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INTEGRAL EQUATIONS OF FRACTIONAL ORDER WITH MULTIPLE TIME DELAYS IN BANACH SPACES

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Abstract. In this article, we give sufficient conditions for the existence of solutions for an integral equation of fractional order with multiple time delays in Banach spaces. Our main tool is a fixed point theorem of Mönch type associated with measures of noncompactness. Our results are illustrated by an example.

1. Introduction

Fractional differential and integral equations play an important role in characterizing many chemical, physical, viscoelasticity, control and engineering problems. For more details, see [6, 11, 13, 16, 19], and references therein. In consequence, the subject of fractional differential and integral equations is gaining much importance and attention; see, for instance, the monograph of Abbas *et al.* [2], Kilbas *et al.* [15], and the papers of Abbas and Benchohra [1], Agarwal *et al.* [3], Banaś and Zając [8], Benchohra and Seba [9, 10], Vityuk and Golushkov [20] and the references therein.

Ibrahim and Jalab [14] studied the existence of solutions of the fractional integral inclusion

$$u(t) - \sum_{i=1}^{m} b_i(t)u(t-\tau_i) \in I^{\alpha}F(t,u(t)), \quad t \in [0,T],$$

where $\tau_i < t \in [0,T]$, $b_i : [0,T] \to \mathbb{R}$, $i = 1,\dots,n$ are continuous functions, and $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a given multivalued map. Motivated by their work, we study the fractional integral equation

$$u(x,y) = \sum_{i=1}^{m} g_i(x,y)u(x-\xi_i, y-\mu_i) + I_{\theta}^{r} f(x,y,u(x,y)), \quad (x,y) \in J := [0,a] \times [0,b]; \tag{1.1}$$

$$u(x,y) = \Phi(x,y), \quad (x,y) \in \tilde{J} := [-\xi,a] \times [-\mu,b] \backslash (0,a] \times (0,b], \tag{1.2}$$

where $a,b > 0$, $\theta = (0,0)$, $\xi_i, \mu_i \geq 0$; $i = 1,\dots,m$, $\xi := \max_{i=1,\dots,m}\{\xi_i\}$, $\mu := \max_{i=1,\dots,m}\{\mu_i\}$, $f : J \times E \to E$ is a function satisfying some assumptions specified

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later, I_θ^r is the left-sided mixed Riemann-Liouville integral of order $r = (r_1, r_2) \in (0,\infty)\times(0,\infty)$, $g_i : J \to E$; $i = 1,\dots m$, are continuous functions, $\Phi : \tilde{J} \to E$ is a continuous function such that

$$\Phi(x,0) = \sum_{i=1}^{m} g_i(x,0)\Phi(x-\xi_i,-\mu_i), \quad x \in [0,a],$$
$$\Phi(0,y) = \sum_{i=1}^{m} g_i(0,y)\Phi(-\xi_i,y-\mu_i), \quad y \in [0,b],$$

and E is a real Banach space with norm $\|\cdot\|$.

Using properties of the Kuratowski measure of noncompactness and a fixed point theorem of Mönch type, we prove the existence of solutions to (1.1)-(1.2). Let us note here that the technique of measures of noncompactness is a very important tool for finding solutions for differential and integral equations; for more details see [4, 9, 10] and references therein.

2. Preliminaries

In this section, we collect a few auxiliary results which will be needed in the sequel. By $C(J,E)$ we denote the Banach space of continuous functions $u : J \to E$, with the norm

$$\|u\|_\infty = \sup_{(x,y)\in J} \|u(x,y)\|.$$

Let $L^1(J,E)$ be the space of Lebesgue integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x,y)\|dxdy.$$

Let $C([-\xi,a]\times[-\mu,b],E)$ be a Banach space endowed with the norm

$$\|u\|_C = \sup_{(x,y)\in[-\xi,a]\times[-\mu,b]} \|u(x,y)\|.$$

Definition 2.1 ([20])**.** Let $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$, $\theta = (0,0)$ and $u \in L^1(J,E)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds.$$

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y),\ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)dtds;$$

for almost all $(x,y) \in J$, where $\sigma = (1,1)$. For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J,E)$. Note also that when $u \in C(J,E)$, then $(I_\theta^r u) \in C(J,E)$, moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0, \quad x \in [0,a],\ y \in [0,b].$$

Now we recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.2 ([5, 7])**.** Let F be a Banach space and let Ω_F be the family of bounded subsets of F. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_F \to [0, \infty]$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } \operatorname{diam}(B_i) \leq \epsilon\}, \quad \text{here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies the following properties (For more details see [5, 7]).

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\alpha(B) = \alpha(\overline{B})$.
(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$
(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$.
(f) $\alpha(\operatorname{conv} B) = \alpha(B)$.

For our purpose we will need the following auxiliary results.

Theorem 2.3 ([17])**.** *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{\operatorname{conv}} N(V) \quad \text{or} \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.4 ([12])**.** *Let $V \subset C(J, E)$ be bounded and equicontinuous on J. Then the map $(s, t) \mapsto \alpha(V(s, t))$ is continuous on J and*

$$\alpha\Big(\int_J V(s,t)\, ds\, dt\Big) \leq \int_J \alpha(V(s,t))\, ds\, dt,$$

where $V(s, t) = \{u(s, t) : u \in V\}$.

3. Main Results

Definition 3.1. A function $u \in C(J, E)$ is said to be a solution of (1.1)-(1.2) if u satisfies equation (1.1) on J and condition (1.2).

Set

$$B = \max_{i=1,\dots m} \Big\{ \sup_{(x,y)\in J} \|g_i(x, y)\| \Big\}.$$

Let us impose two conditions for convenience.

(H1) $f : J \times E \to E$ is a continuous map.
(H2) There exists $p \in C(J, \mathbb{R}_+)$, such that

$$\|f(x, y, u)\| \leq p(x, y)\|u\|, \quad \text{for } (x, y) \in J \text{ and each } u \in E.$$

Let $p^* = \|p\|_\infty$. The main result in this paper reads as follows.

Theorem 3.2. *Assume that assumptions* (H1) *and* (H2) *hold. If*

$$mB + \frac{p^* a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1 \tag{3.1}$$

then the problem (1.1)-(1.2) *has at least one solution.*

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $N : C(J, E) \to C(J, E)$ defined by

$$N(u)(x, y) = \sum_{i=1}^{m} g_i(x, y)u(x - \xi_i, y - \mu_i) + I_\theta^r f(x, y, u(x, y)). \tag{3.2}$$

Since f is continuous, the operator N is well defined; i.e., N maps $C(J, E)$ into itself. The problem of finding the solutions of equation (1.1)-(1.2) is reduced to finding the solutions of the operator equation $N(u) = u$. Let $R > 0$ and consider the set

$$D_R = \{u \in C(J, E) : \|u\|_\infty \le R\}.$$

It is clear that D_R is a closed bounded and convex subset of $C(J, E)$. We shall show that N satisfies the assumptions of Theorem 2.3. The proof will be given in three steps. □

Step 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, E)$, then for each $(x, y) \in J$,

$$\begin{aligned} &\|N(u_n)(x, y) - N(u)(x, y)\| \\ &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1} \|f(s, t, u_n) - f(s, t, u)\|\, ds\, dt. \end{aligned}$$

Let $\rho > 0$ be such that

$$\|u_n\|_\infty \le \rho, \ \|u\|_\infty \le \rho.$$

By (H2) we have

$$(x - s)^{r_1-1}(y - t)^{r_2-1}\|f(s, t, u_n) - f(s, t, u)\| \le 2\rho p^*(x - s)^{r_1-1}(y - t)^{r_2-1}$$

which belongs to $L^1(J, \mathbb{R}_+)$. Since f is continuous, then by the Lebesgue dominated convergence theorem we have

$$\|N(u_n) - N(u)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: N maps D_R into itself. For each $u \in D_R$, by (H2) and (3.1) we have for each $(x, y) \in J$,

$$\begin{aligned} &\|N(u)(x, y)\| \\ &\le \sum_{i=1}^{m} \|g_i(x, y)\|\|u(x - \xi_i, y - \mu_i)\| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1}\|f(s, t, u(s, t))\|\, ds\, dt \\ &\le mB\|u\|_\infty + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1} p(s, t)\|u\|_\infty\, ds\, dt \\ &\le mBR + \frac{p^* R}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1}\, ds\, dt \\ &\le mBR + \frac{p^* R\, a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < R. \end{aligned}$$

Step 3: $N(D_R)$ is bounded and equicontinuous. By Step 2 we have $N(D_R) = \{N(u) : u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_\infty \le R$ which means that $N(D_R)$ is bounded.

For the equicontinuity of $N(D_R)$, let $(x_1, y_1), (x_2, y_2) \in J$, $x_1 < x_2, y_1 < y_2$, and $u \in D_R$. Then

$$
\begin{aligned}
&\|N(u)(x_2,y_2) - N(u)(x_1,y_1)\| \\
&= \Big\| \sum_{i=1}^{m} \big[g_i(x_2,y_2)u(x_2-\xi_i, y_2-\mu_i) - g_i(x_1,y_1)u(x_1-\xi_i, y_1-\mu_i)\big] \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \\
&\times f(s,t,u(s,t))\,ds\,dt \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_0^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} f(s,t,u)\,ds\,dt \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} f(s,t,u)\,ds\,dt \Big\| \\
&\le \sum_{i=1}^{m} \|g_i(x_2,y_2)u(x_2-\xi_i, y_2-\mu_i) - g_i(x_1,y_1)u(x_1-\xi_i, y_1-\mu_i)\| \\
&\quad + \frac{p^*R}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\,ds\,dt \\
&\quad + \frac{p^*R}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_0^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,ds\,dt \\
&\quad + \frac{p^*R}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}\,ds\,dt \\
&\le \sum_{i=1}^{m} \|g_i(x_2,y_2)u(x_2-\xi_i, y_2-\mu_i) - g_i(x_1,y_1)u(x_1-\xi_i, y_1-\mu_i)\| \\
&\quad + \frac{p^*R}{\Gamma(r_1+1)\Gamma(r_2+1)} [(x_2-x_1)^{r_1}(y_2-y_1)^{r_2} + {x_1}^{r_1}{y_1}^{r_2} - {x_2}^{r_1}{y_2}^{r_2}] \\
&\quad + \frac{p^*R}{\Gamma(r_1+1)\Gamma(r_2+1)} [{y_2}^{r_2}(x_2-x_1)^{r_1} - (x_2-x_1)^{r_1}(y_2-y_1)^{r_2}] \\
&\le \sum_{i=1}^{m} \|g_i(x_2,y_2)u(x_2-\xi_i, y_2-\mu_i) - g_i(x_1,y_1)u(x_1-\xi_i, y_1-\mu_i)\| \\
&\quad + \frac{p^*R}{\Gamma(r_1+1)\Gamma(r_2+1)} [{y_2}^{r_2}(x_2-x_1)^{r_1} + {x_1}^{r_1}{y_1}^{r_2} - {x_2}^{r_1}{y_2}^{r_2}].
\end{aligned}
$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_R such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $(x,y) \to v(x,y) = \alpha(V(x,y))$ is continuous on J. Using Lemma 2.4 and the properties of the measure α we have for each $(x,y) \in J$,

$$
\begin{aligned}
v(t) &\le \alpha(N(V)(x,y) \cup \{0\}) \\
&\le \alpha(N(V)((x,y)) \\
&\le mB\alpha(V(x-\xi_i, y-\mu_i)) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y p(s,t)\alpha(V(s,t))\,ds\,dt
\end{aligned}
$$

$$\begin{aligned}&\le mBv(x-\xi_i,y-\mu_i)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y p(s,t)v(s,t)\,ds\,dt\\&\le mB\|v\|_\infty+\|v\|_\infty\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y p(s,t)\,ds\,dt\\&\le \|v\|_\infty\Big(mB+\frac{p^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big).\end{aligned}$$

This implies

$$\|v\|_\infty\le\|v\|_\infty\Big(mB+\frac{p^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big).$$

By (3.1) it follows that $\|v\|_\infty=0$; that is, $v(x,y)=0$ for each $(x,y)\in J$, and then $V(x,y)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 2.3 we conclude that N has a fixed point which is a solution of problem (1.1)-(1.2). □

4. An Example

As an application, we consider the infinite system of partial hyperbolic fractional differential equations

$$\begin{aligned}u_n(x,y)=\ &\frac{x^4y}{7}u_n\big(x-\frac12,y-\frac35\big)+\frac{x^5y^2}{12}u_n\big(x-\frac23,y-\frac14\big)\\&+\frac19u_n\big(x-\frac25,y-\frac13\big)+I_\theta^r\big(\frac{1}{3e^{x+y+4}}u_n(x,y)\big),\\&(x,y)\in J:=[0,1]\times[0,1];\end{aligned}\tag{4.1}$$

$$u_n(x,y)=\Phi(x,y),\quad (x,y)\in\tilde J:=[-\frac23,1]\times[-\frac35,1]\backslash(0,1]\times(0,1],\tag{4.2}$$

where $n=1,2,\dots,n,\dots$, $r=(\frac12,\frac15)$, and $\Phi:\tilde J\to E$ is continuous with

$$\Phi(x,0)=\frac19\Phi\big(x-\frac23,-\frac35\big),\quad \Phi(0,y)=\frac19\Phi\big(-\frac23,y-\frac35\big),\quad x,y\in(0,1]\tag{4.3}$$

Let

$$E=l^1=\big\{u=(u_1,u_2,\dots,u_n,\dots):\sum_{n=1}^\infty|u_n|<\infty\big\}$$

with the norm

$$\|u\|_E=\sum_{n=1}^\infty|u_n|.$$

Set $u=(u_1,u_2,\dots,u_n,\dots)$ and $f=(f_1,f_2,\dots,f_n,\dots)$, with

$$\begin{gathered}f_n(x,y,u_n)=\frac{1}{3e^{x+y+4}}u_n,\quad (x,y)\in[0,1]\times[0,1],\\ g_1(x,y)=\frac{x^4y}{7},\quad g_2(x,y)=\frac{x^5y^2}{12},\quad g_3(x,y)=\frac19.\end{gathered}$$

Then problem (4.1)–(4.2) can be written as (1.1)–(1.2). In which case, we have

$$|f_n(x,y,u_n)|\le\frac{1}{3e^{x+y+4}}|u_n|,\quad\text{for } (x,y)\in[0,1]\times[0,1],\ \text{and } u_n\in\mathbb{R}.\tag{4.4}$$

Hence conditions (H1) and (H2) are satisfied with $p(x,y) = \frac{1}{3e^{x+y+4}}$. Condition (3.1) holds with $a = b = 1$. Indeed

$$mB + \frac{p^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{3}{7} + \frac{1}{3e^4\Gamma(r_1+1)\Gamma(r_2+1)} < 1$$

which is satisfied for each $(r_1, r_2) \in (0,1]\times(0,1]$. Consequently, Theorem 3.2 implies that (4.1)–(4.2) has a solution defined on $[-\frac{2}{3}, 1] \times [-\frac{3}{5}, 1]$.

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