which we denote by M_3, another one on the y-axis denoted by M_3 and a fourth singularity denoted by M_4 which is not located on anyone of the two axes. The specific values od the coordinates are:

$$M_1(0,0), \quad M_2(-c/g, 0), \quad M_3(0, -f/n), \quad M_4\left(\frac{cn - fh}{hm - gn}, \frac{fg - cm}{hm - gn}\right) \tag{3.2}$$

in case when the corresponding denominators are different from zero.

The systems (3.1) possess in the generic case three infinite singular points which are:

$$R_1(g - m, n - h, 0), \quad R_2(1, 0, 0), \quad R_3(0, 1, 0).$$

Observation 3.1. The lines $y = 0$ and $x = 0$ intersect the line at infinity at $(1, 0, 0)$ and $(0, 1, 0)$) respectively. We denote by R_2 and R_3 these points so as to have M_2 and R_2 (respectively M_3 and R_3) on the same invariant line.

Notation 3.2. Whenever for a system (3.1) we have a multiple singularity we indicate which of the singularities coalesced,for example $M_3 \equiv M_1$ or $M_4 \equiv R_1$.

For each finite singular point M_j, $j \in \{1,2,3,4\}$ we denote its basic invariants: the trace ρ_j, the determinant Δ_j and the discriminant δ_j.

Similarly for each infinite singular point R_j, $j \in \{1,2,3\}$ we denote the respective basic invariants by $\tilde{\rho}_j$, $\tilde{\Delta}_j$ and $\tilde{\delta}_j$.

Lemma 3.3. *Assume that a quadratic system belongs to the family* LV*. Then $B_3 = 0$ is a necessary condition for the system to have a center. Moreover, if this system possesses a focus, then this focus could only be a strong one, i.e. its trace is not zero.*

Proof. Since the points M_1, M_2 and M_3 are placed on the invariant lines it is clear that only the point M_4 (which exists if $hm - gh \neq 0$) could be of the focus-center type. As it is known this occurs only if the discriminant δ_4 of the equation for the eigenvalues corresponding to the point M_4 is negative. Moreover it could be a center only if the corresponding trace $\rho_4 = 0$, where

$$\rho_4 = \big[n(cm - fg) + g(fh - cn))\big]/(gn - hm),$$
$$\delta_4 = \Big\{\big[n(cm - fg) - g(fh - cn))\big]^2 + 4hm(cm - fg)(fh - cn)\Big\}/(gn - hm)^2.$$

On the other hand for systems (3.1) calculation yields

$$B_3 = 3(f - c)\big[n(cm - fg) + g(fh - cn))\big]x^2y^2 = 3(f - c)\rho_4 x^2y^2$$

and therefore the condition $\rho_4 = 0$ implies $B_3 = 0$, i.e. the last condition is necessary for the existence of a center.

Suppose now that a system from the family LV-systems possesses a weak focus. For this it is necessary to have $\delta_4 < 0$ and $\rho_4 = 0$. But as we show below a singular point satisfying these conditions is necessarily a center. Indeed placing the singular point M_4 at the origin of coordinates we get the systems

$$\dot{x} = \Big(\frac{cn - fh}{hm - gn} + x\Big)(gx + hy), \quad \dot{y} = \Big(\frac{fg - cm}{hm - gn} + y\Big)(mx + ny). \tag{3.3}$$

Considering Lemma 2.25 for these systems calculations yield

$$\begin{gathered} I_1 = \frac{n(fg - cm) + g(cn - fh)}{hm - gn}, \quad 2I_2 = I_1^2 + \delta_4, \\ I_3 = I_1(mn - gh)/2, \quad I_6 = \frac{I_1}{4(hm - gn)} W(c,f,g,h,m,n), \end{gathered} \tag{3.4}$$

where $W(c,f,g,h,m,n)$ is a polynomial in the coefficients of systems. According to Lemma 2.25 the condition $I_1 = 0$ is necessary for $(0,0)$ to be a center. Then $I_6 = I_3 = 0$ and the condition $\delta_4 < 0$ implies $I_2 < 0$. Thus the conditions of Lemma 2.25 are satisfied and hence, $(0,0)$ of systems (3.3) and, consequently, the singular point M_4 of systems (3.1) is a center.

Thus a Lotka-Volterra quadratic system could not possess a weak focus. □

As it will follow from the proof of the Main Theorem the following assertion is valid:

Lemma 3.4. *Assume that a quadratic system belongs to the class LV. Then the system possesses one of the following 18 configurations of finite singularities and each one of them is realizable in this class:*

Table 2

Orbit representative	Necessary and sufficient conditions: $B_2=0$ and	Configuration
$(III.1)\ \begin{cases}\dot x = x[1+gx+(h-1)y],\\ \dot y = y[f+(g-1)x+hy],\\ f,g,h\in\mathbb{R},\quad \text{cond. }(\mathcal{A}_1)\end{cases}$	$\eta>0,\ \mu_0B_3H_9\neq 0$ and either $\theta\neq 0$ or $(\theta=0\ \&\ NH_7\neq 0)$	Config. 3.1
$(III.2)\ \begin{cases}\dot x = x[1+gx+(h-1)y],\\ \dot y = y[(g-1)x+hy],\\ g,h\in\mathbb{R},\quad \text{cond. }(\mathcal{A}_2)\end{cases}$	$\eta>0,\ \mu_0B_3\neq 0,\ H_9=H_{13}=0$ and either $\theta\neq 0$ or $(\theta=0\ \&\ NH_7\neq 0)$	Config. 3.2
$(III.3)\ \begin{cases}\dot x = x[g+gx+(h-1)y],\\ \dot y = y[g-1+(g-1)x+hy],\\ g,h\in\mathbb{R},\quad \text{cond. }(\mathcal{A}_2)\end{cases}$	$\eta>0,\ \mu_0B_3H_{13}\neq 0,\ H_9=0$ and either $\theta\neq 0$ or $(\theta=0\ \&\ NH_7\neq 0)$	Config. 3.3
$(III.4)\ \begin{cases}\dot x = x[1+(h-1)y],\\ \dot y = y(f-x+hy),\\ f,h\in\mathbb{R},\quad \text{cond. }(\mathcal{A}_3)\end{cases}$	$\eta>0,\theta B_3H_9\neq 0,\mu_0=H_{14}=0$	Config. 3.4
$(III.5)\ \begin{cases}\dot x = x[1+(1-h)(x-y)],\\ \dot y = y(f-hx+hy),\\ f,h\in\mathbb{R},\quad \text{cond. }(\mathcal{A}_3)\end{cases}$	$\eta>0,\ \theta B_3H_9H_{14}\neq 0,\ \mu_0=0$	Config. 3.5
$(III.6)\ \begin{cases}\dot x = x[1+(h-1)y],\\ \dot y = y(-x+hy),\\ h\in\mathbb{R},\ h(h-1)\neq 0\end{cases}$	$\eta>0,\ \theta B_3\neq 0,\ \mu_0=H_9=0,$ $H_{13}=H_{14}=0$	Config. 3.6
$(III.7)\ \begin{cases}\dot x = x[h-1+(h-1)y],\\ \dot y = y(h-x+hy),\\ h\in\mathbb{R},\ h(h-1)\neq 0\end{cases}$	$\eta>0,\ \theta B_3H_{13}\neq 0,$ $\mu_0=H_9=H_{14}=0$	Config. 3.7
$(III.8)\ \begin{cases}\dot x = x[1+(1-h)(x-y)],\\ \dot y = hy(y-x),\\ h\in\mathbb{R},\ h(h-1)\neq 0\end{cases}$	$\eta>0,\theta B_3H_{14}\neq 0,\mu_0=H_9=0$	Config. 3.8
$(III.9)\ \begin{cases}\dot x = x(1+gx+y),\\ \dot y = y(f-x+gx+y),\\ f,g\in\mathbb{R},\quad \text{cond. }(\mathcal{A}_4)\end{cases}$	$\eta=0,\ \theta H_4B_3\mu_0H_9\neq 0$	Config. 3.9
$(III.10)\ \begin{cases}\dot x = x(g+gx+y),\\ \dot y = y[g-1+(g-1)x+y],\\ g\in\mathbb{R},\quad g(g-1)\neq 0\end{cases}$	$\eta=0,\theta H_4B_3\mu_0H_{13}\neq 0,H_9=0$	Config. 3.10
$(III.11)\ \begin{cases}\dot x = x(1+gx+y),\\ \dot y = y(-x+gx+y),\\ g\in\mathbb{R},\quad g(g-1)\neq 0\end{cases}$	$\eta=0,\ \theta H_4B_3\mu_0\neq 0,$ $H_9=H_{13}=0$	Config. 3.11
$(III.12)\ \begin{cases}\dot x = x(1+y),\\ \dot y = y(f+x+y),\\ f\in\mathbb{R},\ f(f-1)\neq 0\end{cases}$	$\eta=0,\ \theta H_4B_3H_9\neq 0,\ \mu_0=0$	Config. 3.12
$(III.13)\ \begin{cases}\dot x = x(1+y),\\ \dot y = y(x+y),\end{cases}$	$\eta=0,\ \theta H_4B_3\neq 0,\ \mu_0=H_9=0$	Config. 3.13

$$gh(g+h-1)(g-1)(h-1)f(f-1)(fg+h)(1-g+fg)(f+h-fh)\neq 0; \qquad (\mathcal{A}_1)$$

$$gh(g+h-1)(g-1)(h-1)\neq 0; \qquad (\mathcal{A}_2)$$

$$h(h-1)f(f-1)(f+h-fh)\neq 0. \qquad (\mathcal{A}_3)$$

$$g(g-1)f(f-1)(1-g+fg)\neq 0. \qquad (\mathcal{A}_4)$$

$(a)\ \tilde s,\tilde s,\tilde s,\tilde n;$ $(\overset{*}{a})\ \tilde s,\tilde s,\tilde s,\tilde f;$ $(b)\ \tilde s,\tilde n,\tilde n,\tilde n;$ $(\overset{*}{b})\ \tilde s,\tilde n,\tilde n,\tilde f;$ $(c)\ \tilde s,\tilde s,\tilde n,\tilde n;$
$(\overset{*}{c})\ \tilde s,\tilde s,\tilde n,\tilde f;$ $(d)\ \tilde s\tilde n,\tilde s,\tilde s;$ $(e)\ \tilde s\tilde n,\tilde n,\tilde n;$ $(\overset{*}{e})\ \tilde s\tilde n,\tilde n,\tilde f;$ $(f)\ \tilde s\tilde n,\tilde s,\tilde n;$
$(\overset{*}{f})\ \tilde s\tilde n,\tilde s,\tilde f;$ $(g)\ \tilde s,\tilde s,\tilde n;$ $(\overset{*}{g})\ \tilde s,\tilde s,\tilde f;$ $(h)\ \tilde s,\tilde n,\tilde n;$ $(\overset{*}{h})\ \tilde s,\tilde n,\tilde f;$
$(k)\ \tilde s\tilde n,\tilde s;$ $(l)\ \tilde s\tilde n,\tilde n;$ $(\overset{*}{l})\ \tilde s\tilde n,\tilde f.$

Table 3

Configuration	Necessary and sufficient *conditions*	*Additional conditions for phase portraits*	Phase portrait
Config. 4.1	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,\ H_7 \neq 0$	$\mu_0 > 0$	Picture 4.1(a)
		$\mu_0 < 0,\ K < 0$	Picture 4.1(b)
		$\mu_0 < 0,\ K > 0$	Picture 4.1(c)
Config. 4.3	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 \neq 0,\ \mu_0 \neq 0$	$\mu_0 > 0$	Picture 4.3(a)
		$\mu_0 < 0,\ K < 0$	Picture 4.3(b)
		$\mu_0 < 0,\ K > 0$	Picture 4.3(c)
Config. 4.4	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 \neq 0,\ \mu_0 = 0$	$K < 0$	Picture 4.4(a)
		$K > 0$	Picture 4.4(b)
Config. 4.5	$\eta > 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ H_1 = 0$	$\mu_0 > 0$	Picture 4.5(a)
		$\mu_0 < 0,\ K < 0$	Picture 4.5(b)
		$\mu_0 < 0,\ K > 0$	Picture 4.5(c)
Config. 4.9	$\eta > 0,\ B_2 = \theta = H_7 = 0,$ $\mu_0 B_3 H_4 H_9 \neq 0$ *and either* $H_{10} N > 0$ *or* $N = 0,\ H_8 > 0$	$\mathcal{G}_2 > 0, H_4 > 0, \mathcal{G}_3 < 0$	Picture 4.9(a)
		$\mathcal{G}_2 < 0$	Picture 4.9(b)
		$\mathcal{G}_2 > 0, H_4 < 0$	
		$\mathcal{G}_2 > 0, H_4 > 0, \mathcal{G}_3 > 0$	Picture 4.9(c)
Config. 4.10	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 \neq 0, H_7 = H_9 = 0, H_{10} N > 0$	$H_4 > 0, \mathcal{G}_3 > 0$	Picture 4.10(a)
		$H_4 < 0$	Picture 4.10(b)
		$H_4 > 0, \mathcal{G}_3 < 0$	Picture 4.10(c)
	$\eta > 0, B_3 H_4 \neq 0,$ $B_2 = N = H_9 = 0, H_8 > 0$	–	
Config. 4.11	$\eta = 0,\ M B_3 \neq 0,\ B_2 = \theta = 0,$ $H_7 = 0,\ \mu_0 \neq 0,\ H_{10} > 0$	$H_4 > 0$	Picture 4.11(a)
		$H_4 < 0$	Picture 4.11(b)
Config. 4.12	$\eta = 0,\ M \neq 0,\ B_3 = \theta = 0,$ $K H_6 \neq 0, H_7 = \mu_0 = 0, H_{11} > 0$	$\mu_2 > 0,\ L > 0$	Picture 4.12(a)
		$\mu_2 > 0,\ L < 0$	Picture 4.12(b)
		$\mu_2 < 0,\ K < 0$	Picture 4.12(c)
		$\mu_2 < 0, K > 0, L > 0$	Picture 4.12(d)
		$\mu_2 < 0, K > 0, L < 0$	Picture 4.12(e)
Config. 4.16	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 = H_7 = 0,\ H_9 \neq 0$	$\mathcal{G}_2 > 0$	*Portrait 4.16(a)*
		$\mathcal{G}_2 < 0$	*Portrait 4.16(b)*
Config. 4.17	$\eta > 0,\ B_3 \neq 0,\ B_2 = \theta = 0,$ $\mu_0 = H_7 = H_9 = 0,\ H_{10} \neq 0$	–	Picture 4.17
Config. 4.18	$\eta > 0,\ B_3 = \theta = 0,$ $\mu_0 = 0,\ H_7 \neq 0$	$\mu_2 L > 0$	Picture 4.18(a)
		$\mu_2 L < 0$	Picture 4.18(b)
Config. 4.19	$\eta = 0, M \neq 0, B_3 = \theta = K = 0,$ $N H_6 \neq 0,\ \mu_0 = H_7 = 0,\ H_{11} \neq 0$	$\mu_3 K_1 < 0$	Picture 4.19(a)
		$\mu_3 K_1 > 0$	Picture 4.19(b)
Config. 4.20	$\eta = 0,\ M \neq 0,\ B_3 = 0,\ \theta \neq 0,$ $H_7 = 0,\ D = 0$	$\mu_0 > 0$	Picture 4.20(a)
		$\mu_0 < 0$	Picture 4.20(b)

Table 3 (continued)

Configuration	Necessary and sufficient *conditions*	Additional conditions for *phase portraits*	Phase portrait
Config. 4.21	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D\neq 0,\ \mu_0\neq 0$	$\mu_0>0$	Picture 4.21(a)
		$\mu_0<0$	Picture 4.21(b)
Config. 4.22	$\eta>0,\ B_3\neq 0,\ B_2=\theta=0,$ $\mu_0\neq 0,\ N\neq 0,\ H_7=H_{10}=0$	$H_1>0$	Picture 4.22(a)
		$H_1<0$	Picture 4.22(b)
	$\eta>0, B_3H_4\neq 0, B_2=\theta=N=H_8=0$	–	
Config. 4.23	$\eta=0,\ MB_3\neq 0,\ B_2=\theta=0,$ $\mu_0\neq 0,\ H_7=H_{10}=0$	–	Picture 4.23
Config. 4.24	$\eta=0, M\neq 0, B_3=\theta=0,$ $KH_6\neq 0,\ \mu_0=H_7=H_{11}=0$	$L>0$	Picture 4.24(a)
		$L<0$	Picture 4.24(b)
Config. 4.25	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7\neq 0$	$\mu_0>0$	Picture 4.25(a)
		$\mu_0<0$	Picture 4.25(b)
Config. 4.26	$\eta=0,\ M\neq 0,\ B_3=0,\ \theta\neq 0,$ $H_7=0,\ D\neq 0,\ \mu_0=0$	–	Picture 4.26
Config. 5.1	$\eta>0,\ B_3=\theta=0,$ $N\neq 0, \mu_0\neq 0, H_1\neq 0$	–	Picture 5.1
Config. 5.3	$\eta>0, B_2=N=0, B_3\neq 0,$ $H_1>0,\ H_4=0,\ H_5>0$	–	Picture 5.3
Config. 5.7	$\eta>0, B_3=\theta=0,$ $N\neq 0, \mu_0=H_6=0$	–	Picture 5.7
Config. 5.8	$\eta>0, B_3=\theta=0,$ $N\neq 0, \mu_0\neq 0, H_1=0$	–	Picture 5.8
Config. 5.11	$\eta=0, M\neq 0, B_3=\theta=0,$ $\mu_0\neq 0, N\neq 0, D\neq 0$	–	Picture 5.11
Config. 5.12	$\eta>0, B_2=N=0, B_3\neq 0,$ $H_1>0, H_4=H_5=0$	–	Picture 5.12
Config. 5.13	$\eta=0, M\neq 0, B_3=N=0,$ $H=N_1=0, N_2D\neq 0, N_5>0$	–	Picture 5.13
Config. 5.14	$\eta=0, M\neq 0, B_3=\theta=0,$ $NK\neq 0,\ \mu_0=H_6=0$	$L>0$	Picture 5.14(a)
		$L<0$	Picture 5.14(b)
Config. 5.17	$\eta=0, M\neq 0, B_3=N=0,$ $H=N_1=N_5=0, N_2D\neq 0$	–	Picture 5.17
Config. 5.18	$\eta=0, M\neq 0, B_3=\theta=0,$ $N\neq 0, \mu_0=K=H_6=0$	–	Picture 5.18
Config. 5.19	$\eta=0, M\neq 0, B_3=\theta=0,$ $\mu_0\neq 0, N\neq 0, D=0$	–	Picture 5.19
Config. 6.1	$\eta>0,\ B_3=N=0,\ H_1>0$	–	Picture 6.1
Config. 6.5	$\eta>0,\ B_3=N=\ H_1=0$	–	Picture 6.5

Table 3(continued)

Configuration	Necessary and sufficient *conditions*	*Additional conditions for phase portraits*	Phase portrait
Config. 6.7	$MD \neq 0, \eta = B_3 = N = 0,$ $H = N_1 = N_2 = 0$	–	Picture 6.7
Config. 6.8	$MH \neq 0, \quad \eta = B_3 = N = 0,$ $H_2 = 0, H_3 > 0$	–	Picture 6.8
Config. $C_2.1$	$C_2 = 0,\ H_{10} \neq 0,\ H_9 < 0$	–	Picture $C_2.1$
Config. $C_2.3$	$C_2 = 0,\ H_{10} \neq 0, H_9 = 0, H_{12} \neq 0$	–	Picture $C_2.3$
Config. $C_2.5$	$C_2 = 0,\ H_{10} = 0, H_{12} \neq 0, H_{11} > 0$	$\mu_2 < 0$	Picture $C_2.5$(a)
		$\mu_2 > 0$	Picture $C_2.5$(b)
Config. $C_2.7$	$C_2 = 0,\ H_{10} = 0, H_{12} \neq 0, H_{11} = 0$	–	Picture $C_2.7$

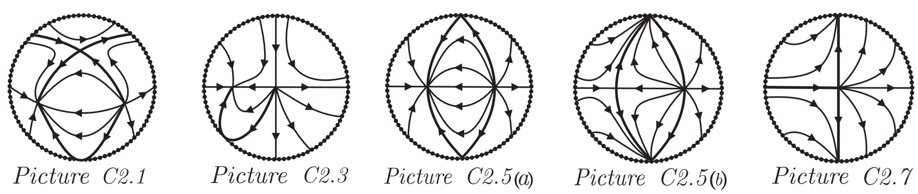


FIGURE 1. Phase portraits of LV-systems with all points at infinity singular

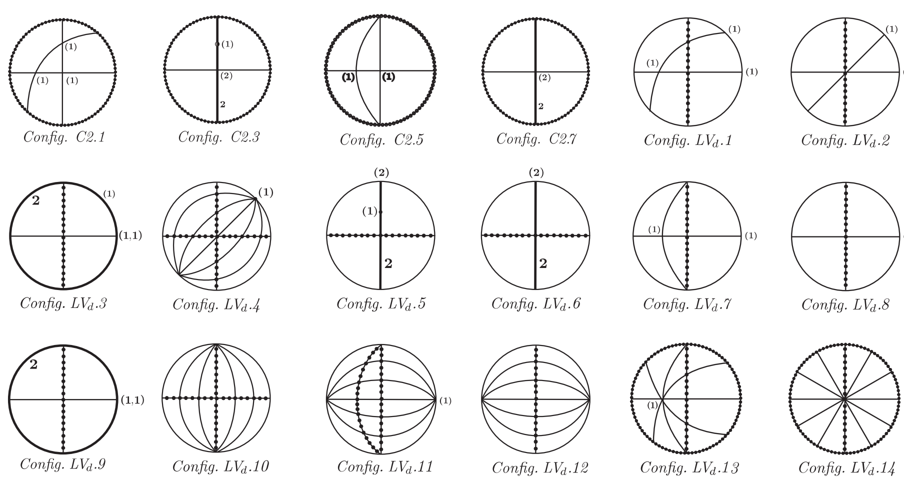


fig2

FIGURE 2. Configurations of LV-systems with infinite number of singularities

Notation 3.5. For the notations of the phase portraits corresponding to Configs. 3.j ($j = 1, 2, \dots, 13$) we shall use the number *3.j* of the configuration and the

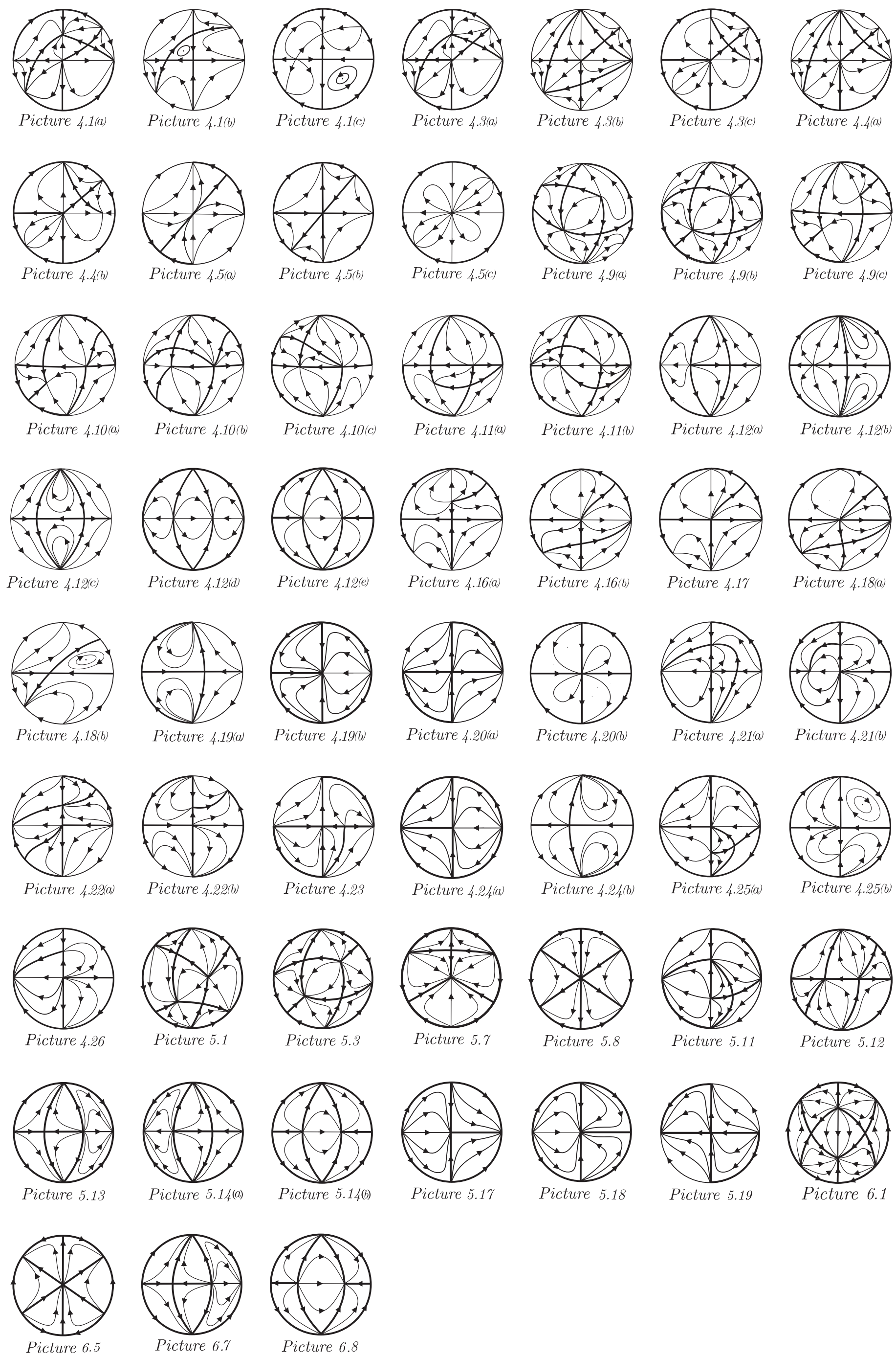


FIGURE 3. Phase portraits of LV-systems with at least 4 invariant lines

corresponding additional couple (ik) (or $(\overset{*}{i}\,k)$) with

$$(i) \in \{(a), (b), (c), (d), (e), (f), (g), (h), (k), (l)\}.$$

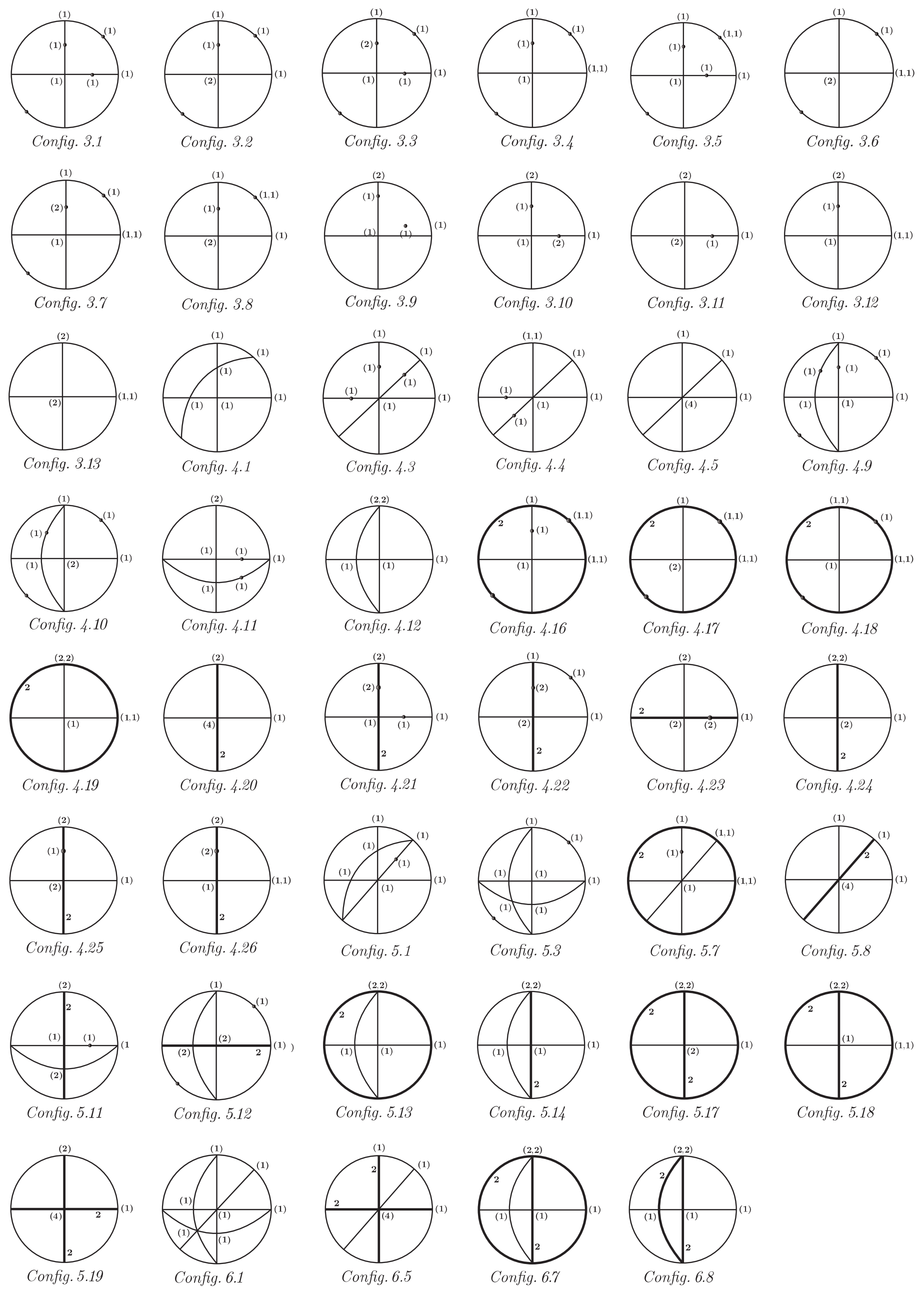


FIGURE 4. Configurations of LV-systems with finite number of singularities

The symbol (i) depends on the configuration of the finite singularities indicated by Lemma 3.4, whereas $k \in \{1, \dots, 5\}$ indicates the number we give to the respective phase portraits with the Configs. 3.j. For example, the notation Picture 3.3(e2) denotes one of the phase portraits associated with Configs. 3.3 having the finite

Table 4

Orbit representative	Necessary and sufficient conditions	Configuration
$(LV_d.1)\ \begin{cases} \dot{x}=x(1+gy-y), \\ \dot{y}=(g-1)xy, \\ g(g-1)\neq 0 \end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7\neq 0$	Config. $LV_d.1$
$(LV_d.2)\ \begin{cases} \dot{x}=x(gx-y), \\ \dot{y}=(g-1)xy, \\ g(g-1)\neq 0 \end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7=0$	Config. $LV_d.2$
$(LV_d.3)\ \begin{cases} \dot{x}=x(1+y), \\ \dot{y}=xy, \end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $H_4=0,\ H_7\neq 0$	Config. $LV_d.3$
$(LV_d.4)\ \begin{cases} \dot{x}=xy, \\ \dot{y}=xy, \end{cases}$	$\eta>0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $H_4=0,\ H_7=0$	Config. $LV_d.4$
$(LV_d.5)\ \begin{cases} \dot{x}=xy, \\ \dot{y}=y(1-x+y), \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7\neq 0$	Config. $LV_d.5$
$(LV_d.6)\ \begin{cases} \dot{x}=xy, \\ \dot{y}=y(-x+y), \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta\neq 0,\ H_7=0$	Config. $LV_d.6$
$(LV_d.7)\ \begin{cases} \dot{x}=x(1+gx), \\ \dot{y}=(g-1)xy, \\ g(g-1)\neq 0 \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K\neq 0,\ H_2\neq 0$	Config. $LV_d.7$
$(LV_d.8)\ \begin{cases} \dot{x}=gx^2, \\ \dot{y}=(g-1)xy, \\ g(g-1)\neq 0 \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K\neq 0,\ H_2=0$	Config. $LV_d.8$
$(LV_d.9)\ \begin{cases} \dot{x}=x, \\ \dot{y}=xy, \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K=0,\ N\neq 0,\ H_7=0,\ H_2\neq 0$	Config. $LV_d.9$
$(LV_d.10)\ \begin{cases} \dot{x}=0, \\ \dot{y}=xy, \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,\ K=0,$ $N\neq 0,\ H_7=0,\ H_2=0,\ D=0$	Config. $LV_d.10$
$(LV_d.11)\ \begin{cases} \dot{x}=x(x+2), \\ \dot{y}=0, \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K=0,\ N=0,\ D=N_1=0,\ N_5>0$	Config. $LV_d.11$
$(LV_d.12)\ \begin{cases} \dot{x}=x^2, \\ \dot{y}=0, \end{cases}$	$\eta=0,\ \mu_{0,1,2,3,4}=0,\ \theta=0,$ $K=0,\ N=0,\ D=N_1=0,\ N_5=0$	Config. $LV_d.12$
$(LV_d.13)\ \begin{cases} \dot{x}=x(1+x), \\ \dot{y}=xy, \end{cases}$	$C_2=0,\ \mu_{0,1,2,3,4}=0,\ H_2\neq 0$	Config. $LV_d.13$
$(LV_d.14)\ \begin{cases} \dot{x}=x^2, \\ \dot{y}=xy, \end{cases}$	$C_2=0,\ \mu_{0,1,2,3,4}=0,\ H_2=0$	Config. $LV_d.14$

singularities $\tilde{s}\tilde{n}, \tilde{n}, \tilde{n}$. We note that we keep the same letter, only adding a star (i.e. $(\overset{*}{i})$) in the case when one node is substituted by a focus (which are locally topologically equivalent).

Proof of the main theorem. To prove the main theorem we first notice that we can split the class of all LV-systems into six distinct subclasses: (i) the class of all LV-systems having exactly three simple real invariant lines; (ii) the three classes of LV- systems possessing invariant lines of total multiplicity 4, respectively 5 and 6;

(iii) the class of all LV systems with the line at infinity filled up with singularities; (iv) the class of all LV systems which are degenerate. We recall that in [33] and [35] the phase portraits of the quadratic systems with invariant lines of total multiplicity at least four are constructed. Moreover in [36] the topological classification of the whole family of quadratic systems with the infinite line filled up with singularities (the case $C_2 = 0$) is done and hence the phase portraits for the cases (ii) and (iii) are already done. So it remains firstly to examine the cases (i) which have the configurations given by Configs. 3.j with $j = 1, 2, \dots, 13$ (see Subsection 3.1) and the case (iv) of the degenerate LV-systems with the configurations given by Configs. LV_d.j with $j = 1, 2, \dots, 14$ (see Subsection 3.2); and secondly to prove in Subsection 3.3 the statements (v) of the main theorem.

3.1. **Phase portraits of LV-systems with exactly three simple real invariant straight lines.** The result concerning the normal forms, the configurations and the respective invariant criteria for this class is encapsulated in Table 2. In this Table we observe that for any system with the configuration of invariant lines given by Configs. 3.j ($j = 1, 2, \dots, 13$) the condition $B_3 \neq 0$ holds. Therefore by Lemma 3.3 this system could not have a center.

Theorem 3.6. *The phase portraits of Lotka-Volterra quadratic differential systems possessing exactly three real invariant straight lines all simple, correspond to 13 configurations. Adding up the numbers of topologically distinct phase portraits for each of the 13 configurations we end up with a total of 65 phase portraits (Fig. 5), only 60 of which are topologically distinct. In Table 2 are listed in columns 2 and 3 the necessary and sufficient conditions for the realization of each one of the portraits appearing in column four.*

Proof. We shall consider step by step each one of the configurations Configs. 3.j ($j = 1, 2, \dots, 13$). For all the configurations the proof follows the same pattern, which we describe in the steps below.

- We take for a configuration Configs. 3.j its normal form from Table 2 and we calculate the coordinates of its singularities finite and infinite.
- We evaluate for each finite singularity M_i the basic invariants: the trace ρ_i, the determinant Δ_i and the discriminant δ_i.
- For each infinite singularity R_k it suffices to evaluate the determinant $\tilde{\Delta}_k$ and the trace $\tilde{\rho}_k$.
- In order to use Table 1 from [3] we evaluate for the normal form the invariant polynomials which we need and which occur in the third column of this table. We translate the inequalities given by these polynomials into inequalities of the coefficients.
- We relate the signs of the invariant polynomials to the signs of the basic invariants Δ_j, $\tilde{\Delta}_k$ and δ_j. These give us the number of saddles and anti-saddles.
- In order to determine the types of the anti-saddles, we evaluate for the normal form the invariant polynomials we need and which occur in column four of Table 1 in [3]. The inequalities involved in that column, translate into inequalities of the coefficients of the normal form.
- We use the invariant conditions in column 5 of Table 5 from this article and we show that they lead to the phase portrait given in column 6 of Table 5.

□

3.1.1. *The phase portraits associated with Config. 3.1.* According to Table 2 we consider the family of systems

$$\dot{x} = x[1 + gx + (h-1)y], \quad \dot{y} = y[f + (g-1)x + hy], \tag{3.5}$$

for which the condition

$$gh(g+h-1)(g-1)(h-1)f(f-1)(fg+h)(g-1-fg)(fh-f-h) \neq 0 \tag{3.6}$$

holds. For all four distinct finite singularities of systems (3.5) with condition (3.6), we have

$$\begin{gathered}
M_1(0,0): \quad \Delta_1 = f, \quad \rho_1 = f+1, \quad \delta_1 = (f-1)^2;\\
M_2(-1/g,0): \quad \Delta_2 = (g-1-fg)/g, \quad \delta_2 = (1+fg)^2/g^2;\\
M_3(0,-f/h): \quad \Delta_3 = f(fh-f-h)/h, \quad \delta_3 = (f+h)^2/h^2;\\
M_4\Big(\frac{fh-f-h}{g+h-1}, \frac{g-1-fg}{g+h-1}\Big): \quad \Delta_4 = \frac{(fh-f-h)(g-1-fg)}{g+h-1},\\
\rho_4 = \frac{(fg+h)}{1-g-h}, \quad \delta_4 = \rho_4^2 - 4\Delta_4.
\end{gathered} \tag{3.7}$$

and for the three infinite singular points we obtain

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 1-g-h; \quad R_2(1,0,0): \ \widetilde{\Delta}_2 = g; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h. \tag{3.8}$$

Remark 3.7. Observe that substituting (x, y, t, f, g, h) by $(fy, fx, t/f, 1/f, h, g)$ keeps systems (3.5) and interchanges the two invariant lines and therefore also interchanges the corresponding singularities.

Considering (3.7) the condition (3.6) is equivalent to the condition

$$\Delta_1\Delta_2\Delta_3\Delta_4\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3(\widetilde{\Delta}_1+\widetilde{\Delta}_2)(\widetilde{\Delta}_1+\widetilde{\Delta}_3)\delta_1\rho_4 \neq 0 \tag{3.9}$$

Taking into account (3.7) and (3.8) we evaluate for systems (3.5) the invariant polynomials we need:

$$\begin{gathered}
\mu_0 = gh(g+h-1) = -\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3,\\
K = 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2,\\
W_4 = \mu_0^2\,\delta_1\delta_2\delta_3\delta_4, \quad W_3 = \mu_0^2 \sum_{1\le i<j<l\le 4} \delta_i\,\delta_j\,\delta_l,\\
H_9 = -576f^2(1-g+fg)^2(f+h-fh)^2 = -576\Delta_2^2\Delta_3^2,\\
H_{14} = 30(1-f)gh(fg+h) = 30\rho_4\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3(1-f),\\
B_3 = 3(1-f)(fg+h)x^2y^2 = 3\widetilde{\Delta}_1\rho_4(1-f)x^2y^2,\\
U_1 = \frac{1}{8}f(1-f)(g-1)^2(h-1)^2(fg+h) = \frac{1}{8}\Delta_1\widetilde{\Delta}_1\rho_4(1-f)(g-1)^2(h-1)^2,\\
U_2 = 3f(g-1-fg)(fh-f-h) = 3\Delta_2\Delta_3\widetilde{\Delta}_2\widetilde{\Delta}_3,\\
U_3 = \frac{1}{2}(1+f)(g-1)(h-1)(fg+h) = \frac{1}{2}\widetilde{\Delta}_1\rho_1\rho_4(g-1)(h-1),\\
U_4 = f(g-1-fg)(fh-f-h)(fg-2f-f^2g-h+fh).
\end{gathered} \tag{3.10}$$

Table 5

Configu–ration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*			*Phase portrait*
Config. 3.1	$\eta > 0, \mu_0 B_3 H_9 \neq 0$, $B_2 = 0$ *and* *either* $\theta \neq 0$ *or* $(\theta = 0$ & $NH_7 \neq 0)$	$\mu_0 < 0$, $K < 0$	$W_4 \geq 0$	$B_3U_1 < 0, U_2 < 0$	Picture 3.1(a1)
				$B_3U_1 < 0, U_2 > 0$	Picture 3.1(a2)
				$B_3U_1 > 0$	Picture 3.1(a3)
			$W_4 < 0$	–	Picture 3.1($\overset{*}{a}$2)
		$\mu_0 < 0$, $K > 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$B_3U_1 < 0$	Picture 3.1(b1)
				$B_3U_1 > 0, U_2 > 0$, $U_4 > 0, U_3 > 0$	
				$B_3U_1 > 0, U_2 < 0$, $B_3H_{14} > 0$	Picture 3.1(b2)
				$B_3U_1 > 0, U_2 > 0$, $U_4 > 0, U_3 < 0$	
				$B_3U_1 > 0, U_2 < 0$, $B_3H_{14} < 0$	Picture 3.1(b3)
				$B_3U_1 > 0, U_2 > 0$, $U_4 < 0$	
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3U_1 < 0, U_2 < 0$	Picture 3.1($\overset{*}{b}$1)
				$B_3U_1 > 0, U_2 > 0$, $U_4 > 0, U_3 > 0$	
				$B_3U_1 > 0, U_2 < 0$	Picture 3.1($\overset{*}{b}$2)
				$B_3U_1 > 0, U_2 > 0$, $U_4 > 0, U_3 < 0$	
				$B_3U_1 > 0, U_2 > 0$, $U_4 < 0$	Picture 3.1($\overset{*}{b}$3)
				$B_3U_1 < 0, U_2 > 0$	Picture 3.1($\overset{*}{b}$4)
		$\mu_0 > 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$U_2 < 0, B_3H_{14} < 0$	Picture 3.1(c1)
				$U_2 > 0, U_4 > 0$, $B_3U_1 > 0$	
				$U_2 < 0, B_3H_{14} > 0$ $B_3U_1 < 0$	Picture 3.1(c2)
				$U_2 < 0, B_3H_{14} > 0$ $B_3U_1 > 0$	Picture 3.1(c3)
				$U_2 > 0, U_4 < 0$	
				$U_2 > 0, U_4 > 0$, $B_3U_1 < 0$	Picture 3.1(c4)
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3U_1 > 0, U_2 < 0$	Picture 3.1($\overset{*}{c}$1)
				$B_3U_1 < 0, U_2 < 0$	Picture 3.1($\overset{*}{c}$2)
				$B_3U_1 > 0, U_2 > 0$	Picture 3.1($\overset{*}{c}$3)
				$B_3U_1 < 0, U_2 > 0$	Picture 3.1($\overset{*}{c}$4)

Denoting $\mathcal{A} = f(g - 1 - fg) = \Delta_1\Delta_2\widetilde{\Delta}_2$ and $\mathcal{B} = (fh - f - h) = \Delta_3\widetilde{\Delta}_3/\Delta_1$ the invariants U_2 and U_4 could be represented in the following forms, respectively:

$$U_2 = 3\mathcal{A}\mathcal{B}, \quad U_4 = \mathcal{A}\mathcal{B}(\mathcal{A} + \mathcal{B}).$$

Table 5 (continued)

Configu–ration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*			*Phase portrait*
Config. 3.2	$\eta > 0, \mu_0 B_3 \neq 0$ $B_2 = H_9 = H_{13} = 0$ *and either* $\theta \neq 0$ *or* $(\theta = 0$ & $NH_7 \neq 0)$	$\mu_0 < 0, K < 0$			Picture 3.2(d1)
		$\mu_0 < 0,$ $K > 0$	$W_4 \geq 0$	$H_1 > 0$	Picture 3.2(e1)
				$H_1 < 0$	Picture 3.2(e2)
			$W_4 < 0$	$H_1 > 0$	Picture 3.2($\overset{*}{e}1$)
				$H_1 < 0, H_5 < 0$	Picture 3.2($\overset{*}{e}2$)
				$H_1 < 0, H_5 > 0$	Picture 3.2($\overset{*}{e}3$)
		$\mu_0 > 0$	$W_4 \geq 0$	$B_3 H_{14} < 0, H_5 < 0$	Picture 3.2(f1)
				$B_3 H_{14} < 0, H_5 > 0, H_1 < 0$	Picture 3.2(f2)
				$B_3 H_{14} < 0, H_5 > 0, H_1 > 0$	Picture 3.2(f3)
				$B_3 H_{14} > 0, H_5 < 0$	Picture 3.2(f4)
				$B_3 H_{14} > 0, H_5 > 0$	Picture 3.2(f5)
			$W_4 < 0$	$H_1 < 0$	Picture 3.2($\overset{*}{f}2$)
				$H_1 > 0$	Picture 3.2($\overset{*}{f}5$)
Config. 3.3	$\eta > 0, \mu_0 B_3 \neq 0$ $B_2 = H_9 = 0, H_{13} \neq 0$ *and either* $\theta \neq 0$ *or* $(\theta = 0$ & $NH_7 \neq 0)$	$\mu_0 < 0, K < 0$			Picture 3.3(d1)
		$\mu_0 < 0,$ $K > 0$	$H_5 < 0$		Picture 3.3(e1)
			$H_5 > 0$		Picture 3.3(e2)
		$\mu_0 > 0$	$B_3 H_{14} < 0, H_5 < 0$		Picture 3.3(f1)
			$B_3 H_{14} < 0, H_5 > 0, H_1 < 0$		Picture 3.3(f2)
			$B_3 H_{14} < 0, H_5 > 0, H_1 > 0$		Picture 3.3(f3)
			$B_3 H_{14} > 0, H_5 < 0$		Picture 3.3(f4)
			$B_3 H_{14} > 0, H_5 > 0$		Picture 3.3(f5)
Config. 3.4	$\eta > 0, \theta B_3 H_9 \neq 0,$ $B_2 = \mu_0 = H_{14} = 0$	$K < 0$	$W_4 \geq 0$	$B_3 U_1 < 0, U_2 < 0$	Picture 3.4(g1)
				$B_3 U_1 < 0, U_2 > 0$	Picture 3.4(g2)
				$B_3 U_1 > 0$	Picture 3.4(g3)
			$W_4 < 0$		Picture 3.4($\overset{*}{g}2$)
		$K > 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$B_3 U_1 < 0$	Picture 3.4(h1)
				$B_3 U_1 > 0, U_2 < 0, H_5 < 0$	Picture 3.4(h2)
				$B_3 U_1 > 0, U_2 > 0$	Picture 3.4(h2)
				$B_3 U_1 > 0, U_2 < 0, H_5 > 0$	Picture 3.4(h3)
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3 U_1 < 0, U_2 < 0$	Picture 3.4($\overset{*}{h}1$)
				$B_3 U_1 > 0, U_2 > 0$	Picture 3.4($\overset{*}{h}2$)
				$B_3 U_1 > 0, U_2 < 0$	Picture 3.4($\overset{*}{h}3$)
				$B_3 U_1 < 0, U_2 > 0$	Picture 3.4($\overset{*}{h}4$)

Remark 3.8. We note that provided the condition (3.6) is satisfied, by (3.10)the following relations hold:

- $B_3 U_1 = \frac{3}{8}\Delta_1 \rho_4^2 \widetilde{\Delta}_1^2 (g-1)^2(h-1)^2(f-1)^2 x^2 y^2 \Rightarrow \text{sign}(B_3 U_1) = \text{sign}(\Delta_1);$
- $\text{sign}(U_2) = \text{sign}(\Delta_2 \Delta_3 \widetilde{\Delta}_2 \widetilde{\Delta}_3);$

Table 5 (continued)

Configu–ration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*			*Phase portrait*
Config. 3.5	$\eta > 0, \theta B_3 H_9 H_{14} \neq 0$, $B_2 = \mu_0 = 0$	$K < 0$	$B_3U_1 < 0$		Picture 3.5(g1)
			$B_3U_1 > 0$		Picture 3.5(g2)
		$K > 0$	$B_3U_1 < 0$		Picture 3.5(h1)
			$B_3U_1 > 0, U_3 > 0$		
			$B_3U_1 > 0, U_3 < 0$		Picture 3.5(h2)
Config. 3.6	$\eta > 0, \theta B_3 \neq 0$, $B_2 = \mu_0 = H_9 = 0$, $H_{13} = H_{14} = 0$	$K < 0$			Picture 3.6(k1)
		$K > 0$	$W_4 \geq 0$		Picture 3.6(l2)
			$W_4 < 0$	$H_1 < 0$	Picture 3.6($\overset{*}{l}$ 1)
				$H_1 > 0$	Picture 3.6($\overset{*}{l}$ 2)
Config. 3.7	$\eta > 0, \theta B_3 H_{13} \neq 0$, $B_2 = \mu_0 = 0$, $H_9 = H_{14} = 0$	$K < 0$			Picture 3.7(k1)
		$K > 0$	$H_5 > 0$		Picture 3.7(l1)
			$H_5 < 0$		Picture 3.7(l2)
Config. 3.8	$\eta > 0, \theta B_3 H_{14} \neq 0$, $B_2 = \mu_0 = H_9 = 0$	$K < 0$			Picture 3.8(k1)
		$K > 0$	$H_5 < 0$		Picture 3.8(l1)
			$H_5 > 0$		Picture 3.8(l2)
Config. 3.9	$\eta = 0, B_2 = 0$, $M\theta\mu_0 B_3 H_9 \neq 0$	$\mu_0 < 0$	$W_4 > 0$ *or* $W_4 = 0$ & $W_3 \geq 0$	$U_4 < 0, H_4 < 0$	Picture 3.9(b1)
				$U_4 < 0, H_4 > 0$	Picture 3.9(b2)
				$U_4 > 0$	Picture 3.9(b3)
			$W_4 < 0$ *or* $W_4 = 0$ & $W_3 < 0$	$B_3U_1 > 0, U_4 < 0$	Picture 3.9($\overset{*}{b}2$)
				$B_3U_1 > 0, U_4 > 0$	Picture 3.9($\overset{*}{b}3$)
				$B_3U_1 < 0$	Picture 3.9($\overset{*}{b}4$)
		$\mu_0 > 0$	$W_4 \geq 0$	$B_3U_1 < 0, U_4 < 0$	Picture 3.9(c1)
				$B_3U_1 > 0, H_4 > 0$	
				$B_3U_1 < 0, U_4 > 0$	Picture 3.9(c2)
				$B_3U_1 > 0, H_4 < 0$	Picture 3.9(c3)
			$W_4 < 0$	$B_3U_1 > 0$	Picture 3.9($\overset{*}{c}1$)
				$B_3U_1 < 0$	Picture 3.9($\overset{*}{c}2$)
Config. 3.10	$\eta = 0, B_2 = H_9 = 0$, $M\theta B_3 \mu_0 H_{13} \neq 0$	$\mu_0 < 0$			Picture 3.10(e1)
		$\mu_0 > 0$	$B_3U_1 < 0$		Picture 3.10(f1)
			$B_3U_1 > 0$		Picture 3.10(f2)
Config. 3.11	$\eta = 0, M\theta B_3 \mu_0 \neq 0$, $B_2 = H_9 = H_{13} = 0$	$\mu_0 < 0$			Picture 3.11($\overset{*}{e}1$)
		$\mu_0 > 0$	$W_4 \geq 0$	$H_5 > 0$	Picture 3.11(f1)
				$H_5 < 0$	Picture 3.11(f2)
			$W_4 < 0$		Picture 3.11($\overset{*}{f}1$)

Table 5 (continued)

Configu–ration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*		*Phase portrait*
Config. 3.12	$\eta=0, M\theta B_3H_9\neq 0$, $B_2=\mu_0=0$	$W_4>0$	$H_5<0$	Picture 3.12(h3)
			$H_5>0$	Picture 3.12(h2)
		$W_4<0$	$B_3U_1>0$	Picture 3.12($\overset{*}{h}2$)
			$B_3U_1<0$	Picture 3.12($\overset{*}{h}1$)
Config. 3.13	$\eta=0, M\theta B_3\neq 0$, $B_2=\mu_0=H_9=0$	–		Picture 3.13($\overset{*}{l}$ 1)

- If $U_2>0 \quad \Rightarrow \quad \text{sign}(U_4)=\text{sign}(\Delta_1\Delta_2\widetilde{\Delta}_2)=\text{sign}(\Delta_1\Delta_3\widetilde{\Delta}_3)$;
- $B_3H_{14}=90\rho_4^2\widetilde{\Delta}_2\widetilde{\Delta}(f-1)^2\widetilde{\Delta}_1^2x^2y^2 \quad \Rightarrow \quad \text{sign}(B_3H_{14})=\text{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3)$.

The case $\mu_0<0$. As $\mu_0=\text{Discrim}(K)/16$ we conclude that $K(x,y)$ ie a binary form with well defined sign and we shall consider two subcases: $K<0$ and $K>0$. The subcase $K<0$. Then according to [3] (see Table 1) on the finite part of the phase plane, systems (3.5) possess three saddles and one anti-saddle. Moreover the anti-saddle is a node if $W_4\geq 0$ and it is of the center-focus type if $W_4<0$. In the second case, by Lemma 3.3 we have a strong focus.

Since at infinity there exist three real distinct singularities, according to the index theory all of them must be nodes.

I. Assume first $W_4\geq 0$. We claim that in this case the phase portraits of systems (3.5) correspond to one of those indicated below if and only if the conditions indicated on the right are satisfied:

$$\text{Picture 3.1(a1)} \Leftrightarrow B_3U_1<0,\ U_2<0;$$
$$\text{Picture 3.1(a2)} \Leftrightarrow B_3U_1<0,\ U_2>0;$$
$$\text{Picture 3.1(a3)} \Leftrightarrow B_3U_1>0.$$

Indeed, since by Remark 3.8 we have $\text{sign}(\Delta_1)=\text{sign}(B_3U_1)$ we need to consider two cases.

(1) If $B_3U_1<0$ then $\Delta_1<0$ and hence the singular point $M_1(0,0)$ is a saddle. Since all infinite singular points are nodes this implies $\widetilde{\Delta}_i>0$, $i=1,2,3$. By Remark 3.8 we get $\text{sign}(U_2)=\text{sign}(\Delta_2\Delta_3)$.

Thus, if $U_2<0$ then $\Delta_2\Delta_3<0$ and one of the points M_2 or M_3 is a node and the remaining points are saddles. This univocally leads to Picture 3.1(a1).

Assuming $U_2>0$ we obtain $\Delta_2\Delta_3>0$, i.e. both points are saddles and the point M_4 is a node. In this case we clearly have Picture 3.1(a2).

(2) Suppose $B_3U_1>0$. Then $\Delta_1>0$ and hence the singular point $M_1(0,0)$ is a node, whereas the remaining tree finite singularities are saddles. As the infinite singular points are nodes we get Picture 3.1(a3). This completes the proof of our claim.

II. Assume $W_4<0$. Then on the phase plane, apart from the three saddles there exists a focus which clearly could only be the singular point M_4. It is known that this point must be located inside the triangle formed by other three saddle points (see for instance [9]).

We claim that there could not be a separatrix connecting M_2 with M_3. Indeed, suppose that such a connection exists. This connection could not belong to an

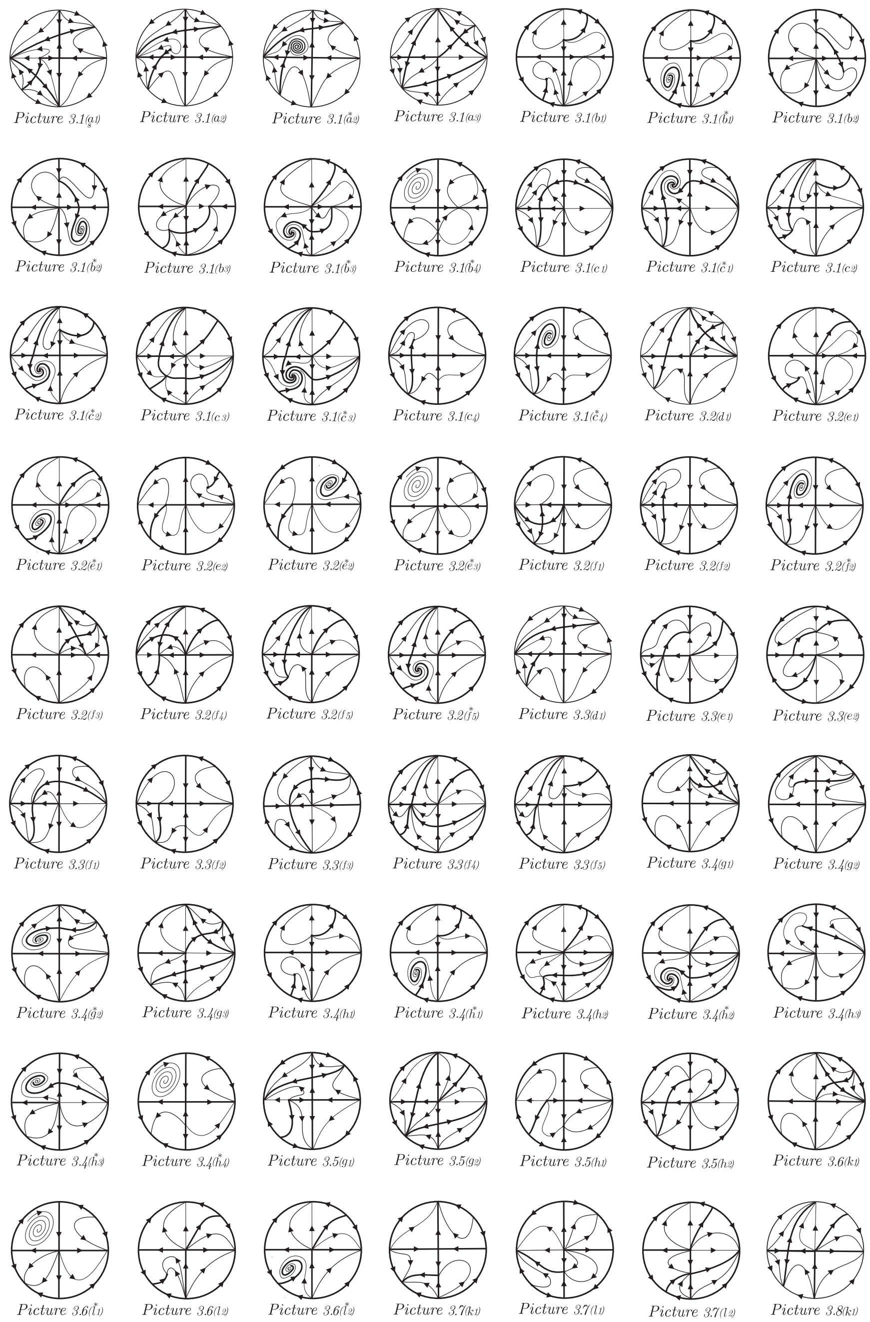


Figure 5. Phase portraits of the family of LV-systems with exactly three invariant lines

invariant line, otherwise systems (3.5) possess four invariant lines and we get a contradiction. Suppose that this connection is different from the segment M_2M_3 of a line. Then we obtain a closed domain bordered by this segment and the separatrix

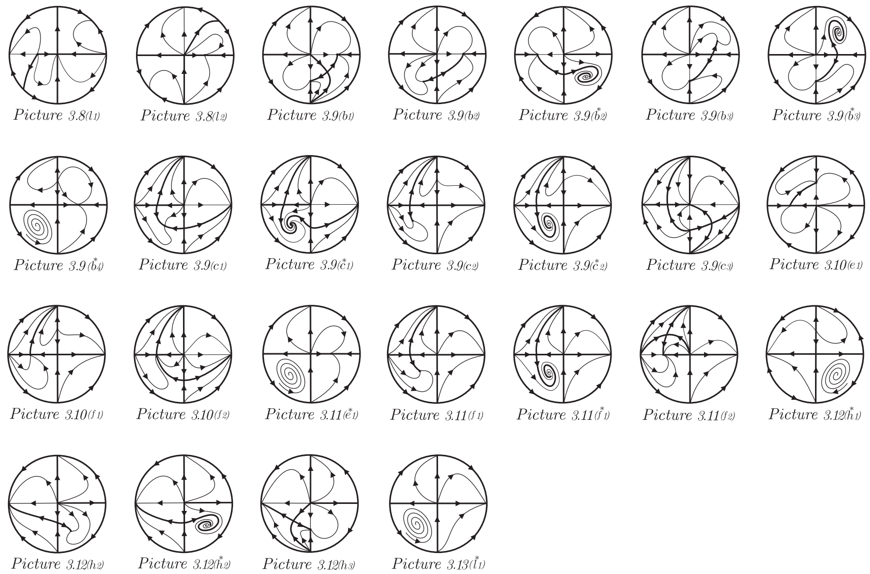


FIGURE 5 (CONT.). Phase portraits of the family of LV-systems with exactly three invariant lines

connecting M_2 and M_3 (at which we have two saddles) and clearly on the segment of the straight line M_2M_3 there must be at least one point of contact. Therefore according to [11, Theorem 2.5] the straight line passing through these singular points must be an invariant line and we again get the contradiction mentioned above. These arguments lead univocally to a phase portrait which is topologically equivalent to Picture 3.1($\overset{*}{a}2$).

The subcase $K > 0$. Then according to [3] (see Table 1) systems (3.5) possess one saddle and three anti-saddles. Clearly only one anti-saddle could be a focus and considering [3] (see Table 1) besides the saddle we have three nodes if either $W_4 > 0$ or $W_4 = 0$ and $W_3 \geq 0$; and we have two nodes and a focus if either $W_4 < 0$ or $W_4 = 0$ and $W_3 < 0$.

I. Assume first $W_4 > 0$ *or* $W_4 = 0$ *and* $W_3 \geq 0$. We claim that in this case the phase portraits of systems (3.5) correspond to one of those indicated below if and only if the conditions indicated on the right are respectively satisfied:

$$\text{Picture 3.1(b1)} \Leftrightarrow \text{either } B_3U_1 < 0, \text{ or } B_3U_1 > 0,\ U_2 > 0,\ U_4 > 0,\ U_3 > 0;$$

$$\text{Picture 3.1(b2)} \Leftrightarrow B_3U_1 > 0 \text{ and either } U_2 < 0, B_3H_{14} > 0, \\ \text{or } U_2 > 0, U_4 > 0, U_3 < 0;$$

$$\text{Picture 3.1(b3)} \Leftrightarrow B_3U_1 > 0 \text{ and either } U_2 < 0, B_3H_{14} < 0, \text{ or } U_2 > 0, U_4 < 0.$$

Indeed, first of all we observe that at infinity we must have one node and two saddles (the sum of the indexes must be -1). Then due to Remark 3.7 without loss of generality we may assume that $R_2(1,0,0)$ is a saddle, i.e. $\widetilde{\Delta}_2 < 0$.

As it was mentioned previously, the type of the singular point $M_1(0,0)$ depends on the sign of the invariant polynomial B_3U_1. So we consider two cases.

(1) If $B_3U_1 < 0$ then $\Delta_1 < 0$ and hence the singular point $M_1(0,0)$ is a saddle and the other three singularities are nodes. As $R_2(1,0,0)$ is a saddle, by 3.8 we have $g < 0$ and this fixes the position of the finite node $M_2(-1/g,0)$. We claim that in this case the singular point $R_3(0,1,0)$ could not be a saddle. Indeed, assuming the contrary we have $\widetilde{\Delta}_3 < 0$ (i.e. $h < 0$). Since $f < 0$, the domain in the second quadrant bordered by the three invariant lines (one being the line at infinity) has on its border exactly three singularities, which are saddles. Moreover the singular point M_4 inside this domain is forced in this case to be a focus or a center. So we obtain a contradiction which proves our claim.

Thus $R_3(0,1,0)$ is a node (i.e. $h > 0$) and we get univocally a phase portrait topologically equivalent to Picture 3.1(b1).

(2) Assume now $B_3U_1 > 0$, i.e. the singular point $M_1(0,0)$ is a node. As by our assumption $R_2(1,0,0)$ is a saddle (i.e. $\widetilde{\Delta}_2 < 0$), considering Remark 3.8 we obtain $\operatorname{sign}(U_2) = -\operatorname{sign}(\Delta_2\Delta_3\widetilde{\Delta}_3)$.

(a) If $U_2 < 0$ then we obtain $\Delta_2\Delta_3\widetilde{\Delta}_3 > 0$. On the other hand by Remark 3.8 we have $\operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3)$ and we shall consider two subcases: $B_3H_{14} > 0$ and $B_3H_{14} < 0$.

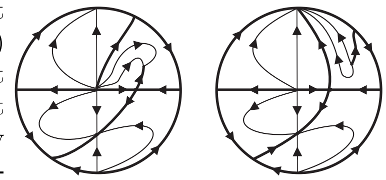
(α) Suppose first $B_3H_{14} > 0$. Since $\widetilde{\Delta}_2 < 0$ we obtain $\widetilde{\Delta}_3 < 0$, i.e. $h < 0$. Therefore the condition $U_2 < 0$ implies $\Delta_2\Delta_3 < 0$, i.e. one of the singularities $M_2(-1/g,0)$ or $M_3(0,-f/h)$ is a saddle and the other is a node. As both infinite singular points $R_2(1,0,0)$ and $R_3(0,1,0)$ are of the same types (saddles), due to Remark 3.7 without loss of generality we may assume that $M_2(-1/g,0)$ is a saddle. Taking into account that the finite saddle must be inside the triangle formed by three finite nodes we obviously get a phase portrait topologically equivalent to Picture 3.1(b2).

(β) In the case $B_3H_{14} < 0$ we obtain $\widetilde{\Delta}_3 > 0$, i.e. $h > 0$ and hence, besides the saddle R_2 at infinity we have the node R_3 and the saddle R_1. In this case considering the condition $U_2 < 0$ we get $\Delta_2\Delta_3 > 0$ and this means that both singular points M_2 and M_3 are nodes. Clearly the remaining point M_4 is a saddle. Then considering the location of the singularities we get a phase portrait topologically equivalent to Picture 3.1(b3).

(b) Suppose now $U_2 > 0$. Then due to the Remark 3.8 and $\widetilde{\Delta}_2 < 0$ we get $\Delta_2\Delta_3\widetilde{\Delta}_3 < 0$. Moreover, since $U_2 > 0$ we have $\operatorname{sign}(U_4) = \operatorname{sign}(\Delta_1\Delta_3\widetilde{\Delta}_3)$ and since $\Delta_1 > 0$ then clearly we obtain $\operatorname{sign}(U_4) = \operatorname{sign}(\Delta_3\widetilde{\Delta}_3) = -\operatorname{sign}(\Delta_2)$.

(α) Admit first $U_4 > 0$, i.e. $\Delta_3\widetilde{\Delta}_3 > 0$. So, we have $\Delta_2 < 0$ (i.e. M_2 is a saddle) and then M_3 and M_4 are nodes. Hence $\Delta_3 > 0$ and this implies $\widetilde{\Delta}_3 > 0$, i.e. R_3 is a node. Taking into consideration the node R_3 and the saddles R_2 and R_1 at infinity, we arrive at the two possibilities given by the *Portraits A* and *B*. We observe that for the first (respectively second) phase portraits the nodes M_1 and M_4 have different (respectively the same) stability. As $\rho_1 = 1 + f > 0$ we conclude that the realization of each portrait depends on the sign of ρ_4. We point out that the singularity M_4 changes its stability when ρ_4 change the sign (passing through zero) and since $\rho_4 \neq 0$ (due to $B_3 \neq 0$) we could not have a separatrix connection M_2R_1.

On the other hand, as R_1 is a saddle (i.e. $1-g-h < 0$) considering $g < 0$ we get $h > 1-g > 1$. Therefore according to (3.10) we obtain $\operatorname{sign}(U_3) = \operatorname{sign}(\rho_1\rho_4)$. Thus, we get Portrait A if $U_3 < 0$ and Portrait B if $U_3 > 0$. It remains to note, that the phase portrait given by Portrait A (respectively by Portrait B) is topologically equivalent to Picture 3.1(b2) (respectively Picture 3.1(b1)).


Portrait A Portrait B

(β) Assuming $U_4 < 0$ we have $\Delta_2 > 0$ (i.e. M_2 is a node). So we have two possibilities: either M_3 is a saddle and M_4 is a node, or M_3 is a node and M_4 is a saddle. In the first (respectively the second) case due to $\Delta_3\widetilde{\Delta}_3 < 0$ we obtain that R_3 is a node (respectively a saddle) and hence R_1 is a saddle (respectively a node). So in both cases we get phase portraits which are topologically equivalent to Picture 3.1(b3).

II. Suppose now $W_4 < 0$ *or* $W_4 = 0$ *and* $W_3 < 0$. It was mentioned earlier that systems (3.5) possess as finite singularities a saddle, two nodes and a focus. We claim that in this case the phase portraits of these systems correspond to one of the portraits indicated below if and only if the conditions given on the right are respectively satisfied:

$$\text{Picture } 3.1(\overset{*}{b}1) \Leftrightarrow \text{either } B_3U_1 < 0, U_2 < 0, \text{ or } B_3U_1 > 0, U_2 > 0, U_4 > 0, U_3 > 0;$$
$$\text{Picture } 3.1(\overset{*}{b}2) \Leftrightarrow B_3U_1 > 0 \text{ and either } U_2 < 0, \text{ or } U_2 > 0, U_4 > 0, U_3 < 0;$$
$$\text{Picture } 3.1(\overset{*}{b}3) \Leftrightarrow B_3U_1 > 0, U_2 > 0, U_4 < 0;$$
$$\text{Picture } 3.1(\overset{*}{b}4) \Leftrightarrow B_3U_1 < 0, U_2 > 0.$$

Indeed, first of all we recall that the singular point of the focus type could be only M_4.

(1) *The subcase* $B_3U_1 < 0$. Then by Remark 3.8 we have $\Delta_1 < 0$ (yielding $f < 0$), i.e. $M_1(0,0)$ is a saddle and M_2 and M_3 are nodes (M_4 being a focus). At infinity we have the same singularities: two saddles and one node. Thus similarly to the previous case, due to Remark 3.7 we may assume that $R_2(1,0,0)$ is a saddle. So one of the remaining infinite points R_3 and R_1 is a saddle and other one is a node. Moreover, since $\Delta_2 > 0$ and $\Delta_3 > 0$, by Remark 3.8 we find out that $\operatorname{sign}(U_2) = -\operatorname{sign}(\widetilde{\Delta}_3)$. So we shall examine two cases: $U_2 < 0$ and $U_2 > 0$.

(a) If $U_2 < 0$ then $\widetilde{\Delta}_3 > 0$ and hence R_3 is node and R_1 is a saddle. Considering the focus $M_4(x_4, y_4)$ we univocally arrive to the Picture $3.1(\overset{*}{b}1)$.

(b) Admit now $U_2 > 0$. Then $\widetilde{\Delta}_3 < 0$ and therefore the singular point R_3 is a saddle and R_1 is a node. Taking into account the location of all the singularities in this case we get Picture $3.1(\overset{*}{b}4)$.

(2) *The subcase* $B_3U_1 > 0$. Considering Remark 3.8 we have $\Delta_1 > 0$ (i.e. $f > 0$) and therefore $M_1(0,0)$ is a node. Hence $\Delta_2\Delta_3 < 0$ (since M_4 is a focus) and as $\widetilde{\Delta}_2 < 0$ according to Remark 3.8 we obtain $\operatorname{sign}(U_2) = \operatorname{sign}(\widetilde{\Delta}_3)$.

(a) Suppose $U_2 < 0$. Then $\widetilde{\Delta}_3 < 0$ and hence $R_3(0,1,0)$ is a saddle. Therefore since the infinite singularity $R_2(1,0,0)$ is also a saddle, then without loss of generality we may assume that the point $M_2(-1/g,0)$ is a saddle due to Remark 3.7.

Then considering the positions of the focus M_4 and of the node R_1 we univocally arrive to the Picture 3.1($\overset{*}{b}2$).

(b) Assume now $U_2 > 0$. Then we get $\widetilde{\Delta}_3 > 0$ (i.e. $h > 0$) and in this case the singular point R_3 is a node and the third infinite point R_1 is a saddle. Hence according to Remark 3.8 we have $\mathrm{sign}(U_4) = \mathrm{sign}(\Delta_1\Delta_3\widetilde{\Delta}_3) = \mathrm{sign}(\Delta_3)$.

($\boldsymbol{\alpha}$) Admit first $U_4 < 0$. Then $\Delta_3 < 0$ (i.e. M_3 is a saddle) and therefore M_2 is a node. Taking into consideration the location of all the singularities of a system (3.5) in the case under consideration we obtain univocally the phase portrait given by Picture 3.1($\overset{*}{b}3$).

($\boldsymbol{\beta}$) If $U_4 > 0$ then $\Delta_3 > 0$, i.e. M_3 is a node and hence M_2 is a saddle. So, in the same manner as in the case with one saddle and three nodes, we obtain two different phase portraits given by *Portrait A* and *Portrait B* (see page 35) but with M_4 as focus instead of a node. Moreover, it is clear, that the stabilities of the node M_1 and of the focus M_4 distinguish these two phase portraits. More exactly, we obtain *Portrait A* if $\rho_1\rho_4 < 0$ and *Portrait B* if $\rho_1\rho_4 > 0$. It remains to remark, that *Portrait A* (respectively *Portrait B*) is topologically equivalent to Picture 3.1($\overset{*}{b}2$) (respectively Picture 3.1($\overset{*}{b}1$)) and according to (3.10), in this case we have $\mathrm{sign}(\rho_1\rho_4) = \mathrm{sign}(U_3)$.

The case $\mu_0 > 0$. Then according to [3] (see Table 1) for systems (3.5), on the finite part of the phase plane there are two saddles and two anti-saddles. Moreover as for systems in this family one anti-saddle is always a node, by [3] we conclude that both anti-saddles are nodes if either $W_4 > 0$ or $W_4 = 0$ and $W_3 \geq 0$; and one of them is a focus if either $W_4 < 0$ or $W_4 = 0$ and $W_3 < 0$.

On the other hand since the three singular points at infinity are simple, then by the index theory we must have two nodes and a saddle. So due to Remark 3.7 we may assume that the point $R_2(1, 0, 0)$ is a node, i.e. $\tilde{\Delta}_2 > 0$ (this yields $g > 0$).

Assume first $W_4 > 0$ *or* $W_4 = 0$ *and* $W_3 \geq 0$. So systems (3.5) have two saddles and two nodes. We claim that the phase portrait of a system in this family is given by one of the portraits indicated below if and only if the conditions on the right are respectively satisfied:

$$\text{Picture 3.1(c1)} \Leftrightarrow \text{ either } U_2 < 0, B_3H_{14} < 0, \text{ or } U_2 > 0, U_4 > 0, B_3U_1 > 0;$$
$$\text{Picture 3.1(c2)} \Leftrightarrow U_2 < 0, B_3H_{14} > 0, B_3U_1 < 0;$$
$$\text{Picture 3.1(c3)} \Leftrightarrow \text{ either } U_2 < 0, B_3H_{14} > 0,\ B_3U_1 > 0,$$
$$\text{or } U_2 > 0, U_4 < 0;$$
$$\text{Picture 3.1(c4)} \Leftrightarrow U_2 > 0, U_4 > 0, B_3U_1 < 0.$$

Indeed first we recall that the type of the singular point $M_1(0, 0)$ is governed by the invariant polynomial B_3U_1.

I. The subcase $B_3U_1 < 0$. Then by Remark 3.8 it follows $\Delta_1 < 0$ (yielding $f < 0$), i.e. $M_1(0, 0)$ is a saddle.

(1) *Suppose first* $U_2 < 0$. Then by Remark 3.8 we get $\Delta_2\Delta_3\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$ and $\mathrm{sign}(B_3H_{14}) = \mathrm{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3)$.

(a) If $B_3H_{14} < 0$ then $\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$ and this implies $\Delta_2\Delta_3 > 0$. Therefore (as M_1 is a saddle) both points M_2 and M_3 are nodes. On the other hand since $\widetilde{\Delta}_2 > 0$ we obtain $\widetilde{\Delta}_3 < 0$, i.e. R_3 is a saddle. So taking into considerations the locations of the finite singularities we arrive univocally at a phase portrait topologically equivalent to Picture 3.1(c1).

(b) In the case $B_3H_{14} > 0$ we obtain $\widetilde{\Delta}_2\widetilde{\Delta}_3 > 0$ (i.e. R_2 and R_3 are both nodes) and this implies $\Delta_2\Delta_3 < 0$. Therefore one of the points M_2 or M_3 is a saddle and due to Remark 3.7, without loss of generality we may assume that such a saddle is M_2. So in this case we get univocally the Picture 3.1(c2).

(2) *Assume now* $U_2 > 0$. Then $\Delta_2\Delta_3\widetilde{\Delta}_2\widetilde{\Delta}_3 > 0$ and according to Remark 3.8 due to the condition $\Delta_1 < 0$ we obtain $\operatorname{sign}(U_4) = -\operatorname{sign}(\Delta_2\widetilde{\Delta}_2) = -\operatorname{sign}(\Delta_3\widetilde{\Delta}_3)$.

(a) If $U_4 < 0$ then we obtain $\Delta_2\widetilde{\Delta}_2 > 0$ and $\Delta_3\widetilde{\Delta}_3 > 0$. Since by assumption $\widetilde{\Delta}_2 > 0$ we obtain $\Delta_2 > 0$, i.e. the singular point M_2 is a node. We claim that in this case the singular point R_3 could not be a saddle. Indeed, supposing the contrary, we obtain $\widetilde{\Delta}_3 = h < 0$ and since $f < 0$ this implies $\Delta_3 = f(fh - f - g)/h > 0$. However this contradicts $\Delta_3\widetilde{\Delta}_3 > 0$ and our claim is proved.

Thus we get the conditions $\Delta_2 > 0$, $\Delta_3 > 0$, $\widetilde{\Delta}_2 > 0$ and $\widetilde{\Delta}_3 > 0$. In other words all the singularities M_2, M_3, R_2 and R_3 are nodes. Considering the position of the saddles M_4 and R_1 we univocally get a phase portrait which is topologically equivalent to Picture 3.1(c3).

(b) Assuming $U_4 > 0$ we obtain $\Delta_2\widetilde{\Delta}_2 < 0$ and $\Delta_3\widetilde{\Delta}_3 < 0$. As $\widetilde{\Delta}_2 > 0$ we obtain $\Delta_2 < 0$, i.e. the singular point M_2 is a saddle. Therefore the other two finite singularities are nodes. Hence $\Delta_3 > 0$ and this implies $\widetilde{\Delta}_3 < 0$, i.e. the infinite point R_3 is a saddle. i.e. the singular point M_2 is a node. Considering the location of the singularities we obtain univocally Picture 3.1(c4).

II. The subcase $B_3U_1 > 0$. Considering Remark 3.8 we obtain $\Delta_1 > 0$, i.e. $M_1(0,0)$ is a node. Since by assumption R_2 is a node (i.e. $\widetilde{\Delta}_2 > 0$), by Remark 3.8 we get $\operatorname{sign}(U_2) = \operatorname{sign}(\Delta_2\Delta_3\widetilde{\Delta}_3)$ and $\operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_3)$.

(1) *Assume first* $U_2 < 0$. Then $\Delta_2\Delta_3\widetilde{\Delta}_3 < 0$ and we shall consider two subcases: $B_3H_{14} < 0$ and $B_3H_{14} > 0$.

(a) If $B_3H_{14} < 0$ then we have $\widetilde{\Delta}_3 < 0$, i.e. R_3 is a saddle and then R_1 is a node. In this case the condition $U_2 < 0$ implies $\Delta_2\Delta_3 > 0$ and as M_1 is a node, both singular points M_2 and M_3 must be saddles. Therefore the fourth point M_4 is a node. So considering the location of the singular points and their respective types we univocally obtain Picture 3.1(c1).

(b) Suppose now $B_3H_{14} > 0$, i.e. $\widetilde{\Delta}_3 > 0$. Hence both points R_2 and R_3 are nodes and R_1 is a saddle and the condition $U_2 < 0$ yields $\Delta_2\Delta_3 < 0$, i.e. one of the points M_2 or M_3 is a node and another one is a saddle. Since R_2 and R_3 are nodes due to a substitution (see Remark 3.7) we may consider that M_2 is a saddle. So M_4 is a saddle and we arrive univocally at a phase portrait topologically equivalent to Picture 3.1(c3).

(2) *Admit now* $U_2 > 0$. Then $\Delta_2\Delta_3\widetilde{\Delta}_3 > 0$ and according to Remark 3.8 we obtain $\operatorname{sign}(U_4) = \operatorname{sign}(\Delta_1\Delta_2\widetilde{\Delta}_2)$ and as $\Delta_1 > 0$ and $\widetilde{\Delta}_2 > 0$ we get $\operatorname{sign}(U_4) = \operatorname{sign}(\Delta_2)$.

(a) If $U_4 < 0$ then we have $\Delta_2 < 0$ and $\Delta_3\widetilde{\Delta}_3 < 0$. So we have two possibilities: *(i)* $\Delta_3 < 0$ (in which case systems (3.5) possess three saddles (M_2, M_3, R_1) and four nodes (M_1, M_4, R_2, R_3) and *(ii)* $\Delta_3 > 0$ (in this case the three saddles are M_2, M_4 and R_3, and the four nodes are M_1, M_3, R_1 and R_2). Considering the location of these singularities and their types, in both cases we arrive at phase portraits which are topologically equivalent to Picture 3.1(c3).

(b) Assuming $U_4 > 0$ we obtain $\Delta_2 > 0$ and $\Delta_3\widetilde{\Delta}_3 > 0$. So M_2 is a node and then M_3 and M_4 are saddles. This implies $\Delta_3 < 0$ and then $\widetilde{\Delta}_3 < 0$, i.e. R_3 is a saddle and the remaining infinite singular point R_1 must be a node. In the same manner as above, considering the types and the location of the singularities we arrive at a phase portrait which is topologically equivalent to Picture 3.1(c1).

Summarizing the sets of conditions given above for each one of the Pictures 3.1(c j), $j = 1, 2, 3, 4$ we conclude that our claim is proved.

Suppose now $W_4 < 0$ *or* $W_4 = 0$ *and* $W_3 < 0$. So on the phase plane systems (3.5) possess two saddles, one node and one focus, and at infinity they have two nodes and a saddle. We assume again that R_2 is a node, i.e. $\widetilde{\Delta}_2 > 0$.

We claim that the phase portrait of a system in this family is given by one of the ones indicated below if and only if the corresponding conditions are satisfied, respectively:

$$\text{Picture } 3.1(\overset{*}{c}1) \Leftrightarrow B_3U_1 > 0, U_2 < 0;$$
$$\text{Picture } 3.1(\overset{*}{c}2) \Leftrightarrow B_3U_1 < 0, U_2 < 0;$$
$$\text{Picture } 3.1(\overset{*}{c}3) \Leftrightarrow B_3U_1 > 0, U_2 > 0;$$
$$\text{Picture } 3.1(\overset{*}{c}4) \Leftrightarrow B_3U_1 < 0, U_2 > 0.$$

Indeed to convince ourselves we shall examine again both cases: $B_3U_1 < 0$ and $B_3U_1 > 0$.

I. The case $B_3U_1 < 0$. Then by Remark 3.8 it follows $\Delta_1 < 0$ (yielding $f < 0$) (i.e. $M_1(0, 0)$ is a saddle).

Since $\widetilde{\Delta}_2 > 0$ according to Remark 3.8 we obtain $\operatorname{sign}(U_2) = \operatorname{sign}(\Delta_2\Delta_3\widetilde{\Delta}_3)$. Moreover, since M_4 is a focus then either M_2 or M_3 must be a saddle, i.e. $\Delta_2\Delta_3 < 0$ and this implies $\operatorname{sign}(U_2) = -\operatorname{sign}(\widetilde{\Delta}_3)$.

(1) *Assume first* $U_2 < 0$. Then R_3 is a node and R_1 is a saddle. Taking into account that R_2 and R_3 are both nodes then without loss of generality, due to Remark 3.7 we may assume that M_2 is a saddle (then M_3 is a node). So considering the types and the location of all the singularities we arrive at Picture 3.1($\overset{*}{c}2$).

(2) *Admit now* $U_2 > 0$. In this case $\widetilde{\Delta}_3 < 0$ ($h < 0$), i.e. R_3 is a saddle and hence R_1 is a node. We observe that the conditions $f < 0$ and $h < 0$ imply $fh - f - h > 0$ and therefore $\Delta_3 = f(fh - f - h)/h > 0$, i.e. M_3 is a node. Therefore M_2 is a saddle and as M_4 is a focus we arrive at Picture 3.1($\overset{*}{c}4$).

II. The case $B_3U_1 > 0$. Herein by Remark 3.8 it follows that $\Delta_1 > 0$ ($f > 0$), i.e. $M_1(0, 0)$ is a node. Since M_4 is a focus then clearly M_2 and M_3 should be saddles, i.e. $\Delta_2 < 0$ and $\Delta_3 < 0$. Hence taking into account our assumption (i.e. that R_2 is a node) according to Remark 3.8 in this case we have $\operatorname{sign}(U_2) = \operatorname{sign}(\widetilde{\Delta}_3)$.

If $U_2 < 0$ we get $\widetilde{\Delta}_3 < 0$ (i.e. R_3 is a saddle) and then R_1 is a node. This leads to Picture 3.1($\overset{*}{c}1$).

The condition $U_2 > 0$ implies that both points R_2 and R_3 are nodes (then R_1 is a saddle) and we get Picture 3.1($\overset{*}{c}3$). This completes the proof of our claim and thus all the phase portraits associated with Config. 3.1 as well as the respective invariant criteria for the realization of each of them are determined.

3.1.2. *The phase portraits associated with Config. 3.2.* According to Table 2 we consider the family of systems

$$\dot{x} = x[1 + gx + (h-1)y], \quad \dot{y} = y[(g-1)x + hy], \tag{3.11}$$

for which the condition

$$gh(g+h-1)(g-1)(h-1) \neq 0 \tag{3.12}$$

holds. We observe that this family of systems is a particular case of the family (3.5) when the parameter f equals zero (and in this case the point M_3 has coalesced with $M_1(0,0)$). For the finite singularities of systems (3.11) with the condition (3.12) we have

$$\begin{gathered} M_3 \equiv M_1(0,0): \quad \Delta_{1,3} = 0, \quad \rho_{1,3} = 1, \ \delta_{1,3} = 1; \\ M_2(-1/g, 0): \quad \Delta_2 = (g-1)/g, \quad \rho_2 = (1-2g)/g, \ \delta_2 = 1/g^2; \\ M_4\left(\frac{h}{1-g-h}, \frac{1-g}{1-g-h}\right): \quad \Delta_4 = \frac{-h(g-1)}{g+h-1}, \ \rho_4 = \frac{h}{1-g-h}, \ \delta_4 = \rho_4^2 - 4\Delta_4. \end{gathered} \tag{3.13}$$

and for the three infinite singular points we have again

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 1-g-h; \quad R_2(1,0,0): \ \widetilde{\Delta}_2 = g; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h. \tag{3.14}$$

We observe that due to the relation $\rho_1 = 1$ (i.e. only one of the respective eigenvalues vanishes) the double singular point $M_1(0,0)$ is a saddle-node.

Taking into account (3.13) and (3.14) we evaluate for systems (3.11) the invariant polynomials we need:

$$\begin{gathered} \mu_0 = gh(g+h-1) = -\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3, \\ K = 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2, \\ W_4 = h^3[4(g-1)^2 - 3h + 4gh] = \mu_0^2\,\delta_1\delta_2\delta_3\delta_4, \\ H_1 = 288h = 288\widetilde{\Delta}_3, \\ H_5 = 384(1-g)h^2 = -384\Delta_2\widetilde{\Delta}_2\widetilde{\Delta}_3^2, \\ H_{14} = 30gh^2 = 30\widetilde{\Delta}_2\widetilde{\Delta}_3^2, \\ B_3 = 3hx^2y^2 = 3\widetilde{\Delta}_1\rho_4x^2y^2 = 3\widetilde{\Delta}_3x^2y^2. \end{gathered} \tag{3.15}$$

The case $\mu_0 < 0$. In the same manner as in the previous section (in the case of Config. 3.1) we shall consider two subcases: $K < 0$ and $K > 0$.
The subcase $K < 0$. In this case for systems (3.11), according to [3] (see Table 1) on the finite part of the phase plane besides the saddle-node there are two saddles. We claim that in this case we obtain the unique phase portrait given by Picture 3.2(d1).

Indeed, due to the index theory all three simple infinite singularities (3.14) are nodes. Considering the position of the saddles M_2 and M_4 and of the saddle-node M_1 we get univocally the phase portrait given by Picture 3.2(d1).
The subcase $K > 0$. Following [3] (see Table 1) we find that besides the saddle-node there are two anti-saddles. Moreover, by (3.13) and (3.15) we observe that the relation $W_4 = 0$ holds if and only if $\delta_4 = 0$. So according to [3] (see Table 1) we have two nodes if $W_4 \geq 0$ and we have one node and one focus if $W_4 < 0$.

On the other hand in both cases we have two saddles and a node at infinity.

I. Assume first $W_4 \geq 0$. According to (3.15) we have $\text{sign}(H_1) = \text{sign}(\widetilde{\Delta}_3)$.

(1) If $H_1 < 0$ then $\widetilde{\Delta}_3 < 0$ (i.e. $h < 0$) and hence the singular point R_3 is a saddle. We claim that the condition $W_4 \geq 0$ implies $\widetilde{\Delta}_2 > 0$, i.e. $g > 0$. Indeed assuming the contrary we have $g < 0$ and due to $h < 0$ we obtain $h[4(g-1)^2 - 3h + 4gh] < 0$ which contradicts $W_4 \geq 0$ (see (3.15)).

Thus $g > 0$ (i.e. $\widetilde{\Delta}_2 > 0$) and then R_2 is a node and consequently R_3 must be a saddle. Considering the position and the types of the singularities we arrive at Picture 3.2(e2).

(2) Suppose now $H_1 > 0$. Then $\widetilde{\Delta}_3 > 0$ (i.e. $h > 0$) and hence the singular point R_3 is a node and consequently the other two infinite points are saddles. Obviously we get Picture 3.2(e1).

II. Admit now $W_4 < 0$. Then the point M_4 is a focus and M_2 is a node, i.e. $\Delta_2 > 0$.

(1) If $H_1 < 0$ then $\widetilde{\Delta}_3 < 0$ (i.e. $h < 0$) and hence the singular point R_3 is a saddle. According to (3.15) due to $\Delta_2 > 0$ we obtain $\operatorname{sign}(H_5) = -\operatorname{sign}(\widetilde{\Delta}_2)$.

(a) Assume $H_5 < 0$. Then $\widetilde{\Delta}_2 > 0$ and hence R_2 is a node. So the remaining infinite point R_1 must be a saddle. As $M_1(0,0)$ is a saddle-node and M_2 is a node we get Picture 3.2($\overset{*}{e}2$).

(b) If $H_5 > 0$ then $\widetilde{\Delta}_2 < 0$ and the singular point R_2 is a saddle, whereas R_1 is a node. As $g < 0$, $h < 0$ and $1 - g - h > 0$ considering (3.13) we obtain

$$\rho_2\rho_4 = \frac{(1-2g)h}{g(1-g-h)} > 0.$$

Hence we conclude that the node M_2 and the focus M_4 have the same stability and we univocally obtain Picture 3.2($\overset{*}{e}3$).

(2) Assume now $H_1 > 0$. Then we have $\widetilde{\Delta}_3 > 0$ (i.e. $h > 0$) and the singular point R_3 is a node. Therefore the remaining two infinite singular points are saddles and considering the location of the singularities we obtain univocally Picture 3.2($\overset{*}{e}1$).

The case $\mu_0 > 0$. According to [3] (see Table 1) the systems (3.11),besides the saddle-node possess one saddle and in addition either one node if $W_4 \geq 0$, or one focus if $W_4 < 0$. On the other hand at infinity we have a saddle and two nodes.

The subcase $W_4 \geq 0$. We claim that in this case the phase portrait of a system (3.11) is necessarily one of those indicated below if and only if the conditions on the right side are satisfied:

$$\text{Picture 3.2(f1)} \Leftrightarrow B_3H_{14} < 0, H_5 < 0;$$
$$\text{Picture 3.2(f2)} \Leftrightarrow B_3H_{14} < 0, H_5 > 0, H_1 < 0;$$
$$\text{Picture 3.2(f3)} \Leftrightarrow B_3H_{14} < 0, H_5 > 0, H_1 > 0;$$
$$\text{Picture 3.2(f4)} \Leftrightarrow B_3H_{14} > 0, H_5 < 0;$$
$$\text{Picture 3.2(f5)} \Leftrightarrow B_3H_{14} > 0, H_5 > 0.$$

Indeed, first of all we observe that according to (3.15) we have:

$$\operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3), \quad \operatorname{sign}(H_1) = \operatorname{sign}(\widetilde{\Delta}_3), \quad \operatorname{sign}(H_5) = -\operatorname{sign}(\Delta_2\widetilde{\Delta}_2), \tag{3.16}$$

so we can control the signs of each of the determinants $\widetilde{\Delta}_2$, $\widetilde{\Delta}_3$ and Δ_2.

I. The possibility $B_3H_{14} < 0$. Then $\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$ and we shall consider two subcases: $H_5 < 0$ and $H_5 > 0$.

(1) If $H_5 < 0$ then considering (3.16) and (3.13) we obtain $g^2(g-1) > 0$, i.e. $g > 1$. Consequently $\widetilde{\Delta}_2 > 0$ and $\widetilde{\Delta}_3 < 0$. Hence the singular points R_2, R_1 and M_2 are nodes, whereas R_3 and M_4 are saddles. Considering the position of these singularities we univocally get Picture 3.2(f1).

(2) Assume $H_5 > 0$. Then $\Delta_2\widetilde{\Delta}_2 < 0$ and we consider two possibilities: $H_1 < 0$ and $H_1 > 0$.

(a) Assume first $H_1 < 0$. According to (3.16) we have $\widetilde{\Delta}_3 < 0$ and hence we obtain $\widetilde{\Delta}_2 > 0$ which implies $\Delta_2 < 0$. So besides the saddle-node $M_1(0,0)$ systems (3.11) have the nodes R_2, R_1 and M_4 and the saddles R_3 and M_2. This obviously leads to the phase portrait given by Picture 3.2(f2).

(b) If $H_1 > 0$, similarly as above, we get the nodes R_3, R_1 and M_2 and the saddles R_2 and M_4. Considering the location of these singularities we univocally obtain Picture 3.2(f3).

II. The possibility $B_3H_{14} > 0$. In this case we have $\widetilde{\Delta}_2\widetilde{\Delta}_3 > 0$ and as there could not be two saddles at infinity, both points R_2 and R_3 are nodes (i.e. $\widetilde{\Delta}_2 > 0$ and $\widetilde{\Delta}_3 > 0$), whereas the point R_1 is a saddle.

(1) If $H_5 < 0$ then by (3.16) we obtain $\Delta_2 > 0$ (then $g > 1$) and therefore M_2 is a node and M_4 is a saddle. Considering the location of all the singularities we univocally obtain Picture 3.2(f4).

(2) Assume $H_5 > 0$. This implies $\Delta_2 < 0$ (then $0 < g < 1$) and hence M_2 is a saddle and M_4 is a node. Thus we get Picture 3.2(f5).

The subcase $W_4 < 0$. As we mentioned above, in this case systems (3.11) have a saddle and a focus besides the saddle-node. Clearly a focus could only be the singularity M_4 and hence M_2 is a saddle. Therefore we have $\Delta_2 < 0$ which implies $0 < g < 1$ by (3.13). Consequently $\widetilde{\Delta}_2 > 0$, i.e. R_2 is a node. It remains to distinguish the possibilities when R_3 is a node or a saddle. According to (3.16), these situations are governed by the invariant polynomial H_1. More precisely, we obtain the saddle R_3 and the node R_1 if $H_1 < 0$, and we have the saddle R_1 and the node R_3 if $H_1 > 0$. Considering the location of the singularities in the first case we obtain Picture 3.2($\overset{*}{f}2$), whereas in the second case we have Picture 3.2($\overset{*}{f}5$).

3.1.3. *The phase portraits associated with Config. 3.3.* According to Table 2 we shall consider the family of systems

$$\dot{x} = x[g + gx + (h-1)y], \quad \dot{y} = y[g - 1 + (g-1)x + hy], \tag{3.17}$$

for which the condition

$$gh(g+h-1)(g-1)(h-1) \neq 0 \tag{3.18}$$

holds and we keep the same notations for the singularities. For systems (3.17) we have:

$$\begin{gathered} M_1(0,0): \Delta_1 = g(g-1);\ \delta_1 = g^2; \\ M_4 \equiv M_2(-1,0):\ \Delta_2 = 0, \rho_2 = -g,\ \delta_2 = g^2; \\ M_3\big(0,(1-g)/h\big):\ \Delta_3 = (g-1)(1-g-h)/h, \quad \delta_3 = (gh-g-1)^2/h^2 \end{gathered} \tag{3.19}$$

for the finite singularities and

$$R_1(1,1,0):\ \widetilde{\Delta}_1 = 1-g-h; \quad R_2(1,0,0):\ \widetilde{\Delta}_2 = g; \quad R_3(0,1,0):\ \widetilde{\Delta}_3 = h. \tag{3.20}$$

for the infinite ones. We observe that due to the condition (3.18) we have $\rho_2 = -g \neq 0$ (i.e. only one of the corresponding eigenvalues vanishes) and hence the double singular point $M_1(0,0)$ is a saddle-node.

Considering the expressions above for systems (3.17) we obtain

$$\begin{gathered}\mu_0 = gh(g+h-1) = -\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3,\\ K = 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2,\\ B_3 = 3g(g+h-1)x^2y^2 = -3\widetilde{\Delta}_1\widetilde{\Delta}_2x^2y^2,\\ H_1 = 288g(g+h-1) = -288\widetilde{\Delta}_1\widetilde{\Delta}_2,\\ H_5 = 384(1-g)g^2(1-g-h)^2 = -384\Delta_1\widetilde{\Delta}_2\widetilde{\Delta}_1^2.\\ H_{14} = g^2h(g+h-1) = -30\widetilde{\Delta}_1\widetilde{\Delta}_2^2\widetilde{\Delta}_3,\end{gathered} \tag{3.21}$$

Clearly the next relations hold:

$$\begin{gathered}\operatorname{sign}(B_3H_{14}) = \operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3), \quad \operatorname{sign}(H_1) = -\operatorname{sign}(\widetilde{\Delta}_1\widetilde{\Delta}_2),\\ \operatorname{sign}(H_5) = -\operatorname{sign}(\Delta_1\widetilde{\Delta}_2).\end{gathered} \tag{3.22}$$

Since $\mu_0 = -\widetilde{\Delta}_1\widetilde{\Delta}_2)\widetilde{\Delta}_3)$ we can control the signs of each one of the determinants $\widetilde{\Delta}_2$, $\widetilde{\Delta}_3$ and Δ_1.

Since in this case we have one double (a saddle-node) and two simple finite singularities, the types of these points are determined by the same conditions indicated in the previous section (for systems (3.11)). So we shall consider the same cases as for Config. 3.2.

The case $\mu_0 < 0$. According to [3] (see Table 1) the simple singular points M_1 and M_3 are of the same type.

The subcase $K < 0$. Then M_1 and M_3 are saddles and all the infinite points are nodes and this univocally leads to the phase portrait given by Picture 3.3(d1).

The subcase $K > 0$. Then M_1 and M_3 are nodes and at infinity we have two saddles and one node. In order to distinguish which one among the infinite points is a node, we apply the invariant polynomial H_5 considering its sign given in (3.22).

I. Assume first $H_5 < 0$. Then due to $\Delta_1 > 0$ (as M_1 is a node) we obtain $\widetilde{\Delta}_2 > 0$ and hence R_2 is a node. Consequently R_3 and R_1 are saddles and we univocally obtain Picture 3.3(e1).

II. Suppose now $H_5 > 0$. In this case $\widetilde{\Delta}_2 < 0$ (i.e. $g < 0$) and the singular point R_2 is a saddle. So we have two possibilities: $h < 0$ (when R_3 is a saddle and R_1 is a node) and $h > 0$ (when R_3 is a node and R_1 is a saddle). It easily could be determined that in both cases we obtain phase portraits topologically equivalent to Picture 3.3(e2).

The case $\mu_0 > 0$. Since both singular points M_1 and M_3 are located on the invariant lines we conclude that one of them is a saddle and another one is a node. Regarding the infinite singular points, clearly by the index theory there should be a saddle and two nodes. We claim that in this case the phase portrait of a system (3.17) is one of those indicated below if and only if the corresponding conditions on the right side are satisfied:

$$\text{Picture 3.3(f1)} \Leftrightarrow B_3H_{14} < 0, H_5 < 0;$$
$$\text{Picture 3.3(f2)} \Leftrightarrow B_3H_{14} < 0, H_5 > 0, H_1 < 0;$$

$$\text{Picture 3.3(f3)} \Leftrightarrow B_3H_{14} < 0, H_5 > 0, H_1 > 0;$$
$$\text{Picture 3.3(f4)} \Leftrightarrow B_3H_{14} > 0, H_5 < 0;$$
$$\text{Picture 3.3(f5)} \Leftrightarrow B_3H_{14} > 0, H_5 > 0.$$

Indeed in order to prove this claim, considering (3.22) we shall examine two cases. The subcase $B_3H_{14} < 0$. Then $\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$ and as $\mu_0 > 0$ according to (3.21) it follows $\widetilde{\Delta}_1 > 0$, i.e. R_1 is a node.

I. Assume first $H_5 < 0$. In this case by (3.22) we obtain $\Delta_1\widetilde{\Delta}_2 > 0$ and this implies $g^2(g-1) > 0$. Therefore $g > 1$ and consequently $\widetilde{\Delta}_1 > 0$ and $\widetilde{\Delta}_3 < 0$. Hence the singular points R_2, R_1 and M_1 are nodes, whereas R_3 and M_3 are saddles. Considering the location of these singularities we univocally get Picture 3.3(f1).

II. Suppose now $H_5 > 0$. Then $g - 1 < 0$ and we consider two possibilities: $H_1 < 0$ and $H_1 > 0$.

(1) If $H_1 < 0$, then considering (3.22) we obtain $\operatorname{sign}(B_3H_{14}H_1) = \operatorname{sign}(\mu_0\widetilde{\Delta}_2) < 0$ and as $\mu_0 > 0$ (i.e. $\widetilde{\Delta}_1\widetilde{\Delta}_2\widetilde{\Delta}_3 < 0$) we get $\widetilde{\Delta}_2 > 0$. Therefore the condition $H_1 < 0$ yields $\widetilde{\Delta}_1 > 0$ and the condition $H_5 > 0$ implies $\Delta_1 < 0$. Thus besides the saddle-node $M_2(0,0)$, systems (3.17) have the nodes R_2, R_1 and M_3 and the saddles R_3 and M_1. This obviously leads to the phase portrait given by Picture 3.3(f2).

(2) If $H_1 > 0$ then in a similar way as above we get the nodes R_3, R_1 and M_1 and the saddles R_2 and M_3. Considering the locations of these singularities we univocally obtain Picture 3.3(f3).

The subcase $B_3H_{14} > 0$. Then $\widetilde{\Delta}_2\widetilde{\Delta}_3 > 0$ and as at infinity there could not be two saddles we obtain that both points R_2 and R_3 are nodes (i.e. $\widetilde{\Delta}_2 > 0$ and $\widetilde{\Delta}_3 > 0$), whereas the point R_1 is a saddle. Therefore considering (3.22) we have $\operatorname{sign}(H_5) = -\operatorname{sign}(\Delta_1)$, i.e. H_5 governs the types of the finite singularities. It is clear that we get Picture 3.3(f4) if $H_5 < 0$ and Picture 3.3(f5) if $H_5 > 0$.

3.1.4. *The phase portraits associated with Config. 3.4.* Considering Table 2 we examine the family of systems

$$\dot{x} = x[1 + (h-1)y], \quad \dot{y} = y(f - x + hy), \tag{3.23}$$

for which the condition

$$h(h-1)f(f-1)(f+h-fh) \neq 0 \tag{3.24}$$

holds. We observe that this family of systems is a particular case of the family (3.5) when the parameter g equals zero and in this case the point M_2 has gone to infinity and has coalesced with $R_2(1,0,0)$. So these systems possess three simple finite singularities and three infinite singularities (one of which being double). For all the singularities of these systems with the condition (3.24) we have:

$$M_1(0,0):\ \Delta_1 = f,\ \delta_1 = (1-f)^2;$$
$$M_3(0,-f/h):\ \Delta_3 = f(fh-f-h)/h,\ \delta_3 = (f+h)^2/h^2;$$
$$M_4\left(\frac{f+h-fh}{1-h}, \frac{1}{1-h}\right):\ \Delta_4 = \frac{fh-f-h}{1-h},\ \rho_4 = \frac{h}{1-h},\ \delta_4 = \rho_4^2 - 4\Delta_4 \tag{3.25}$$

for the finite singularities and

$$R_1(1,1,0):\ \widetilde{\Delta}_1=1-h;\ \ R_2(1,0,0):\ \widetilde{\Delta}_2=0,\rho_2=1;\ \ R_3(0,1,0):\ \widetilde{\Delta}_3=h \quad (3.26)$$

for the infinite ones. As $\rho_2 \neq 0$ the singular point $R_2(1,0,0)$ is a saddle-node (both hyperbolic sectors being on the same part of the infinite line).

Considering (3.25) and (3.26) for systems (3.23) we evaluate the invariant polynomials which we need:

$$\begin{gathered}
\mu_0=0, \quad K=2h(h-1)y^2, \quad \eta=1,\\
B_3=3h(1-f)x^2y^2=3(1-f)\widetilde{\Delta}_3x^2y^2,\\
W_3=\widetilde{\Delta}_1^2\widetilde{\Delta}_3^2(\delta_1\,\delta_3+\delta_1\,\delta_4+\delta_3\delta_4),\\
W_4=\widetilde{\Delta}_1^2\widetilde{\Delta}_3^2\,\delta_1\delta_3\delta_4,\\
U_1=\frac{1}{8}fh(1-f)(1-h)^2=\frac{1}{8}\Delta_1\widetilde{\Delta}_1^2\widetilde{\Delta}_3(1-f),\\
U_2=3f(f+h-fh)=-3\Delta_3\widetilde{\Delta}_3,\\
H_5=-384\big[h(fh-f-h)+f^2(h-1)\big],\\
G_9=h(h-1)/8=-\widetilde{\Delta}_1\widetilde{\Delta}_3/8.
\end{gathered} \tag{3.27}$$

From this, considering (3.25) and (3.26) we get the following relations:

$$\operatorname{sign}(B_3U_1)=\operatorname{sign}(\Delta_1);\quad \operatorname{sign}(U_2)=-\operatorname{sign}(\Delta_3\widetilde{\Delta}_3). \tag{3.28}$$

For systems (3.23) with three finite simple singularities we have $\mu_0 = 0$ and $G_9 = h(h-1)/8$. Therefore considering the fact that we could not have two foci, according to [3] (see Table 1) the types of the finite singularities are determined by the corresponding affine invariant conditions on the right side:

$$\begin{gathered}
(g)\ \tilde{s},\tilde{s},\tilde{n} \quad\Leftrightarrow\quad K<0, W_4\geq 0;\\
(\overset{*}{g})\ \tilde{s},\tilde{s},\tilde{f} \quad\Leftrightarrow\quad K<0, W_4<0;\\
(h)\ \tilde{s},\tilde{n},\tilde{n} \quad\Leftrightarrow\quad K>0 \text{ and either } W_4>0, \text{ or } W_4=0, W_3\geq 0;\\
(\overset{*}{h})\ \tilde{s},\tilde{n},\tilde{f} \quad\Leftrightarrow\quad K>0 \text{ and either } W_4<0, \text{ or } W_4=0, W_3<0;
\end{gathered} \tag{3.29}$$

So we consider the two cases: $K<0$ and $K>0$.

The case $K<0$. Then by (3.27) we have $0<h<1$ and considering (3.26) we get $\widetilde{\Delta}_1>0$ and $\widetilde{\Delta}_3>0$. So apart from the saddle-node $R_2(1,0,0)$, systems (3.23) possess at the infinity two nodes.

The subcase $W_4\geq 0$. According to (3.29) we have two saddles and one node. We claim, that in this case the phase portrait of a system (3.23) corresponds to one of those indicated below if and only if the corresponding conditions on the right side are satisfied:

$$\begin{gathered}
\text{Picture 3.4(g1)} \Leftrightarrow B_3U_1<0, U_2<0;\\
\text{Picture 3.4(g2)} \Leftrightarrow B_3U_1<0, U_2>0;\\
\text{Picture 3.4(g3)} \Leftrightarrow B_3U_1>0.
\end{gathered}$$

Indeed, considering (3.28) we examine two possibilities: $B_3U_1<0$ and $B_3U_1>0$.

I. The possibility $B_3U_1<0$. Then $\Delta_1<0$ and the singular point $M_1(0,0)$ is a saddle.

If $U_2 < 0$ then considering (3.28) and the relation $\widetilde{\Delta}_3 > 0$ we obtain $\Delta_3 > 0$. Hence M_3 is a node and consequently M_4 is a saddle and this leads univocally to Picture 3.4(g1).

Assume now $U_2 > 0$. In this case we get $\Delta_3 < 0$ and then M_3 is a saddle whereas M_4 is a node. In this case we univocally get the phase portrait given by Picture 3.4(g2).

II. The possibility $B_3U_1 > 0$. In this case we obtain that the singular point $M_1(0,0)$ is a node (as $\Delta_1 > 0$). Hence the other two singularities are saddles and considering also the infinite singularities we obtain Picture 3.4(g3).

The subcase $W_4 < 0$. As a focus could only be at the singular point M_4 we obtain that M_1 and M_3 are saddles and considering the nodes R_1 and R_3 at infinity this univocally leads to Picture 3.4($\overset{*}{g}2$).

The case $K > 0$. Then $h(h-1) > 0$ and considering (3.26) we get $\widetilde{\Delta}_1\widetilde{\Delta}_3 < 0$, i.e. at infinity besides the saddle-node we have one saddle and one node.

On the other hand for systems (3.23) according to (3.29), on the phase plane there exist one saddle and two anti-saddles and these two possibilities are distinguished by the invariant polynomials W_4 and W_3 as it is indicate in (3.29).

The subcase $W_4 > 0$ or $W_4 = 0$ and $W_3 \geq 0$. Then we have one saddle and two nodes. We claim that in this case the phase portrait of a system (3.23) is given by one of those indicated below if and only if the corresponding conditions are satisfied, respectively:

$$\text{Picture 3.4(h1)} \;\Leftrightarrow\; B_3U_1 < 0;$$
$$\text{Picture 3.4(h3)} \;\Leftrightarrow\; B_3U_1 > 0, U_2 < 0, H_5 > 0;$$
$$\text{Picture 3.4(h2)} \;\Leftrightarrow\; B_3U_1 > 0 \text{ and either } U_2 > 0, \text{ or } U_2 < 0, H_5 < 0.$$

To prove this claim we shall consider again two possibilities: $B_3U_1 < 0$ and $B_3U_1 > 0$.

I. The possibility $B_3U_1 < 0$. Then $\Delta_1 < 0$ (i.e. $f < 0$) and M_1 is a saddle whereas the other two points are nodes. We observe that due to $\delta_4 \geq 0$ the condition $K > 0$ (i.e. $h(h-1) > 0$) implies $h > 1$. Indeed since $f < 0$, supposing $h < 0$ we clearly obtain a contradiction: $\delta_4 = \frac{4f(h-1)^2 - 3h^2 + 4h}{(h-1)^2} < 0$. Thus, $h > 1$ and then R_3 is a node and R_1 is a saddle. This immediately leads to the Picture 3.4(h1).

II. The possibility $B_3U_1 > 0$. In this case we obtain $\Delta_1 > 0$ (i.e. $f > 0$) and M_1 is a node. Since by (3.28) the invariant polynomial U_2 governs the sign of the product $\Delta_3\widetilde{\Delta}_3$, we examine two subcases: $U_2 < 0$ and $U_2 > 0$.

(1) Assume first $U_2 > 0$. Then by (3.28) we obtain $\Delta_3\widetilde{\Delta}_3 < 0$, i.e. the singular points M_3 and R_3 are of different types. Fixing first $\Delta_3 > 0$ and, secondly $\Delta_3 < 0$, the types of all the singularities, as well as their location become well determined. In both cases we get phase portraits which are topologically equivalent to Picture 3.4(h2).

(2) Admit now $U_2 < 0$. Then we obtain $\Delta_3\widetilde{\Delta}_3 > 0$ and we have to distinguish via invariant polynomials when these determinants are both negative and when they are positive. Due to the condition $f > 0$ and considering (3.27), (3.25) and the relation $\widetilde{\Delta}_1\widetilde{\Delta}_3 < 0$ we obtain

$$\begin{aligned} &\operatorname{sign}(\Delta_3) = \operatorname{sign}\big(h(fh - f - h)\big) = \operatorname{sign}(\widetilde{\Delta}_3) = -\operatorname{sign}(\widetilde{\Delta}_1) = \operatorname{sign}\big(f^2(h-1)\big)\\ &\Rightarrow\; \operatorname{sign}(\Delta_3) = \operatorname{sign}\big(h(fh - f - h) + f^2(h-1)\big) = -\operatorname{sign}(H_5). \end{aligned}$$

So if $H_5 < 0$ then $\Delta_3 > 0$, $\widetilde{\Delta}_3 > 0$ and $\widetilde{\Delta}_1 < 0$. Hence systems (3.23) possess the nodes M_1, M_3 and R_3 and the saddles M_4 and R_1 (R_2 being a saddle-node). Therefore we get a phase portrait which is topologically equivalent to Picture 3.4(h2).

In the case $H_5 > 0$ in the same manner as above we get the nodes M_1, M_4 and R_1 and the saddles M_3 and R_3. This leads univocally to Picture 3.4(h3).
The subcase $W_4 < 0$ or $W_4 = 0$ and $W_3 < 0$. Then systems (3.23) have one saddle, one node and one focus. We claim that the phase portrait of a system in this family corresponds to one of those indicated below if and only if the corresponding conditions are satisfied:

$$\begin{aligned} &\text{Picture } 3.4(\overset{*}{h}1) \ \Leftrightarrow \ B_3U_1 < 0, U_2 < 0;\\ &\text{Picture } 3.4(\overset{*}{h}2) \ \Leftrightarrow \ B_3U_1 > 0, U_2 > 0;\\ &\text{Picture } 3.4(\overset{*}{h}3) \ \Leftrightarrow \ B_3U_1 > 0, U_2 < 0;\\ &\text{Picture } 3.4(\overset{*}{h}4) \ \Leftrightarrow \ B_3U_1 < 0, U_2 > 0. \end{aligned} \tag{3.30}$$

Indeed, first we mention that the focus could only be the singularity M_4 and that the type of the singularity M_1 is governed again by the invariant polynomial B_3U_1. Moreover, we observe that in the case under examination we have $\widetilde{\Delta}_1\widetilde{\Delta}_3 < 0$ and $\Delta_1\Delta_3 < 0$ (since one of the singularities M_1 and M_3 is a saddle and another one is a node). Therefore considering (3.28) we obtain the relations:

$$\operatorname{sign}(\Delta_1) = -\operatorname{sign}(\Delta_3) = \operatorname{sign}(B_3U_1); \quad \operatorname{sign}(\widetilde{\Delta}_3) = -\operatorname{sign}(\widetilde{\Delta}_1) = \operatorname{sign}(U_2B_3U_1).$$

Assuming that the conditions on the right side of (3.30) are satisfied and considering the equalities above it is easy to convince ourselves that we get univocally the corresponding phase portrait, except in the case Picture 3.4($\overset{*}{h}4$). For this case it is necessary to consider the stability of the focus M_4 and of the node M_3 of systems (3.23). Calculations yield: $\rho_3\rho_4 = \frac{f+h-2fh}{1-h}$. Therefore in the case $h < 0$ and $f < 0$ we obtain $\rho_3\rho_4 < 0$, i.e. these singularities have different stabilities. This completes the proof of the claim.

3.1.5. *The phase portraits associated with Config. 3.5.* According to Table 2 we shall consider the family of systems

$$\dot{x} = x[1 + (1-h)(x-y)], \quad \dot{y} = y(f - hx + hy), \tag{3.31}$$

for which the condition

$$h(h-1)f(f-1)(f+h-fh) \neq 0 \tag{3.32}$$

holds. We observe that this family of systems is a particular case of the family (3.5) when we have $g = 1-h$ and in this case the singular point M_4 has coalesced with $R_1(1,1,0)$. So these systems possess three simple finite singularities and three infinite singularities (one of which being double). For the singularities of these systems with the condition (3.24) we have for the finite singularities:

$$\begin{gathered} M_1(0,0): \ \Delta_1 = f, \quad \delta_1 = (1-f)^2;\\ M_2(1/(h-1),0): \ \Delta_2 = (fh-f-h)/(1-h), \quad \delta_3 = (fh-f-1)^2/(h-1)^2;\\ M_3(0,-f/h): \ \Delta_3 = f(fh-f-h)/h, \quad \delta_3 = (f+h)^2/h^2 \end{gathered} \tag{3.33}$$

and for the infinite ones

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 0, \ \widetilde{\rho}_1 = -1; R_2(1,0,0): \ \widetilde{\Delta}_2 = 1-h; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h. \tag{3.34}$$

Since $\widetilde{\rho}_1 \neq 0$ the singular point $R_1(1,1,0)$ is a saddle-node (both hyperbolic sectors being on the same part of the infinite line).

Evaluating the invariant polynomials we need for systems (3.31) we obtain:

$$\begin{gathered}
\mu_0 = 0, \quad K = 2h(h-1)(x-y)^2 = -2\widetilde{\Delta}_2\widetilde{\Delta}_3(x-y)^2, \\
B_3 = 3(f-1)(fh-f-h)x^2y^2 = 3(f-1)\Delta_2\widetilde{\Delta}_2x^2y^2, \\
U_1 = \frac{1}{8}f(f-1)(fh-f-h)h^2(1-h)^2 = \frac{1}{8}(f-1)\Delta_1\Delta_2\widetilde{\Delta}_2^3\widetilde{\Delta}_3^2, \\
U_3 = \frac{1}{2}(f+1)h(h-1)(fh-f-h) = -\frac{1}{2}(f+1)\Delta_2\widetilde{\Delta}_2^2\widetilde{\Delta}_3.
\end{gathered} \tag{3.35}$$

Herein considering (3.33) and (3.34) we get the following relations:

$$\operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1); \quad \text{if} \ \ B_3U_1 > 0 \ \Rightarrow \ \operatorname{sign}(U_3) = -\operatorname{sign}(\Delta_2\widetilde{\Delta}_3). \tag{3.36}$$

We observe that systems (3.31) possess three finite simple singularities. Therefore considering the fact that these systems possess neither a focus nor a center, according to [3] (see Table 1) we obtain two saddles and one node if $K < 0$, and one saddle and two nodes if $K > 0$.

On the other hand by (3.35) we have $\operatorname{sign}(K) = -\operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3)$. So clearly besides the saddle-node $R_1(1,1,0)$ systems (3.31) possess at the infinity two nodes if $K < 0$ and they have a node and a saddle if $K > 0$.

Remark 3.9. Without loss of generality we assume that the infinite singular point $R_2(1,0,0)$ is a node due to the substitution $(x,y,t,f,h) \mapsto (x/f, y/f, ft, 1/f, 1-h)$, which leads us to the systems (3.31) but interchanges the points R_2 and R_3.

The case $K < 0$. Then besides the saddle-node $R_1(1,1,0)$, systems (3.31) possess at the infinity two nodes.

The subcase $B_3U_1 < 0$. Then $\Delta_1 < 0$ (i.e. M_1 is a saddle) and hence one of the singular points M_2 or M_3 is a saddle and another one is a node. Therefore we have either $\Delta_2 > 0$ and $\Delta_3 < 0$ or $\Delta_2 < 0$ and $\Delta_3 > 0$. Considering the relations $f > 0$ and $0 < h < 1$ (which fix the position of the singularities) as well as the two nodes and the saddle-node at infinity in both cases we get the phase portraits topologically equivalent to Picture 3.5(g1).

The subcase $B_3U_1 > 0$. In this case we have $\Delta_1 > 0$ (i.e. $f > 0$) and M_1 is a node. Then the remaining two finite singularities are saddles and this leads to Picture 3.5(g2).

The case $K > 0$. Systems (3.31) possess as finite singularities a saddle and two nodes and at infinity they have a saddle and a node besides the saddle-node. According to Remark 3.9 we may assume that R_2 is a node and R_3 is a saddle (i.e. $h < 0$).

The case $B_3U_1 < 0$. We have $\Delta_1 < 0$ (i.e. $f < 0$) and therefore M_1 is a saddle whereas the remaining points are nodes. Considering the saddle R_3 and the node R_2 we obtain the phase portrait given by Picture 3.5(h1).

The case $B_3U_1 > 0$. In this case we obtain $\Delta_1 > 0$ (i.e. $f > 0$) and hence M_1 is a node. Since $\widetilde{\Delta}_3 < 0$ (as R_3 is a saddle) considering (3.36) we obtain $\operatorname{sign}(U_3) = \operatorname{sign}(\Delta_2)$.

I. The subcase $U_3 < 0$. Then $\Delta_2 < 0$, i.e. M_2 is a saddle and M_3 is a node. This leads univocally to the phase portrait given Picture 3.5(h2).

II. The subcase $U_3 > 0$. In this case M_2 is a node and M_3 is a saddle and considering the location of all the singularities we get a phase portrait which is topologically equivalent to Picture 3.5(h1).

3.1.6. *The phase portraits associated with Config. 3.6.* According to Table 2 we consider the one-parameter family of systems

$$\dot{x} = x[1 + (h-1)y], \quad \dot{y} = y(-x + hy), \quad h(h-1) \neq 0. \tag{3.37}$$

We observe that this family of systems is a particular case of the family (3.11) when $g = 0$. Hence in this case we have two pairs of singularities such that in each pair the two singularities have coalesced: M_3 with M_1 and M_2 with infinite point $R_2(1, 0, 0)$. So for the singularities of these systems we have

$$\begin{gathered} M_3 \equiv M_1(0,0) : \Delta_1 = 0, \quad \rho_1 = 1; \\ M_4\Big(\frac{h}{1-h}, \frac{1}{1-h}\Big) : \Delta_4 = \frac{h}{h-1} = -\rho_4, \quad \delta_4 = \frac{h(4-3h)}{(h-1)^2} \end{gathered} \tag{3.38}$$

for the finite singularities and

$$R_1(1,1,0): \ \widetilde{\Delta}_1 = 1 - h; \quad R_2(1,0,0): \ \widetilde{\Delta}_2 = 0, \ \widetilde{\rho}_2 = 1; \quad R_3(0,1,0): \ \widetilde{\Delta}_3 = h \tag{3.39}$$

for the infinite ones. Clearly both double points $M_1(0,0)$ and $R_2(1,0,0)$ are saddle-nodes. Moreover, for the second point both hyperbolic sectors are on the same part of the infinite line.

For systems (3.37) we have:

$$\begin{gathered} K = 2h(h-1)y^2 = 2\Delta_4\widetilde{\Delta}_1^2 y^2 = -2\widetilde{\Delta}_1\widetilde{\Delta}_3 y^2, \\ H_1 = 288h = 288\widetilde{\Delta}_3, \\ W_4 = h^3(4-3h) = \delta_4\widetilde{\Delta}_1^2\widetilde{\Delta}_3^2. \end{gathered} \tag{3.40}$$

Herein we observe that the invariant polynomials above govern the types of the simple singular points of systems (3.37). More precisely the types of the singularities M_4, R_1 and R_3 are determined by the following conditions, respectively:

$$\begin{array}{lll} \text{(i)} & K < 0 & \Rightarrow \quad M_4 - \text{saddle}, R_1, R_3 - \text{nodes}; \\ \text{(ii)} & K > 0, W_4 < 0, H_1 < 0 & \Rightarrow \quad M_4 - \text{focus}, R_1 - \text{node}, R_3 - \text{saddle}; \\ \text{(iii)} & K > 0, W_4 \geq 0 & \Rightarrow \quad M_4 - \text{node}, R_1 - \text{saddle}, R_3 - \text{node}; \\ \text{(iv)} & K > 0, W_4 < 0, H_1 > 0 & \Rightarrow \quad M_4 - \text{focus}, R_1 - \text{saddle}, R_3 - \text{node}. \end{array}$$

Then considering the location of the singular point M_4 in each one of the cases above we arrive at a phase portrait given by: Picture 3.6(k1) in the case (i); Picture 3.6($\overset{*}{l}1$) in the case (ii); Picture 3.6(l2) in the case (iii) and Picture 3.6($\overset{*}{l}2$) in the case (iv).

We stress that in the case of Picture 3.6($\overset{*}{l}1$) the behaviour of the trajectories in the vicinity of the focus M_4 is determined univocally due to the relation $\rho_1\rho_4 = -\Delta_4 < 0$ and this means that the stability of the focus is opposite to the stability of the parabolic sector of the saddle-node M_1.

3.1.7. *The phase portraits associated with Config. 3.7.* According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x[h-1+(h-1)y], \quad \dot{y} = y(h-x+hy), \quad h(h-1)\neq 0. \tag{3.41}$$

Comparing the singularities of these systems with those of the systems (3.5) we observe that in this case we have two pairs of singularities such that in each pair the two singularities have coalesced: M_4 with M_3 and M_2 with the infinite point $R_2(1,0,0)$. So for the singularities of these systems we obtain:

$$M_1(0,0): \Delta_1 = h(h-1); \quad M_4 \equiv M_3(0,-1):\ \Delta_3 = 0,\ \rho_3 = -h \tag{3.42}$$

for the finite singularities, and

$$R_1(1,1,0):\ \widetilde{\Delta}_1 = 1-h;\ R_2(1,0,0):\ \widetilde{\Delta}_2 = 0,\ \widetilde{\rho}_2 = 1;\ R_3(0,1,0):\ \widetilde{\Delta}_3 = h \tag{3.43}$$

for the infinite ones. Clearly both double points $M_3(0,0)$ and $R_2(1,0,0)$ are saddle-nodes and the hyperbolic sectors of the second saddle-node are located on the same part of the infinite line.

For systems (3.41) we calculate:

$$\begin{gathered} K = 2h(h-1)y^2 = 2\Delta_1 y^2 \quad \Rightarrow \quad \operatorname{sign}(K) = \operatorname{sign}(\Delta_1) = -\operatorname{sign}(\widetilde{\Delta}_1\widetilde{\Delta}_3);\\ H_5 = 384h^2(1-h)^3 = 384\widetilde{\Delta}_1^3\widetilde{\Delta}_3^2 \quad \Rightarrow \quad \operatorname{sign}(H_5) = \operatorname{sign}(\widetilde{\Delta}_1). \end{gathered} \tag{3.44}$$

Herein we observe that the invariant polynomials K and H_5 govern the types of the simple singular points of systems (3.41). More precisely the types of the singularities M_1, R_1 and R_3 are determined by the following conditions, respectively:

$$\begin{array}{lll} \text{(i)} & K<0 & \Rightarrow \quad M_1 - \text{saddle}, R_1, R_3 - \text{nodes};\\ \text{(ii)} & K>0, H_5<0 & \Rightarrow \quad M_1 - \text{node}, R_1 - \text{saddle}, R_3 - \text{node};\\ \text{(iii)} & K>0, H_5>0 & \Rightarrow \quad M_1 - \text{node}, R_1 - \text{node}, R_3 - \text{saddle}. \end{array}$$

Since the coordinates of all positions of the singularities are determined, in each one of the cases above we arrive univocally at the phase portrait given by: Picture 3.7(k1) in the case (i); Picture 3.7(l2) in the case (ii) and Picture 3.7(l1) in the case (iii).

3.1.8. *The phase portraits associated with Config. 3.8.* According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x[1+(1-h)(x-y)], \quad \dot{y} = hy(y-x), \quad h(h-1)\neq 0. \tag{3.45}$$

Comparing the singularities of these systems with those of the systems (3.5) we observe that in this case we have again two pairs of singularities such that in each pair the two singularities have coalesced: M_3 with M_1 and M_4 with the infinite point $R_1(1,1,0)$. So for the singularities of these systems we obtain:

$$M_3 \equiv M_1(0,0):\ \Delta_1 = 0,\ \rho_1 = 1; \quad M_2(1/(h-1),0):\ \Delta_2 = h/(h-1) \tag{3.46}$$

for the finite singularities and

$$R_1(1,1,0):\ \widetilde{\Delta}_1 = 0,\ \widetilde{\rho}_1 = -1;\ \ R_2(1,0,0):\ \widetilde{\Delta}_2 = 1-h;\ \ R_3(0,1,0):\ \widetilde{\Delta}_3 = h \tag{3.47}$$

for the infinite ones. Clearly both double points $M_1(0,0)$ and $R_1(1,0,0)$ are saddle-nodes and the hyperbolic sectors of the second saddle-node are located on the same part of the infinite line.

For systems (3.45) we calculate:

$$\begin{gathered} K = 2h(h-1)(x-y)^2 = 2\Delta_1\widetilde{\Delta}_2^2(x-y)^2 \\ \Rightarrow\ \operatorname{sign}(K) = \operatorname{sign}(\Delta_1) = -\operatorname{sign}(\widetilde{\Delta}_2\widetilde{\Delta}_3); \\ H_5 = 384h^3 = 384\widetilde{\Delta}_3^3\ \Rightarrow\ \operatorname{sign}(H_5) = \operatorname{sign}(\widetilde{\Delta}_3). \end{gathered} \tag{3.48}$$

So we again obtain that these invariant polynomials determine completely the types of the simple singularities. Thus applying the same arguments as above we get for the systems (3.45) the following phase portraits: Picture 3.8(k1) if $K<0$; Picture 3.8(l1) if $K>0$ and $H_5<0$; and Picture 3.8(l2) if $K>0$ and $H_5>0$.

3.1.9. *The phase portraits associated with Config. 3.9.* According to Table 2 we consider the family of systems

$$\dot{x} = x(1+gx+y), \quad \dot{y} = y(f-x+gx+y), \tag{3.49}$$

for which the condition

$$g(g-1)f(f-1)(1-g+fg) \neq 0 \tag{3.50}$$

holds. For the all four distinct finite singularities of systems (3.49) with the condition (3.50) we have

$$\begin{gathered} M_1(0,0):\quad \Delta_1 = f,\quad \rho_1 = f+1,\quad \delta_1 = (f-1)^2; \\ M_2(-1/g,0):\quad \Delta_2 = (g-1-fg)/g,\quad \delta_2 = (1+fg)^2/g^2; \\ M_3(0,-f):\quad \Delta_3 = f(f-1),\quad \delta_3 = 1-2f; \\ M_4(f-1,g-1-fg): \\ \Delta_4 = (f-1)(g-1-fg), \rho_4 = -1,\ \delta_4 = 4g(f-1)^2+4f-3. \end{gathered} \tag{3.51}$$

and for the two infinite singular points we obtain

$$R_2(1,0,0):\ \widetilde{\Delta}_2 = g;\quad R_1 \equiv R_3(0,1,0):\ \widetilde{\Delta}_3 = 0, \widetilde{\rho}_3 = 1. \tag{3.52}$$

We note that in this case the infinite singularity $R_3(0,1,0)$ is a saddle-node for which the infinite line serves as a separatrix for the hyperbolic sectors.

Taking into consideration (3.51) and (3.52) we evaluate for systems (3.49) the invariant polynomials we need:

$$\begin{gathered} \mu_0 = g = \widetilde{\Delta}_2,\ B_3 = 3(1-f)x^2y^2, \\ K = 2g(g-1)x^2 + 4gxy + 2y^2, \\ U_1 = \frac{1}{8}f(1-f)(g-1)^2 = \frac{1}{8}\Delta_1(1-f)(g-1)^2, \\ U_4 = f(1-f)^2(g-1-fg) = \Delta_1\Delta_2\widetilde{\Delta}_2(f-1)^2, \\ H_4 = 48(1-f) = -48\Delta_3/\Delta_1. \end{gathered} \tag{3.53}$$

Herein considering the condition (3.50) we evidently obtain the relations

$$\begin{gathered} \operatorname{sign}(\mu_0) = \operatorname{sign}(\widetilde{\Delta}_2),\quad \operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1), \\ \operatorname{sign}(U_4) = \operatorname{sign}(\Delta_1\Delta_2\widetilde{\Delta}_2),\quad \operatorname{sign}(H_4) = -\operatorname{sign}(\Delta_1\Delta_3). \end{gathered} \tag{3.54}$$

The case $\mu_0 < 0$. As $\mu_0 = \text{Discrim}(K)/16$ by (3.53) we conclude that $K > 0$. Therefore according to [3] (see Table 1) on the finite part of the phase plane systems (3.49) possess one saddle and three anti-saddles. Since three singularities are on the invariant lines, clearly only one anti-saddle could be a focus. Considering [3] (see Table 1), apart from the saddle we have three nodes if either $W_4 > 0$ or $W_4 = 0$ and $W_3 \geq 0$; and we have two nodes and a focus if either $W_4 < 0$ or $W_4 = 0$ and $W_3 < 0$.

On the other hand, by (3.54) we get $\widetilde{\Delta}_2 < 0$, i.e. the infinite singularity $R_2(1,0,0)$ is a saddle.

Remark 3.10. We note that in the case $\mu_0 < 0$ and $\delta_4 \geq 0$ (i.e. when M_4 is a node) the singular point M_1 should be a node.

Indeed suppose that M_1 is a saddle. Considering (3.51) the conditions $g < 0$ and $f < 0$ imply $\delta_4 < 0$, i.e. we get a contradiction.

Herein considering (3.54), we obtain that the types of the finite singularities of systems (3.49) are determined by the following conditions, respectively:

			M_1	M_2	M_3	M_4
(i)	$(W_4 > 0) \vee (W_4 = 0, W_3 \geq 0), U_4 < 0, H_4 < 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{n}$	$\tilde{n}$	$\tilde{s}$;
(ii)	$(W_4 > 0) \vee (W_4 = 0, W_3 \geq 0), U_4 < 0, H_4 > 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{n}$	$\tilde{s}$	$\tilde{n}$;
(iii)	$(W_4 > 0) \vee (W_4 = 0, W_3 \geq 0), U_4 > 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{s}$	$\tilde{n}$	$\tilde{n}$;
(iv)	$(W_4 < 0) \vee (W_4 = 0, W_3 < 0), B_3U_1 > 0, U_4 < 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{n}$	$\tilde{s}$	$\tilde{f}$;
(v)	$(W_4 < 0) \vee (W_4 = 0, W_3 < 0), B_3U_1 > 0, U_4 > 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{s}$	$\tilde{n}$	$\tilde{f}$;
(vi)	$(W_4 < 0) \vee (W_4 = 0, W_3 < 0), B_3U_1 < 0$	$\Rightarrow$	$\tilde{s}$	$\tilde{n}$	$\tilde{n}$	$\tilde{f}$.

We note that in the case (vi) we have $\rho_3\rho_4 < 0$, i.e. the node M_3 and the focus M_4 are of the opposite stabilities. So considering the infinite singularities $R_2(1,0,0)$ (a saddle) and $R_3(0,1,0)$ (a saddle-node) we arrive in each of the mentioned cases to the following phase portrait, respectively:

(i) 3.9(b1); (ii) 3.9(b2); (iii) 3.9(b3);

(iv) 3.9($\overset{*}{b}$2); (v) 3.9($\overset{*}{b}$3); (vi) 3.9($\overset{*}{b}$4).

The case $\mu_0 > 0$. According to [3] (see Table 1) on the finite part of the phase plane, systems (3.49) possess two saddles and two anti-saddles. Moreover as only one anti-saddle could be a focus, besides the saddles we have two nodes if $W_4 \geq 0$ and we have a node and a focus if $W_4 < 0$.

On the other hand, by (3.54) we get $\widetilde{\Delta}_2 > 0$, i.e. the infinite singularity $R_2(1,0,0)$ is a node.

Thus considering (3.54) we obtain that the types of the finite singularities of systems (3.49) are determined by the following conditions, respectively:

			M_1	M_2	M_3	M_4
(i)	$W_4 \geq 0, B_3U_1 < 0, U_4 < 0$	$\Rightarrow$	$\tilde{s}$	$\tilde{n}$	$\tilde{n}$	$\tilde{s}$;
(ii)	$W_4 \geq 0, B_3U_1 < 0, U_4 > 0$	$\Rightarrow$	$\tilde{s}$	$\tilde{s}$	$\tilde{n}$	$\tilde{n}$;
(iii)	$W_4 \geq 0, B_3U_1 > 0, H_4 < 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{s}$	$\tilde{n}$	$\tilde{s}$;
(iv)	$W_4 \geq 0, B_3U_1 > 0, H_4 > 0, U_4 < 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{s}$	$\tilde{s}$	$\tilde{n}$;
(v)	$W_4 \geq 0, B_3U_1 > 0, H_4 > 0, U_4 > 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{n}$	$\tilde{s}$	$\tilde{s}$;
(vi)	$W_4 < 0, B_3U_1 > 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{s}$	$\tilde{s}$	$\tilde{f}$;
(vii)	$W_4 < 0, B_3U_1 < 0$	$\Rightarrow$	$\tilde{s}$	$\tilde{s}$	$\tilde{n}$	$\tilde{f}$.

We note that in cases (i), (iv) and (v) we obtain phase portraits which are topologically equivalent to the same portrait, given by Picture 3.9(c1). Therefore this picture occurs if and only if $W_4 \geq 0$ and either $B_3U_1 < 0$ and $U_4 < 0$, or $B_3U_1 > 0$ and $H_4 > 0$.

Examining all the cases above considering the infinite singularities $R_2(1,0,0)$ (a node) and $R_3(0,1,0)$ (a saddle-node) we arrive in each of the remaining cases to one of the phase portraits:

$$\text{(ii) } 3.9(c2);\quad \text{(iii) } 3.9(c3);\quad \text{(vi) } 3.9(\overset{*}{c}1);\quad \text{(vii) } 3.9(\overset{*}{c}2).$$

Thus we arrive exactly at the respective conditions given by Table 5 in this case.

3.1.10. *The phase portraits associated with Config. 3.10.* According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x(g + gx + y),\quad \dot{y} = y[g - 1 + (g-1)x + y],\quad g(g-1) \neq 0. \tag{3.55}$$

These systems possess three finite singularities (one of them being double) and two infinite (one double). For the finite singularities of systems (3.55) we have

$$\begin{gathered} M_1(0,0):\ \Delta_1 = g(g-1);\quad M_3(0,1-g):\ \Delta_3 = 1-g;\\ M_4 \equiv M_2(-1,0):\quad \Delta_2 = 0,\quad \rho_2 = -g \end{gathered} \tag{3.56}$$

and for the two infinite singular points we obtain

$$R_2(1,0,0):\ \widetilde{\Delta}_2 = g;\quad R_1 \equiv R_3(0,1,0):\ \widetilde{\Delta}_3 = 0, \widetilde{\rho}_3 = 1. \tag{3.57}$$

For systems (3.55) calculations yield:

$$\mu_0 = g,\ B_3 = 3gx^2y^2,\quad U_1 = \frac{1}{8}g^2(g-1)^3. \tag{3.58}$$

Herein considering (3.56) and (3.57) we obtain the following relations:

$$\operatorname{sign}(\mu_0) = \operatorname{sign}(\widetilde{\Delta}_2) = -\operatorname{sign}(\Delta_1\Delta_3),\quad \operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1). \tag{3.59}$$

So we observe that the two invariant polynomials μ_0 and B_3U_1 determine completely the types of the simple singularities. More exactly we obtain that the types of all the singularities of systems (3.55) (for infinite points we denote them by capital letters) and they are determined by the following conditions, respectively:

$$\begin{array}{llcccccc} & & & M_1 & M_2 & M_3 & R_2 & R_3\\ \text{(i)} & \mu_0 < 0 & \Rightarrow & \tilde{n} & \tilde{s}\tilde{n} & \tilde{n} & \widetilde{S} & \widetilde{S}\widetilde{N};\\ \text{(ii)} & \mu_0 > 0, B_3U_1 < 0 & \Rightarrow & \tilde{s} & \tilde{s}\tilde{n} & \tilde{n} & \widetilde{N} & \widetilde{S}\widetilde{N};\\ \text{(iii)} & \mu_0 > 0, B_3U_1 > 0 & \Rightarrow & \tilde{n} & \tilde{s}\tilde{n} & \tilde{s} & \widetilde{N} & \widetilde{S}\widetilde{N}. \end{array}$$

These types of singularities univocally lead to the following phase portraits, respectively:

(i) Picture 3.10(e1); (ii) Picture 3.10(f1); Picture 3.10(f2).

3.1.11. *The phase portraits associated with Config. 3.11.* According to Table 2 a system possessing this configuration belongs to the one-parameter family of systems

$$\dot{x} = x(1 + gx + y), \quad \dot{y} = y(-x + gx + y), \quad g(g-1) \neq 0. \tag{3.60}$$

These systems possess the following finite singularities:

$$\begin{gathered} M_3 \equiv M_1(0,0): \ \Delta_1 = 0, \rho_1 = 1; \quad M_2(-1/g, 0): \ \Delta_2 = \frac{g-1}{g}, \rho_2 = \frac{1-2g}{g}; \\ M_4(-1, g-1): \quad \Delta_4 = 1 - g, \ \rho_4 = -1, \ \delta_4 = 4g - 3 \end{gathered} \tag{3.61}$$

and infinite ones:

$$R_2(1,0,0): \ \widetilde{\Delta}_2 = g; \quad R_1 \equiv R_3(0,1,0): \ \widetilde{\Delta}_3 = 0, \widetilde{\rho}_3 = 1. \tag{3.62}$$

For systems (3.60) calculations yield:

$$\mu_0 = g, \quad W_4 = 4g - 3, \quad H_5 = \ 384(1-g). \tag{3.63}$$

Herein we obtain:

$$\operatorname{sign}(\mu_0) = \operatorname{sign}(\widetilde{\Delta}_2) = -\operatorname{sign}(\Delta_2\Delta_4), \ \operatorname{sign}(H_5) = \operatorname{sign}(\Delta_4), \ \operatorname{sign}(W_4) = \operatorname{sign}(\delta_4). \tag{3.64}$$

So we obtain that the types of all the singularities of systems (3.60) are determined by the following conditions, respectively:

			M_1	M_2	M_4	R_2	R_3
(i)	$\mu_0 < 0$	$\Rightarrow$	$\tilde{s}\tilde{n}$	$\tilde{n}$	$\tilde{f}$	$\widetilde{S}$	$\widetilde{S}\widetilde{N}$;
(ii)	$\mu_0 > 0, W_4 \geq 0, H_5 > 0$	$\Rightarrow$	$\tilde{s}\tilde{n}$	$\tilde{s}$	$\tilde{n}$	$\widetilde{N}$	$\widetilde{S}\widetilde{N}$;
(iii)	$\mu_0 > 0, W_4 < 0$	$\Rightarrow$	$\tilde{s}\tilde{n}$	$\tilde{s}$	$\tilde{f}$	$\widetilde{N}$	$\widetilde{S}\widetilde{N}$;
(iv)	$\mu_0 > 0, W_4 \geq 0, H_5 < 0$	$\Rightarrow$	$\tilde{s}\tilde{n}$	$\tilde{n}$	$\tilde{s}$	$\widetilde{N}$	$\widetilde{S}\widetilde{N}$.

We observe that in the case $\mu_0 < 0$ (i.e. $g < 0$) the condition $\rho_2\rho_4 = (2g-1)/g > 0$, i.e. the stabilities of the node M_2 and of the focus M_4 coincide. So considering the types of the singular points above we get univocally the following phase portraits, respectively:

(i) Picture 3.11($\overset{*}{e}1$); (ii) Picture 3.11(f1);

(iii) Picture 3.11($\overset{*}{f}1$); (iv) Picture 3.11(f2).

3.1.12. *The phase portraits associated with Config. 3.12.* According to Table 2 we consider the family of systems

$$\dot{x} = x(1 + y), \quad \dot{y} = y(f + x + y), \quad f(f-1) \neq 0 \tag{3.65}$$

which possess the following five singularities:

$$\begin{gathered} M_1(0,0): \quad \Delta_1 = f, \quad \rho_1 = f + 1; \\ M_3(0,-f): \quad \Delta_3 = f(f-1), \quad \rho_3 = 1 - 2f; \\ M_4\,(1-f,-1): \quad \Delta_4 = 1 - f, \quad \rho_4 = -1, \quad \delta_4 = 4f - 3 \end{gathered} \tag{3.66}$$

and

$$R_1 = R_3(0,1,0): \quad \widetilde{\Delta}_3 = 0, \widetilde{\rho}_3 = 1; \quad R_2(1,0,0): \quad \widetilde{\Delta}_2 = 0, \widetilde{\rho}_2 = -1. \tag{3.67}$$

Remark 3.11. We observe that both infinite points are double and they are saddle-nodes. However for $R_3(0,1,0)$ the infinite line serves as a separatrix for the hyperbolic sectors, whereas both hyperbolic sectors of the saddle-node $R_2(1,0,0)$ are located on the same part of the infinite line.

Considering (3.66) for systems (3.65) we calculate

$$\begin{gathered} B_3 = 3(f-1)x^2y^2 = -3\Delta_4 x^2y^2,\ U_1 = \frac{1}{8}f(f-1) = -\frac{1}{8}\Delta_1\Delta_4, \\ H_5 = 384(1-f) = 384\Delta_4,\ W_4 = (f-1)^2(4f-3) = \Delta_4^2\delta_4. \end{gathered} \tag{3.68}$$

Herein we obtain

$$\operatorname{sign}(B_3U_1) = \operatorname{sign}(\Delta_1);\quad \operatorname{sign}(H_5) = \operatorname{sign}(\Delta_4);\quad \operatorname{sign}(W_4) = \operatorname{sign}(\delta_4). \tag{3.69}$$

So these invariant polynomials determine the types of all the finite singularities of systems (3.65) as follows:

			M_1	M_3	M_4
(i)	$B_3U_1 < 0$	$\Rightarrow$	$\tilde{s}$	$\tilde{n}$	$\tilde{f}$;
(ii)	$B_3U_1 > 0, H_5 > 0, W_4 \geq 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{s}$	$\tilde{n}$;
(iii)	$B_3U_1 > 0, H_5 > 0, W_4 < 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{s}$	$\tilde{f}$;
(iv)	$B_3U_1 > 0, H_5 < 0$	$\Rightarrow$	$\tilde{n}$	$\tilde{n}$	$\tilde{s}$.

We observe that in the case $B_3U_1 < 0$ (i.e. $f < 0$) the condition $\rho_3\rho_4 = 2f-1 < 0$, i.e. the stabilities of the node M_3 and of the focus M_4 are opposite. So considering the types of the singular points above and Remark 3.11 we get univocally the following phase portraits, respectively:

(i) Picture 3.12($\overset{*}{h}1$); (ii) Picture 3.12(h2);

(iii) Picture 3.12($\overset{*}{h}2$); (iv) Picture 3.12(h3).

As we have the one-parameter family of systems the conditions above could be simplified. More precisely as the bifurcation value $f = 3/4$ (respectively $f = 0$; $f = 1$) for the parameter f is given by polynomial W_4 (respectively B_3U_1; H_5), we get, for the respective phase portraits, the conditions given by Table 5.

3.1.13. *The phase portraits associated with Config. 3.13.* According to Table 2 this configuration corresponds to the normal form

$$\dot{x} = x(1+y),\quad \dot{y} = y(x+y), \tag{3.70}$$

which could be viewed as a special case of systems (3.65), when $f = 0$. So considering (3.66) the singular point M_3 has coalesced with $M_1(0,0)$ (becoming a saddle-node) and M_4 in this case is a focus. Therefore considering Remark 3.11 we get univocally Picture 3.13($\overset{*}{l}1$).

3.2. **The phase portraits of degenerate LV-systems.** In this section we examine the phase portraits of the degenerate LV-systems with the configurations Configs. LV$_d$.j with $j = 1, 2, \dots, 14$ (see Table 4 and Fig. 2).

Theorem 3.12. *The degenerate LV-systems have a total of 20 topologically distinct phase portraits which are given in Fig.6. The necessary and sufficient conditions for the realization of each one of these phase portraits are given in columns 2 and 3 of Table 6.*

Table 6

Configu-ration	*Necessary and sufficient conditions*	*Additional conditions for phase portraits*		*Phase portrait*
Config. LV_d.1	$\eta > 0, \mu_{0,1,2,3,4} = 0$, $\theta \neq 0, H_7 \neq 0$	$K < 0$		Picture LV_d.1(a)
		$K > 0$		Picture LV_d.1(b)
Config. LV_d.2	$\eta > 0, \mu_{0,1,2,3,4} = 0$, $\theta \neq 0, H_7 = 0$	$K < 0$		Picture LV_d.2(a)
		$K > 0$		Picture LV_d.2(b)
Config. LV_d.3	$\eta > 0, \mu_{0,1,2,3,4} = 0$, $\theta = H_4 = 0, H_7 \neq 0$	–		Picture LV_d.3
Config. LV_d.4	$\eta > 0, \mu_{0,1,2,3,4} = 0$, $\theta = H_4 = 0, H_7 = 0$	–		Picture LV_d.4
Config. LV_d.5	$\eta = 0, \mu_{0,1,2,3,4} = 0$, $\theta \neq 0, H_7 \neq 0$	–		Picture LV_d.5
Config. LV_d.6	$\eta = 0, \mu_{0,1,2,3,4} = 0$, $\theta \neq 0, H_7 = 0$	–		Picture LV_d.6
Config. LV_d.7	$\eta = 0, \mu_{0,1,2,3,4} = 0$, $\theta = 0, K \neq 0, H_2 \neq 0$	$K < 0$		Picture LV_d.7(a)
		$K > 0$	$L < 0$	Picture LV_d.7(b)
			$L > 0$	Picture LV_d.7(c)
Config. LV_d.8	$\eta = 0, \mu_{0,1,2,3,4} = 0$, $\theta = 0, K \neq 0, H_2 = 0$	$K < 0$		Picture LV_d.8(a)
		$K > 0$	$L < 0$	Picture LV_d.8(b)
			$L > 0$	Picture LV_d.8(c)
Config. LV_d.9	$\eta = 0, \mu_{0,1,2,3,4} = 0, \theta = 0$, $K = H_7 = 0, N \neq 0, H_2 \neq 0$	–		Picture LV_d.9
Config. LV_d.10	$\eta = 0, \mu_{0,1,2,3,4} = 0, \theta = 0$, $K = H_7 = 0, N \neq 0, H_2 = 0$	–		Picture LV_d.10
Config. LV_d.11	$\eta = 0, \mu_{0,1,2,3,4} = 0, \theta = 0$, $K = N = D = N_1 = 0, N_5 > 0$	–		Picture LV_d.11
Config. LV_d.12	$\eta = 0, \mu_{0,1,2,3,4} = 0, \theta = 0$, $K = N = D = N_1 = 0, N_5 = 0$	–		Picture LV_d.12
Config. LV_d.13	$C_2 = 0, \mu_{0,1,2,3,4} = 0, H_2 \neq 0$	–		Picture LV_d.13
Config. LV_d.14	$C_2 = 0, \mu_{0,1,2,3,4} = 0, H_2 = 0$	–		Picture LV_d.14

Proof of Theorem 3.12. We examine each one of the canonical systems (LV_d.j) ($j \in \{1, 2, \dots, 14\}$) given in the Table 6 corresponding to the configurations Config. LV_d.j of the degenerate LV-systems.

Clearly a degenerate real LV-system must posses at least one real affine straight line filled up with singularities. So the phase portraits can easily be detected and in what follows we only indicate for a given configuration: (i) the invariant lines filled up with singularities; (ii) the corresponding linear (or even constant) systems; (iii) the invariant lines of the linear systems; (iv) the topologically distinct phase portraits of the respective quadratic systems; (v) and whenever necessary the affine invariant polynomials which provide the respective conditions.

3.2.1. *The phase portraits associated with Config. LV_d.1.* (i) Singular line: $x = 0$; (ii) Corresponding linear systems: $\dot{x} = 1 + gx - y,\ \dot{y} = (g-1)y,\ g(g-1) \neq 0$; (iii) Invariant lines of the linear systems: $y = 0$ and $g(x - y) + 1 = 0$; (iv) Phase portraits: Picture LV_d.1(a) if $g(g-1) < 0$ and Picture LV_d.1(b) if

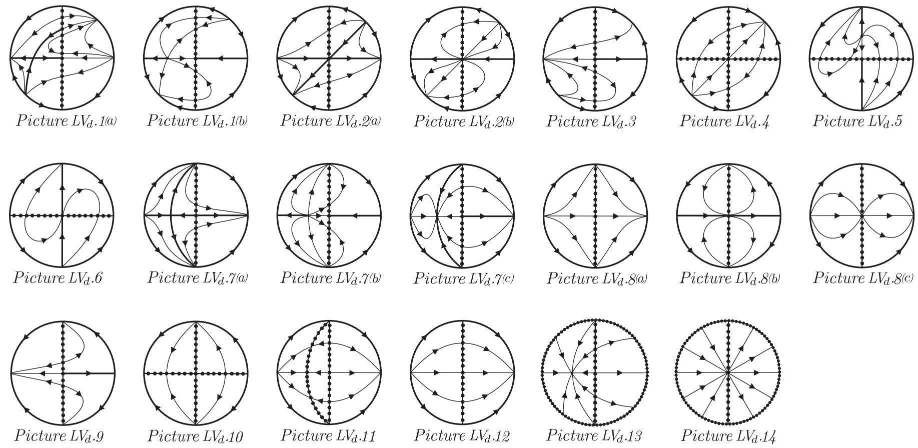


FIGURE 6. Phase portraits of the family of degenerate LV-systems

$g(g-1) > 0$;
(v) Invariant polynomial: $K = 2g(g-1)x^2 \Rightarrow \operatorname{sign}(K) = \operatorname{sign}\big(g(g-1)\big)$.

3.2.2. *The phase portraits associated with Config. LV_d.2.* (i) Singular line: $x = 0$;
(ii) Corresponding linear systems: $\dot{x} = gx - y,\ \dot{y} = (g-1)y,\ g(g-1) \neq 0$;
(iii) Invariant lines of the linear systems: $y = 0$ and $x - y = 0$;
(iv) Phase portraits: Picture LV_d.2(a) if $g(g-1) < 0$ and Picture LV_d.2(b) if $g(g-1) > 0$;
(v) Invariant polynomial: $K = 2g(g-1)x^2 \Rightarrow \operatorname{sign}(K) = \operatorname{sign}\big(g(g-1)\big)$.

3.2.3. *The phase portraits associated with Config. LV_d.3.* (i) Singular line: $x = 0$;
(ii) Corresponding linear system: $\dot{x} = 1 + y,\ \ \dot{y} = y$;
(iii) Invariant lines of the linear system: $y = 0$;
(iv) Phase portrait: Picture LV_d.3.

3.2.4. *The phase portraits associated with Config. LV_d.4.* (i) Singular lines: $x = 0$ and $y = 0$;
(ii) Corresponding constant system: $\dot{x} = 1,\ \dot{y} = 1$;
(iii) Invariant lines of the constant system: $y = x + C$;
(iv) Phase portrait: Picture LV_d.4.

3.2.5. *The phase portraits associated with Config. LV_d.5.* (i) Singular line: $y = 0$;
(ii) Corresponding linear system: $\dot{x} = x,\ \dot{y} = 1 - x + y$;
(iii) Invariant lines of the linear system: $x = 0$ (double);
(iv) Phase portrait: Picture LV_d.5.

3.2.6. *The phase portraits associated with Config. LV_d.6.* (i) Singular line: $y = 0$;
(ii) Corresponding linear system: $\dot{x} = x,\ \dot{y} = -x + y$;
(iii) Invariant lines of the linear system: $x = 0$ (double);
(iv) Phase portrait: Picture LV_d.6.

3.2.7. *The phase portraits associated with Config. LV_d.7.* (i) Singular line: $x = 0$;
(ii) Corresponding linear systems: $\dot{x} = 1 + gx,\ \dot{y} = (g-1)y,\ g(g-1) \neq 0$;
(iii) Invariant lines of the linear systems: $y = 0$ and $gx + 1 = 0$;
(iv) Phase portraits: Pictures: LV_d.7(a) if $g(g-1) < 0$; LV_d.7(b) if $g < 0$; LV_d.7(c) if $g > 1$;
(v) Invariant polynomials: $\begin{cases} K = 2g(g-1)x^2 & \Rightarrow \mathrm{sign}(K) = \mathrm{sign}\big(g(g-1)\big); \\ L = 8gx^2 & \Rightarrow \mathrm{sign}(L) = \mathrm{sign}(g). \end{cases}$

3.2.8. *The phase portraits associated with Config. LV_d.8.* (i) Singular line: $x = 0$;
(ii) Corresponding linear systems: $\dot{x} = gx,\ \dot{y} = (g-1)y,\ g(g-1) \neq 0$;
(iii) Invariant lines of the linear systems: $y = 0$ and $x = 0$;
(iv) Phase portraits: Pictures: LV_d.8(a) if $g(g-1) < 0$; LV_d.8(b) if $g < 0$; LV_d.8(c) if $g > 1$;
(v) Invariant polynomials: $\begin{cases} K = 2g(g-1)x^2 & \Rightarrow \mathrm{sign}(K) = \mathrm{sign}\big(g(g-1)\big); \\ L = 8gx^2 & \Rightarrow \mathrm{sign}(L) = \mathrm{sign}(g). \end{cases}$

3.2.9. *The phase portraits associated with Config. LV_d.9.* (i) Singular line: $x = 0$;
(ii) Corresponding linear system: $\dot{x} = 1,\ \dot{y} = y$;
(iii) Invariant lines of the linear system: $y = 0$;
(iv) Phase portrait: Picture LV_d.9.

3.2.10. *The phase portraits associated with Config. LV_d.10.* (i) Singular lines: $x = 0$ and $y = 0$;
(ii) Corresponding constant system: $\dot{x} = 0,\ \dot{y} = 1$;
(iii) Invariant lines of the constant system: $x = C$, $C \in \mathbb{R}$;
(iv) Phase portrait: Picture LV_d.10.

3.2.11. *The phase portraits associated with Config. LV_d.11.* (i) Singular lines: $x = 0$ and $x + 2 = 0$;
(ii) Corresponding constant system: $\dot{x} = 1,\ \dot{y} = 0$;
(iii) Invariant lines of the constant system: $y = C$, $C \in \mathbb{R}$;
(iv) Phase portrait: Picture LV_d.11.

3.2.12. *The phase portraits associated with Config. LV_d.12.* (i) Singular line: $x^2 = 0$;
(ii) Corresponding constant system: $\dot{x} = 1,\ \dot{y} = 0$;
(iii) Invariant lines of the constant system: $y = C$, $C \in \mathbb{R}$;
(iv) Phase portrait: Picture LV_d.12.

3.2.13. *The phase portraits associated with Config. LV_d.13.* (i) Singular line: $x = 0$;
(ii) Respective linear system: $\dot{x} = 1 + x,\ \dot{y} = y$;
(iii) Invariant lines of the linear system: $y = C(x+1)$, $C \in \mathbb{R}$;
(iv) Phase portrait: Picture LV_d.13.

3.2.14. *The phase portraits associated with Config. LV_d.14.* (i) Singular line: $x = 0$;
(ii) Corresponding linear system: $\dot{x} = x,\ \dot{y} = y$;
(iii) Invariant lines of the linear system: $y = Cx$, $C \in \mathbb{R}$;
(iv) Phase portrait: Picture LV_d.14.

3.3. **Topologically distinct phase portraits of LV-systems.** To find the exact number of topologically distinct phase portraits of LV-systems, we use a number of topological invariants for distinguishing (or identifying) phase portraits. We list below the topological invariants we need and the notation we use.

I. *Singularities, invariant lines, multiplicities and indices:*

- $\mathcal{N}$ = total number of all singularities (they are all real) of the systems;
- $\binom{\mathcal{N}_f}{\mathcal{T}_m}$ = the number $\mathcal{N}_f$ of all distinct finite singularities having a total multiplicity $\mathcal{T}_m$;
- $\deg J$ = the sum of the indices of all finite singularities of the systems;
- $\mathcal{N}_{\rm AIL}^{\rm sing}$ = total number of affine invariant lines filled up with singularities;
- $\mathcal{N}_\infty$ = total number of infinite singularities;

II. *Connections of separatrices:*

- $\#SC_s^s$ = total number of connections of a finite saddle to a finite saddle;
- $\#SC_s^S$ = total number of connections of a finite saddle to an infinite saddle;
- $\#SC_s^{\mathcal{SN}}$ =total number of connections of a finite saddle to an infinite saddle-node;
- $\#SC_{sn}^s$ = total number of connections of a finite saddle-node to a finite saddle;
- $\#SC_{sn}^S$ = total number of connections of a finite saddle-node to an infinite saddle;
- $\#SC_{sn}^{\mathcal{SN}}$ = total number of connections of a finite saddle-node to an infinite saddle-node;
- $\#SC_{sn(hh)}^{\mathcal{S}}$ = total number of separatrices dividing the two hyperbolic sectors of finite saddle-nodes, going to infinite saddles;
- $\#SC_{sn(hh)}^{\mathcal{SN}}$ = total number of separatrices dividing the two hyperbolic sectors of finite saddle-nodes connecting with separatrices of infinite saddle-nodes.
- $\#Sep_{(HH)}^{\mathcal{SN}}$ = total number of separatrices of infinite saddle-nodes located in the finite plane and dividing the two hyperbolic sectors.

III. *The number of separatrices or orbits leaving from or ending at a singular point:*

- $M_{\rm sep}^{\tilde{n}} = \max\{{\rm sep}(\tilde{n})|\,\tilde{n}$ is a node$\}$, where ${\rm sep}(\tilde{n})$ is the number of separatrices leaving from or ending at a finite node $\tilde{n}$;
- $M_{\rm sep}^{\tilde{s}\tilde{n}} = \max\{{\rm sep}(\tilde{s}\tilde{n})|\,\tilde{s}\tilde{n}$ is a node$\}$, where ${\rm sep}(\tilde{s}\tilde{n})$ is the number of separatrices leaving from or ending at a finite saddle-node $\tilde{s}\tilde{n}$;
- $M_{\rm orb} = \max\{{\rm orb}(p)|\,p$ is a finite singularity$\}$, where ${\rm orb}(p)$ is the number of orbits leaving from or arriving at p;
- $M_{\rm ORB} = \max\{{\rm orb}(p_1,p_2)|\,p_1,p_2$ are infinite singularities$\}$, where ${\rm orb}(p_1,p_2)$ is the number of orbits connecting p_1 with p_2.

Using the topological invariants listed above we construct the following global topological invariant $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$, where

$$\mathcal{I}_1 = \Big(\mathcal{N}, \binom{\mathcal{N}_f}{\mathcal{T}_m}, \deg J, \mathcal{N}_{ILA}^{\rm sing}, \mathcal{N}_\infty\Big),$$
$$\mathcal{I}_2 = \Big(\#SC_s^s, \#SC_s^S, \#SC_s^{\mathcal{SN}}, \#SC_{sn}^s, \#SC_{sn}^S, \#SC_{sn}^{\mathcal{SN}}, \#SC_{sn(hh)}^{\mathcal{S}}, \#Sep_{(HH)}^{\mathcal{SN}}\Big),$$
$$\mathcal{I}_3 = \Big(M_{\rm sep}^{\tilde{n}}, M_{\rm sep}^{\tilde{s}\tilde{n}}, M_{\rm orb}, M_{\rm ORB}\Big),$$

which classifies all LV-systems. □

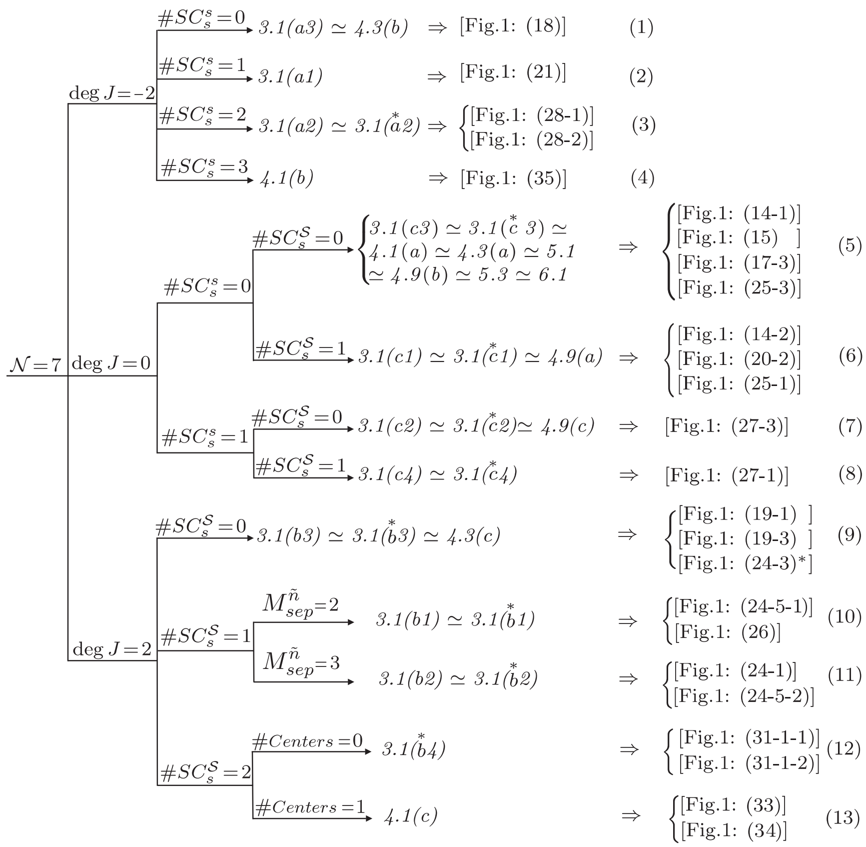


DIAGRAM 1. Global Topological Diagram: phase portraits of LV-systems with $N = 7$

Theorem 3.13. *I. The class of non-degenerate LV-systems have a total of 92 topologically distinct phase portraits. The different phase portraits are contained in the* Global Topological Diagrams *(see Diagrams 1–5) distinguished by the various components of $\mathcal{I}$. In the middle of these diagrams there appear a total of 152 phase portraits for the classes (i)-(iii) of the Main Theorem, and topological equivalences are listed. On the right side of these diagrams the distinct phase portraits are numbered from (1) to (92). Moreover for each phase portrait we indicate on its right side the corresponding phase portraits in the paper* [13]. *More precisely we have the following three cases:*

(a) to the portrait (i) there corresponds only one portrait in [13]*;*

(b) to the portrait (i) there correspond several portraits claimed to be distinct in [13]*.*

(c) to the portrait (i) with $i \in \{68, 81, 86, 87\}$ there is no corresponding phase portrait in [13]. *So from the 92 phase portraits 4 portraits are missing in* [13], *due*

to the different use of the notion of quadratic Lotka-Volterra systems in [13] *(see Observation 1.1).*

II. The class of degenerate LV-systems have a total of 20 topologically distinct phase portraits, distinguished by the topological invariant $\mathcal{I}$. They are numbered from (93) to (112) in the diagram appearing in Fig.6. For each phase portrait we indicate on its right side the corresponding phase portraits in the paper [13] *and the possibilities (a) and (b) above occur also here. Moreover we have*

(c') to the portrait (j) with $j \in \{95, 102, 103, 106\}$ there is no corresponding phase portrait in [13]*. So from the 20 phase portraits of degenerate LV-systems, 4 portraits are missing in* [13] *due to the remark in point (c) above.*

Proof. The phase portraits appearing in the Diagrams 1–6 for which the values of some of the components of the topological invariant I are different, clearly cannot be topologically equivalent. We thus only need to show that whenever for two phase portraits the corresponding values listed in these diagrams of the components of this invariant I coincide and the portraits are indicated as being equivalent, then they are indeed equivalent.

We know that in quadratic systems, inside a limit cycle we have a unique singularity which is a focus. Although a node and a focus are not distinguished by the topological equivalence relation, this distinction is important for the possible presence of limit cycles. So we wanted to keep this distinction in our diagrams. We observe that in the Diagrams 1–6 we have couples of topologically equivalent phase portraits, which are however distinguished because in one we have a focus where in the other portrait we have a node. Hence we do not need to prove the equivalences in (3), (8), (10), (11), (16), (19), (29), (32), (33), (38), (40), (54) and (62).

In the remaining cases whenever a similar repetition occurs we only need to consider one of the two topologically equivalent portraits, for example the one with a node. We then confront it with the remaining portraits listed as being topologically equivalent, and show that they are indeed equivalent.

To prove this we make use of the concept of *separatrix configuration* defined by Markus in [23], called the *completed separatrix skeleton* in [17]. Roughly speaking this is the set of all separatrices together with one orbit from each canonical region. Theorem 1.43 in [17] (Markus-Neumann-Peixoto Theorem) says that two continuous flows on the plane with only isolated singular points are topologically equivalent if and only if their completed separatrix skeletons CSS_1 and CSS_2 are equivalent, i.e. there exists a homeomorphism of the plane mapping the orbits of CSS_1 to the orbits of CSS_2. Furthermore according to [23] instead of having to prove the existence of a homeomorphism of $\mathbb{R}^2$ carrying the orbits of CSS_1 to the orbits of CSS_2, it suffices to check that there is an isomorphism of the two *chordal systems* (see [20]), which are the two completed separatrix skeletons.

To prove equivalence of two portraits, we look at their separatrices and canonical regions. After checking that we have the same number of canonical regions we match them one by one and we check that their bordering separatrices correspond. In some cases, the equivalence is obvious, for example in case (1) where the two portraits, which appear in Fig. 3 and Fig. 5 are identical. We only need to consider the remaining cases. We consider below a case for which we prove the equivalence. The other cases have been treated in an entirely analogous way.

Case (5), portraits 3.1(c3) and 4.1(a). In both portraits we have 7 canonical regions. We start by matching the two canonical regions CR_1 and CR_2 in the

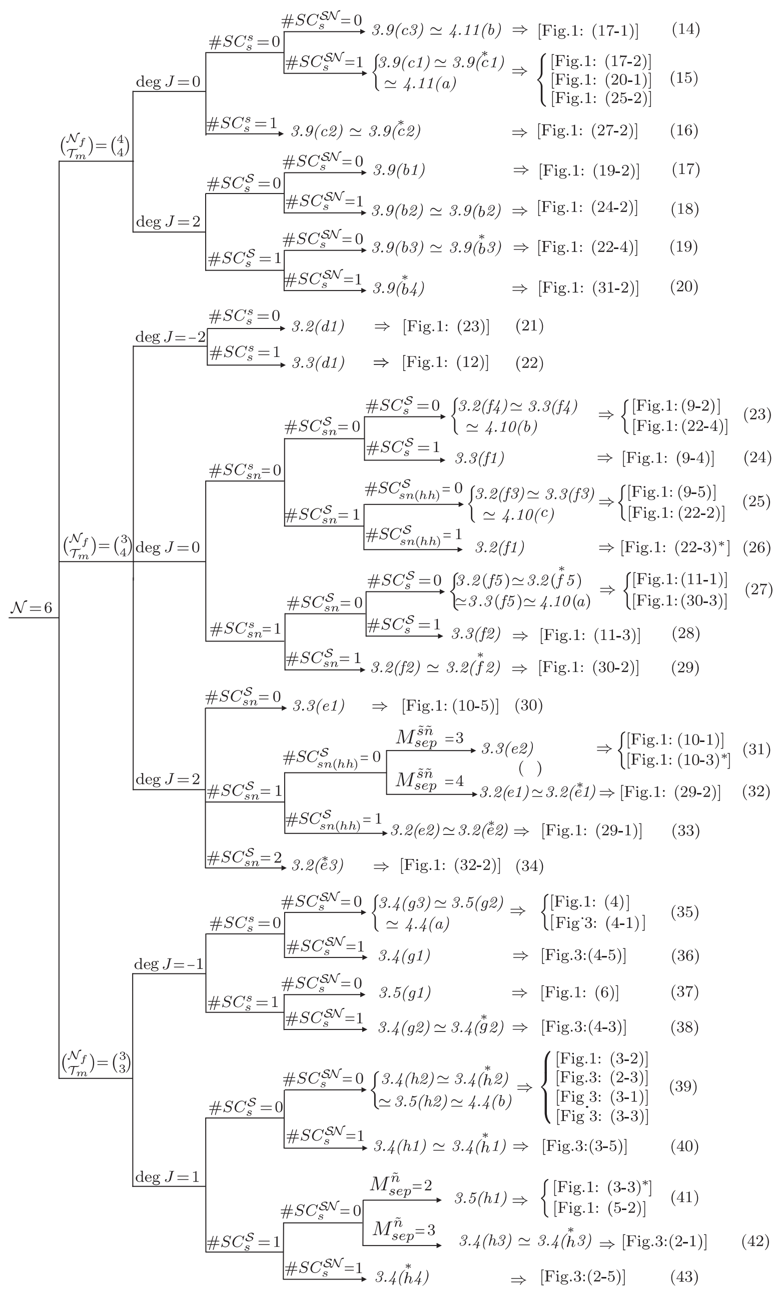


DIAGRAM 2. Global Topological Diagram: phase portraits of LV-systems with $N = 6$

two portraits determined by the separatrices of the finite saddles. These are the only canonical regions which together with their limiting separatrices are bounded in the plane and these regions have equivalent orbit representatives. These two

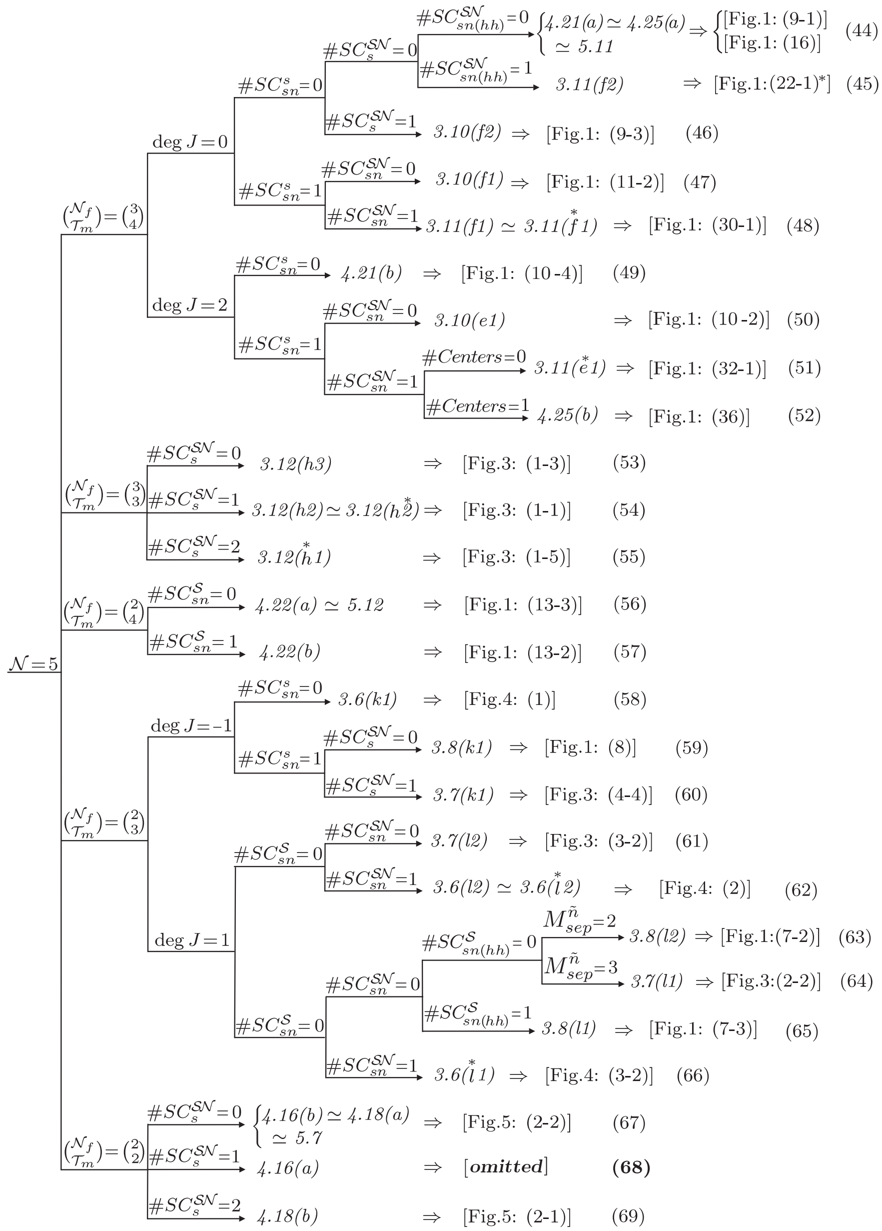


DIAGRAM 3. Global Topological Diagram: phase portraits of LV-systems with $N = 5$

canonical regions are each bordered by 4 separatrices. We next look at the four canonical regions which have a common border separatrix with CR_1, respectively CR_2 and check that they are of the same kind in both portraits which indeed occurs. Finally we look at the remaining two canonical regions whose borders have only one common point (a saddle) with the borders of CR_1, respectively CR_2. Each one of

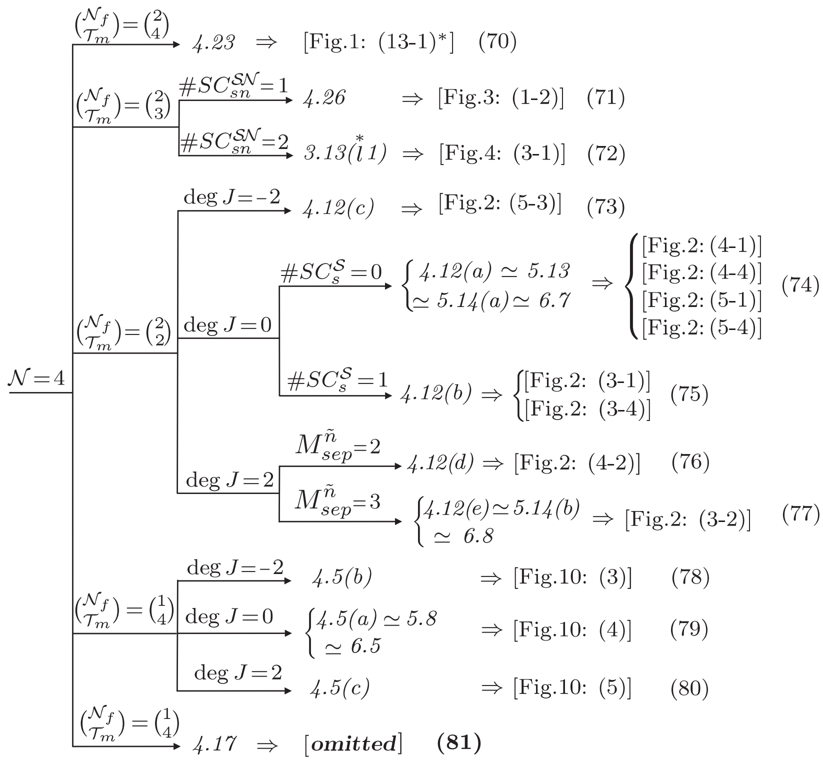


Diagram 4. Global Topological Diagram: phase portraits of LV-systems with $N = 4$

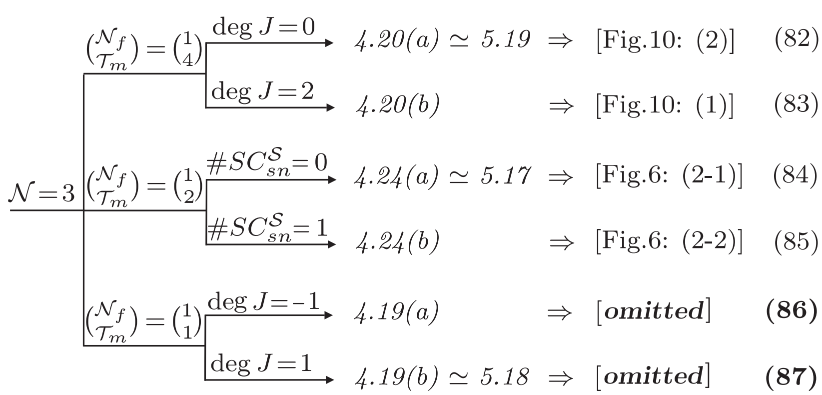


Diagram 5. Global Topological Diagram: phase portraits of LV-systems with $N = 3$

these two regions in CR_1 has a corresponding region in CR_2 and these two regions have equivalent orbit representatives. Similar arguments work for the numerous other equivalences listed in the diagrams.

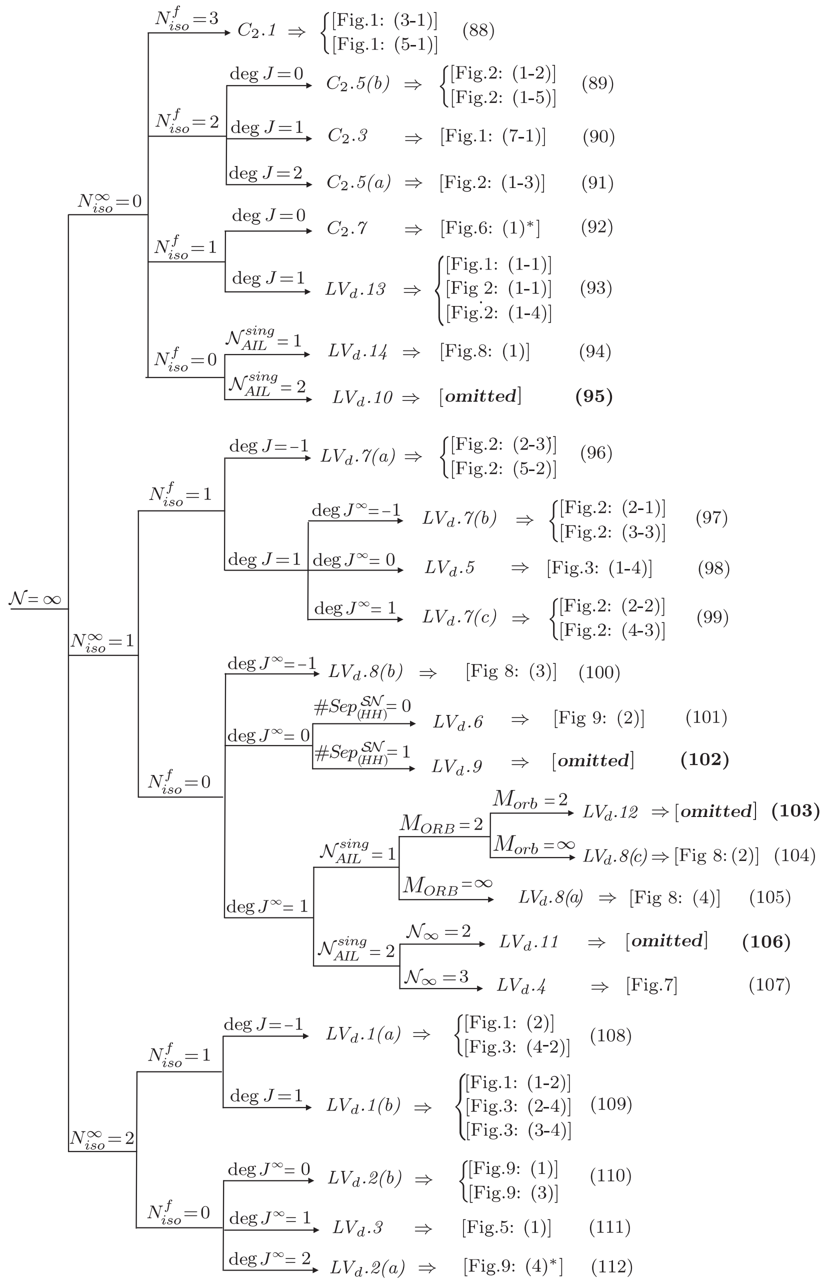


Diagram 6. Global Topological Diagram: phase portraits of LV-systems with $N = \infty$

To prove the point (b) in Theorem 3.3 we consider here two cases:

(1) The phase portraits (14-1) and (25-3) from Fig.1 of [13]. These portraits are claimed to be non-equivalent. Indeed these two portraits occur in Fig.1 which

has the caption: "The 79 non-topologically equivalent phase portraits of vector field X_1." These two phase portraits are however equivalent to the Picture 3.1(c3) (and to the Picture 4.1(a) 4.1(a)) of case (5) discussed above. To convince ourselves of this we repeat the process above starting by counting the number of canonical regions which is again 7 in both cases (14-1) and (25-3). We also have, in (14-1) (respectively in (25-3)) only one canonical region which together with its boundary separatrices is bounded in the plane and the respective orbit representatives have the same behavior. Continuing the process described above for the case (5) of Diagram 1 in this article, we see that (14-1) and (25-3) in Fig.1 of [17] are topologically equivalent and that they are both equivalent with Picture 3.1(c3) (or with Picture 4.1(a)) of Diagram 1 in this paper.

(2) In [13] the authors claimed that in Fig.1 the phase portraits (31-1-1) and (31-1-2) are topologically distinct. More exactly in Remark 6.1 on the page 818 it is mentioned: "the only difference of pictures (31-1-1) and (31-1-2) is the stability of point P" (which is a focus). As it follows from [13] these pictures correspond to systems which belong to the family X_1:

$$\dot{x} = x(1 + x + by), \quad \dot{y} = y(c + dx + y), \quad b, c, d \in \mathbb{R}, \tag{3.71}$$

when some restrictions on the parameters b, c and d are imposed.

We claim that the phase portraits (31-1-1) and (31-1-2) in Fig.1 of [13] are topologically equivalent. To prove this we consider the following two steps:

(a) we take a specific system from the family (3.71) corresponding to a point (b_0, c_0, d_0) fixed in the parameter space, which possesses the phase portrait (31-1-1); and b) we construct a respective rescaling of the variables and time which leads to a system with the phase portrait (31-1-2).

Thus we fix $(b_0, c_0, d_0) = (4, -2, 3)$ and we consider the system

$$\dot{x} = x(1 + x + 4y), \quad \dot{y} = y(-2 + 3x + y). \tag{3.72}$$

It is easy to detect (for example, using the program P4 (see [17]) that the phase portrait of this system is exactly (31-1-1). Now applying the rescaling $(x, y, t) \mapsto (-y, -x, -t)$ we get the system

$$\dot{x} = x(2 + x + 3y), \quad \dot{y} = y(-1 + 4x + y), \tag{3.73}$$

the phase portrait of which correspond to (31-1-2).

We note that a similar rescaling could be applied for the whole family (3.71) in the case of picture (31-1-1) and this leads to the systems with the phase portrait (31-1-2). Thus our claim is proved.

Similar arguments as those encountered in 1) and 2) above hold for all the remaining cases.

We point out that in [13] the authors work with the restriction that both polynomials $p(x, y)$ and $q(x, y)$ in (1.3) are of degree 2. For this reason, in [13] there are some missing portraits which we have here (recall that we ask here only for at least one of the polynomials p and q to be of degree 2). It can be easily checked that the portraits indicated in the points c and c' in the theorem are indeed missing in [13]. For the first part of II of the theorem, the proof is easy as the systems are degenerate and once we remove the common factor of $p(x, y)$ and $q(x, y)$ we have systems which are linear or they have constant right sides. □

Remark 3.14. We observe that in the *Global Topological Diagrams* on the extreme right hand side we have occasionally a star. For example in the diagram from

Diagram 1 for the portrait (9) we have [Fig.1: (24-3)*]. We use the star to indicate those cases where some mistake occurs in that phase portrait, such as for example a wrong orientation of a specific phase curve, or the presence of a phase curve which should not be there or the absence of some separatrices or some other minor error. If such a mistake is corrected, then the resulting phase portrait is equivalent to the corresponding phase portrait in the middle of the diagram.

3.4. **Concluding remarks.** We sum up in the next theorem some basic geometric global properties of the class of LV-systems.

Theorem 3.15. *Consider an LV-system (S). I. Then (S)*

(1) *has only real invariant lines;*
(2) *has only real singularities, at least two of them at infinity;*
(3) *has no finite singularities of multiplicity three;*
(4) *has a focus only if (S) has exactly three invariant lines, all simple;*
(5) *has no weak foci;*
(6) *has no limit cycles.*

II. In the generic case when the system (S) has exactly three invariant lines all simple, (S) has no centers.

Proof. I. The points (1) and (2) easily follow from the definition of LV-systems and from the normal form (1.3). The point (3) was proved in [37, p. 187] (also in [30]).

Point (4). All LV-systems with invariant lines of total multiplicity at least four do not have a focus (see Fig. 3). Similarly the phase portraits with all points at infinity singular (see Fig. 1) as well as the degenerate LV-systems (see Fig. 6) have no foci.

Points (5) and (6) were proved in Lemma 3.3 and Theorem 2.4 respectively.

II. From the Table 2 it follows that for an LV-system with exactly three invariant lines all simple, the condition $B_3 \neq 0$ holds. On the other hand by Lemma 3.3 for the existence of a center the condition $B_3 = 0$ is necessary and this contradiction completes the proof of the statement. □

Theorem 3.16. *I. Of 112 topologically distinct phase portraits of LV-systems only 18 possess graphics and all of them occur in QSL_i, $i \in \{3,4\}$. More precisely we have:*

(i) *8 distinct isolated graphics occur in systems with exactly three invariant lines, all simple. All of them are triangles with an infinite side and they surround a focus.*
(ii) *4 distinct isolated graphics occur in systems in QSL_4 all of them are triangles, one finite and three with an infinite side and they surround a center.*
(iii) *non-isolated graphics occur in 6 topological distinct phase portraits of systems in QSL_4. In each one of them we have two infinite families of graphics. These graphics are: (a) homoclinic loops with either a finite singularity or with an infinite singularity; (b) limiting triangles of families of homoclinic loops.*

II. Infinite families of degenerate graphics occur in: (a) LV-systems with all points at infinity singular, excepting the systems with the phase portrait Picture $C_2.5(a)$, and (b) degenerate LV-systems.

Proof. I. The proof of the points (i) and (ii) results from Fig. 5 and Fig. 3 respectively.

(iii) The proof results from Fig. 3. More precisely the only phase portraits in this figure, which possess non-isolated graphics are Pictures 4.5(c), 4.20(b) (these have homoclinic loops with a finite singularity), Pictures 4.12(b), 4.12(c), 4.19(a) and 4.24(b) (these have homoclinic loops with an infinite singular point).

II. The proof of this part results from Fig. 1 and Fig. 6. □

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