Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 69, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INFINITELY MANY SOLUTIONS FOR NONLOCAL ELLIPTIC SYSTEMS OF $(p_1,\dots,p_n)$-KIRCHHOFF TYPE

SHAPOUR HEIDARKHANI, JOHNNY HENDERSON

Abstract. We establish the existence of infinitely many solutions for a class of nonlocal elliptic systems of $(p_1,\dots,p_n)$-Kirchhoff type. Our approach is based on variational methods.

1. Introduction

Let $\Omega\subset\mathbb{R}^N$ $(N\ge 1)$ be a non-empty bounded open set with a smooth boundary $\partial\Omega$, $K_i:[0,+\infty[\to\mathbb{R}$, for $1\le i\le n$, be continuous functions such that there exist positive numbers m_i and M_i, with $m_i\le K_i(t)\le M_i$, for all $t\ge 0$ and for $1\le i\le n$, $a_i\in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a_i(x)\ge 0$, and $p_i>N$, for $1\le i\le n$.

Consider the nonlocal elliptic Kirchhoff type system

$$\begin{gathered}
\begin{aligned}
&-\Big[K_i\big(\int_\Omega(|\nabla u_i(x)|^{p_i}+a_i(x)|u_i(x)|^{p_i})dx\big)\Big]^{p_i-1}\\
&\times\Big(\operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)+a_i(x)|u_i|^{p_i-2}u\Big)\\
&=\lambda F_{u_i}(x,u_1,\dots,u_n)\quad\text{in }\Omega,
\end{aligned}\\
u_i=0\quad\text{on }\partial\Omega,
\end{gathered}\tag{1.1}$$

for $1\le i\le n$, where λ is a positive parameter and $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is a function such that the mapping $(t_1,t_2,\dots,t_n)\to F(x,t_1,t_2,\dots,t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x\in\Omega$, F_{t_i} is continuous in $\Omega\times\mathbb{R}^n$, for $i=1,\dots,n$, and $F(x,0,\dots,0)=0$ for all $x\in\Omega$. Here, F_{t_i} denotes the partial derivative of F with respect to t_i.

We use Ricceri's Variational Principle [26], to ensure the existence of infinitely many weak solutions for (1.1) in $\prod_{i=1}^n W_0^{1,p_i}(\Omega)$. System (1.1) is related to a model given by the equation of elastic strings

$$\rho\frac{\partial^2u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2u}{\partial^2x}=0\tag{1.2}$$

where ρ is the mass density, P_0 is the initial tension, h is the area of the cross-section, E is the Young modulus of the material, and L is the length of the string, was proposed by Kirchhoff [21] as a extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoffs model takes into account

2000 *Mathematics Subject Classification.* 35A05, 35A15.
Key words and phrases. Infinitely many solutions; Kirchhoff type system;
critical point theory; variational methods.

Submitted April 17, 2012. Published May 2, 2012.

the changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process that depends on the average of itself, for example, the population density. Later, the equation (1.2) was extended to the equation

$$\frac{\partial^2 u}{\partial t^2} - K\Big(\int_\Omega |\nabla u(x)|^2 dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with a given $\partial\Omega$ and $K : [0,+\infty[\to \mathbb{R}$ is a continuous function. Some early classical investigations of Kirchhoff equations can be seen in the papers [1, 12, 17, 18, 19, 20, 22, 24, 25, 27, 31] and the references therein. In particular, these papers discuss the historical development of the problem as well as describe situations that can be realistically modelled by (1.1) with a nonconstant K.

For a discussion about the existence of infinitely many solutions for boundary value problems, using Ricceri's Variational Principle [26], we refer the reader to [14, 15, 16, 21, 27]. Applying a smooth version of [4, Theorem 2.1], which is a more precise version of Ricceri's Variational Principle [26], we refer the reader to [2, 3, 5, 6, 7, 8, 9, 11], and employing a non-smooth version of Ricceri's Variational Principle [26] due to Marano and Motreanu [23], we refer the reader to [10]. Here, our motivation comes from the recent article by Bonanno, et al. [6].

2. Preliminaries

First we recall the celebrated Ricceri's Variational Principle [26, Theorem 2.5] which is our primary tool in proving our main result.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r])}\Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

- (a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*
- (b) *If $\gamma < +\infty$, then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: eithre*
  - (b1) *I_λ possesses a global minimum, or*
  - (b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*
- (c) *If $\delta < +\infty$, then, for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*
  - [(c1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
  - (c2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.*

Denote by $W_0^{1,p_i}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u_i\|_{p_i} = \Big(\int_\Omega |\nabla u_i(x)|^{p_i}dx\Big)^{1/p_i} \quad \text{for } 1 \le i \le n.$$

Put

$$c_i := \max\Big\{\sup_{u_i\in W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u_i(x)|}{\|u_i\|_{p_i}},\ 1\le i\le n\Big\}.$$

Since $p_i > N$ for $1 \le i \le n$, one has $c_i < +\infty$. Moreover, from [30, formula (6b)] one has

$$\sup_{u\in W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u_i(x)|}{\|u_i\|_{p_i}} \le \frac{N^{-1/p_i}}{\sqrt{\pi}}[\Gamma(1+\frac{N}{2})]^{1/N}(\frac{p_i-1}{p_i-N})^{1-1/p_i}|\Omega|^{1/N-1/p_i}$$

for $1 \le i \le n$, where $|\Omega|$ is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball.

Let X be the Cartesian product of the n Sobolev spaces $W_0^{1,p_1}(\Omega)$,..., $W_0^{1,p_n}(\Omega)$; i.e., $X = \prod_{i=1}^n W_0^{1,p_i}(\Omega)$ equipped with the norm

$$\|(u_1,u_2,\dots,u_n)\| = \sum_{i=1}^n \|u_i\|_*$$

where

$$\|u_i\|_* = \Big(\int_\Omega (|\nabla u_i(x)|^{p_i} + a_i(x)|u_i(x)|^{p_i})dx\Big)^{1/p_i}$$

is a norm in $W_0^{1,p_i}(\Omega)$ that is equivalent to the usual norm. Put

$$C := \max\big\{\sup_{u_i\in W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}},\ 1\le i\le n\big\}. \tag{2.1}$$

Let $\underline{p} := \min\{p_i;\ 1 \le i \le n\}$, $\overline{p} := \max\{p_i;\ 1 \le i \le n\}$ and $\underline{m} := \min\{m_i;\ 1 \le i \le n\}$. Following the construction given in [6], define

$$\sigma(p_i,N) := \inf_{\mu\in]0,1[} \frac{1-\mu^N}{\mu^N(1-\mu)^{p_i}},$$

and consider $\overline{\mu}_i \in]0,1[$ such that $\sigma(p_i,N) := \frac{1-\overline{\mu}_i^N}{\overline{\mu}_i^N(1-\overline{\mu}_i)^{p_i}}$. Put

$$\overline{\mu} := \max\overline{\mu}_i, \quad \underline{\mu} := \min\overline{\mu}_i, \quad \tau := \sup \operatorname{dist}(x,\partial\Omega).$$

Simple calculations show that there is an $x_0 \in \Omega$ such that $B(x_0,\tau) \subseteq \Omega$, where $B(x_0,s)$ denotes the ball with center at x_0 and radius of s. Further, put

$$g_{\overline{\mu}_i}(p_i,N) := \overline{\mu_i}^N + \frac{1}{(1-\overline{\mu_i})^{p_i}} N B_{(\overline{\mu_i},1)}(N,p_i+1)$$

where $B_{(\overline{\mu_i},1)}(N,p_i+1)$ denotes the generalized incomplete beta function defined as follows:

$$B_{(\overline{\mu_i},1)}(N,p_i+1) := \int_{\overline{\mu_i}}^1 t^{N-1}(1-t)^{(p_i+1)-1}dt.$$

We also denote by $\omega_\tau := \tau^N \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}$ the measure of the N-dimensional ball of radius τ. Set

$$\upsilon := \max_{1\le i\le n}\big\{\frac{\sigma(p_i,N)}{\tau^{p_i}} + \|a_i\|_\infty \frac{g_{\mu_i}(p_i,N)}{\overline{\mu_i}^N}\big\}.$$

Corresponding to K_i we introduce the functions $\tilde{K}_i : [0, +\infty[\to \mathbb{R}$ as follows

$$\tilde{K}_i(t) = \int_0^t K_i(s)ds \quad \text{for all } t \geq 0$$

for $1 \leq i \leq n$. For $\gamma > 0$ we denote the set

$$Q(\gamma) = \big\{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \gamma\big\}. \tag{2.2}$$

By a (weak) solution of system (1.1), we mean $u = (u_1, \dots, u_n) \in X$ such that

$$\begin{aligned}
&\sum_{i=1}^n \Big(\Big[K_i\Big(\int_\Omega (|\nabla u_i(x)|^{p_i} + a_i(x)|u_i(x)|^{p_i})dx\Big)\Big]^{p_i-1} \\
&\times \int_\Omega \Big(|\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x) + |u_i(x)|^{p_i-2}u_i(x)v_i(x)\Big)dx\Big) \\
&- \lambda \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)dx = 0
\end{aligned}$$

for every $v = (v_1, \dots, v_n) \in X$.

3. Main results

We begin by formulating our main result under the assumptions:

(A1) $F(x, t_1, \dots, t_n) \geq 0$, for each $(x, t_1, \dots, t_n) \in \Omega \times \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(t_1, \dots, t_n) \in \mathbb{R}^n : \ t_i \geq 0, \ \text{for } i = 1, \dots, n\}$;

(A2)

$$\begin{aligned}
&\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi)} F(x, t_1, \dots, t_n)dx}{\xi^{\underline{p}}} \\
&< \frac{1}{\Big(\sum_{i=1}^n (p_i \frac{C}{\underline{m}})^{\frac{1}{p_i}}\Big)^{\underline{p}}} \limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)_{(t_1,\dots,t_n)\in\mathbb{R}^n_+}} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x, t_1, \dots, t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau v|t_i|^{p_i})}{p_i}}.
\end{aligned}$$

Theorem 3.1. *Assume* (A1)–(A2)*, and let* Λ *the interval*

$$\Big] \frac{1}{\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau v|t_i|^{p_i})}{p_i}}}, \frac{\frac{1}{\Big(\sum_{i=1}^n (p_i \frac{C}{\underline{m}})^{\frac{1}{p_i}}\Big)^{\underline{p}}}}{\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi)} F(x,t_1,\dots,t_n)dx}{\xi^{\underline{p}}}} \Big[.$$

If $\lambda \in \Lambda$*, then* (1.1) *has an unbounded sequence of weak solutions in* X*.*

Proof. To apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$, for each $u = (u_1, \dots, u_n) \in X$, defined as follows

$$\Phi(u) = \sum_{i=1}^n \frac{\tilde{K}_i(\|u_i\|_*^{p_i})}{p_i}, \quad \Psi(u) = \int_\Omega F(x, u_1(x), \dots, u_n(x))dx.$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx,$$

for every $v = (v_1, \dots, v_n) \in X$, as well as is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $(u_1, \dots, u_n) \in X$, let $(u_{1k}, \dots, u_{nk}) \to (u_1, \dots, u_n)$ weakly in X as $k \to +\infty$. Then we have $(u_{1k}, \dots, u_{nk})$ converges uniformly to $(u_1, \dots, u_n)$ on Ω as $k \to +\infty$(see [32]). Since $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$, the derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \Omega$, so for $1 \le i \le n$, $F_{u_i}(x, u_{1k}, \dots, u_{nk}) \to F_{u_i}(x, u_1, \dots, u_n)$ strongly as $k \to +\infty$, from which follows $\Psi'(u_{1k}, \dots, u_{nk}) \to \Psi'(u_1, \dots, u_n)$ strongly as $k \to +\infty$. Thus we have that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [32]. Moreover, bearing in mind the conditions $0 < m_i \le K_i(t) \le M_i$ for all $t \ge 0$ for $1 \le i \le n$, we see that Φ is continuously differentiable and whose differential at the point $u \in X$ is

$$\begin{aligned} \Phi'(u)(v) = \sum_{i=1}^n \Big(\Big[K_i\Big(\int_\Omega (|\nabla u_i(x)|^{p_i} + a_i(x)|u_i(x)|^{p_i}) dx \Big) \Big]^{p_i - 1} \\ \times \int_\Omega \Big(|\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x) + |u_i(x)|^{p_i-2} u_i(x) v_i(x) \Big) dx \Big) \end{aligned}$$

for every $v \in X$, and Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Indeed, for any $(u_{1k}, \dots, u_{nk}) \in X$ with $(u_{1k}, \dots, u_{nk}) \to (u_1, \dots, u_n)$ weakly in X, then $u_{ik} \to u_i$ in $W_0^{1,p_i}(\Omega)$ for $1 \le i \le n$. Therefore, taking the norm of weakly lower semicontinuity, we have

$$\liminf_{k\to\infty} \|u_{ik}\|_* \ge \|u_i\|_* \quad \text{for } i = 1, \dots, n.$$

Hence, since $\tilde{K}_i$ is continuous and monotone for $1 \le i \le n$, we obtain

$$\tilde{K}_i(\|u_i\|_*^{p_i}) \le \tilde{K}_i(\liminf_{k\to\infty} \|u_{ik}\|_*^{p_i}) \le \liminf_{k\to\infty} \tilde{K}_i(\|u_{ik}\|_*^{p_i})$$

for $1 \le i \le n$, from which it follows that Φ is sequentially weakly lower semicontinuous. Put $I_\lambda := \Phi - \lambda\Psi$. Clearly, the weak solutions of the system (1.1) are exactly the solutions of the equation $I'_\lambda(u_1, \dots, u_n) = 0$. Moreover, since for $1 \le i \le n$, $m_i \le K_i(s)$ for all $s \in [0, +\infty[$, from the definition of Φ, we have

$$\Phi(u) \ge \sum_{i=1}^n \frac{m_i \|u_i\|_*^{p_i}}{p_i} \ge \underline{m} \sum_{i=1}^n \frac{\|u_i\|_*^{p_i}}{p_i} \quad \forall u = (u_1, \dots, u_n) \in X. \tag{3.1}$$

Now, let us verify that $\gamma < +\infty$. Let $\{\xi_k\}$ be a real sequence such that $\xi_k \to +\infty$ as $k \to \infty$ and

$$\begin{aligned} &\lim_{k\to\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi_k)} F(x, t_1, \dots, t_n) dx}{\xi_k^{\underline{p}}} \\ &= \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi)} F(x, t_1, \dots, t_n) dx}{\xi^{\underline{p}}}. \end{aligned} \tag{3.2}$$

Put $r_k = \frac{\xi_k^p}{\left(\sum_{i=1}^n (p_i \frac{C}{\underline{m}})^{\frac{1}{p_i}}\right)^p}$ for all $k \in \mathbb{N}$. Since

$$\sup_{x\in\overline{\Omega}} |u_i(x)|^{p_i} \le C\|u_i\|_{p_i}^{p_i} \quad \text{for all } u_i \in W_0^{1,p_i}(\Omega)$$

for $1 \le i \le n$, we have

$$\sup_{x\in\overline{\Omega}} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \le C \sum_{i=1}^n \frac{\|u_i\|_*^{p_i}}{p_i} \tag{3.3}$$

for each $u = (u_1, \dots, u_n) \in X$. So, from (3.1) and (3.3) we have

$$\begin{aligned}
\Phi^{-1}(]-\infty, r_k]) &= \{u = (u_1, u_2, \dots, u_n) \in X; \Phi(u) \le r_k\} \\
&\subseteq \Big\{u \in X; \underline{m} \sum_{i=1}^n \frac{\|u_i\|_*^{p_i}}{p_i} \le r_k\Big\} \\
&\subseteq \Big\{u \in X; \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \le \frac{Cr_k}{\underline{m}} \text{ for each } x \in \Omega\Big\} \\
&\subseteq \Big\{u \in X; \sum_{i=1}^n |u_i(x)| \le \xi_k \text{ for each } x \in \Omega\Big\}.
\end{aligned}$$

Hence, taking into account that $\Phi(0, \dots, 0) = \Psi(0, \dots, 0) = 0$, we have for every k large enough,

$$\begin{aligned}
\varphi(r_k) &= \inf_{u\in\Phi^{-1}(]-\infty, r_k[)} \frac{(\sup_{v\in\Phi^{-1}(]-\infty, r_k])} \Psi(v)) - \Psi(u)}{r_k - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(]-\infty, r_k])} \Psi(v)}{r_k} \\
&\le \Big(\sum_{i=1}^n (p_i \frac{C}{\underline{m}})^{\frac{1}{p_i}}\Big)^p \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi_k)} F(x, t_1, \dots, t_n)dx}{\xi_k^p}.
\end{aligned}$$

Moreover, from Assumption (A2), we also have

$$\lim_{k\to\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi_k)} F(x, t_1, \dots, t_n)dx}{\xi_k^p} < +\infty.$$

Therefore,

$$\begin{aligned}
\gamma &\le \liminf_{k\to+\infty} \varphi(r_k) \\
&\le \Big(\sum_{i=1}^n (p_i \frac{C}{\underline{m}})^{\frac{1}{p_i}}\Big)^p \lim_{k\to\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi_k)} F(x, t_1, \dots, t_n)dx}{\xi_k^p} < +\infty.
\end{aligned} \tag{3.4}$$

Assumption (A2) in conjunction with (3.4) implies $\Lambda \subseteq]0, 1/\gamma[$. Fix $\lambda \in \Lambda$. The inequality (3.4) yields that the condition (b) of Theorem 2.1 can be applied, and either I_λ has a global minimum or there exists a sequence $\{u_k = (u_{1k}, \dots, u_{nk})\}$ of weak solutions of the system (1.1) such that $\lim_{k\to\infty} \|(u_{1k}, \dots, u_{nk})\| = +\infty$.

The other step is to show that the functional I_λ has no global minimum. For the fixed λ, let us verify that the functional I_λ is unbounded from below. Arguing as in [6], consider n positive real sequences $\{d_{i,k}\}_{i=1}^n$ such that $\sqrt{\sum_{i=1}^n d_{i,k}^2} \to +\infty$ as $k \to \infty$ and

$$\lim_{k\to+\infty} \frac{\int_\Omega F(x,d_{1,k},\dots,d_{n,k})dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i})}{p_i}} = \limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}}. \tag{3.5}$$

Let $\{w_k = (w_{1k},\dots,w_{nk})\}$ be a sequence in X defined by

$$w_{ik}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,\tau) \\ \frac{d_{i,k}}{\tau(1-\overline{\mu}_i)}(\tau - |x-x_0|) & \text{if } x \in B(x_0,\tau)\setminus B(x_0,\overline{\mu}_i\tau) \\ d_{i,k} & \text{if } x \in B(x_0,\overline{\mu}_i\tau) \end{cases} \tag{3.6}$$

for $1 \le i \le n$. For any fixed $k \in \mathbb{N}$, it is easy to see that $w_k \in X$ and, in particular, one has

$$\begin{aligned} \|w_{ik}\|_*^{p_i} &= \int_\Omega (|\nabla w_{ik}(x)|^{p_i} + a_i(x)|w_{ik}(x)|^{p_i})dx \\ &\le |d_{i,k}|^{p_i}\omega_\tau\Big[\frac{1-\overline{\mu}_i^N}{\tau^{p_i}(1-\overline{\mu}_i)^{p_i}} + \|a_i\|_\infty g_{\overline{\mu}_i}(p_i,N)\Big] \\ &\le \overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i} \end{aligned}$$

for $1 \le i \le n$. Taking into account $\inf_{t\ge0} K(t) > 0$, it follows that

$$\Phi(w_k) = \sum_{i=1}^n \frac{\tilde{K}_i(\|w_{ik}\|_*^{p_i})}{p_i} \le \sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i})}{p_i}. \tag{3.7}$$

On the other hand, bearing in mind Assumption (A1) from the definition of Ψ, we infer

$$\Psi(w_k) \ge \int_{B(x_0,\underline{\mu}\tau)} F(x,d_{1,k},\dots,d_{n,k})dx. \tag{3.8}$$

So, according to (3.7) and (3.8) we obtain

$$I_\lambda(w_k) \le \sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i})}{p_i} - \lambda\int_{B(x_0,\underline{\mu}\tau)} F(x,d_{1,k},\dots,d_{n,k})dx$$

for every $k \in \mathbb{N}$. Now, if

$$\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}} < \infty,$$

we fix $\epsilon \in \Big]1/\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}}, 1\Big[$. From (3.5) there exists ϑ_ϵ such that

$$\begin{aligned} &\int_{B(x_0,\underline{\mu}\tau)} F(x,d_{1,k},\dots,d_{n,k})dx \\ &> \epsilon\Big(\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}}\Big)\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i})}{p_i} \end{aligned}$$

for all $k > \vartheta_\epsilon$; therefore,

$I_\lambda(w_k)$

$$\leq \Big(1 - \lambda\epsilon \limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}}\Big) \sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i})}{p_i}$$

for all $k > \vartheta_\epsilon$, and by the choice of ϵ, one then has

$$\lim_{m\to+\infty}[\Phi(w_k) - \lambda\Psi(w_k)] = -\infty.$$

If

$$\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}} = \infty,$$

let us consider $M > 1/\lambda$. From (3.5) there exists ϑ_M such that

$$\int_{B(x_0,\underline{\mu}\tau)} F(x,d_{1,k},\dots,d_{n,k})dx > M\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i})}{p_i} \quad \forall\ k > \vartheta_M,$$

and therefore

$$I_\lambda(w_k) \leq (1-\lambda M)\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|d_{i,k}|^{p_i})}{p_i} \quad \forall k > \vartheta_M,$$

and by the choice of M, one then has

$$\lim_{k\to+\infty}[\Phi(w_k) - \lambda\Psi(w_k)] = -\infty.$$

Hence, our claim is proved. Since all assumptions of Theorem 2.1 are satisfied, the functional I_λ admits a sequence $\{u_k = (u_{1k},\dots,u_{nk})\} \subset X$ of critical points such that

$$\lim_{k\to\infty} \|(u_{1k},\dots,u_{nk})\| = +\infty,$$

and we have the desired conclusion. □

Remark 3.2. We point out that if $K_i(t) = 1$ for each $t \geq 0$ for $1 \leq i \leq n$, Theorem 3.1 gives [6, Theorem 3.1].

Now we want to point out the following existence result, in which instead of Assumption (A2) in Theorem 3.1 a more general condition is assumed.

(A3) there exist a sequence $\{a_k\}$ and n positive real sequence $\{b_{i,k}\}$ with

$$\frac{a_k^{\underline{p}}}{\Big(\sum_{i=1}^n (p_i\frac{C}{\underline{m}})^{\frac{1}{p_i}}\Big)^{\underline{p}}} > \sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|b_{ik}|^{p_i})}{p_i}$$

and $\lim_{k\to\infty} a_k = +\infty$ such that

$$\lim_{k\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(a_k)} F(x,t_1,\dots,t_n)dx - \int_{B(x_0,\underline{\mu}\tau)} F(x,b_{1k},\dots,b_{nk})dx}{\frac{a_k^{\underline{p}}}{\Big(\sum_{i=1}^n (p_i\frac{C}{\underline{m}})^{\frac{1}{p_i}}\Big)^{\underline{p}}} - \sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|b_{ik}|^{p_i})}{p_i}}$$

$$< \limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)_{(t_1,\dots,t_n)\in\mathbb{R}_+^n}} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}}.$$

Theorem 3.3. *Assume* (A1), (A3) *and let* Λ' *be the interval*

$$\Bigg]\frac{1}{\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)}\frac{\int_{B(x_0,\underline{\mu}\tau)}F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n\frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}}},\frac{\frac{a_k^{\underline{p}}}{\left(\sum_{i=1}^n(p_i\frac{C}{\underline{m}})^{\frac{1}{p_i}}\right)^{\underline{p}}}}{\lim_{k\to+\infty}\frac{\int_\Omega\sup_{(t_1,\dots,t_n)\in Q(a_k)}F(x,t_1,\dots,t_n)dx-\int_{B(x_0,\underline{\mu}\tau)}F(x,b_{1k},\dots,b_{nk})dx}{\frac{a_k^{\underline{p}}}{\left(\sum_{i=1}^n(p_i\frac{C}{\underline{m}})^{\frac{1}{p_i}}\right)^{\underline{p}}}-\sum_{i=1}^n\frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|b_{ik}|^{p_i})}{p_i}}}\Bigg[.$$

If $\lambda\in\Lambda'$, *then* (1.1) *has an unbounded sequence of weak solutions in* X.

Proof. Clearly, from (A3) we obtain (A2), by choosing $b_{i,k}=0$ for all $k\in\mathbb{N}$ and for $1\le i\le n$. Moreover, if we assume (A3) instead of (A2) and set

$$r_k=\frac{a_k^{\underline{p}}}{\left(\sum_{i=1}^n(p_i\frac{C}{\underline{m}})^{\frac{1}{p_i}}\right)^{\underline{p}}}$$

for all $k\in\mathbb{N}$, by the same argument as in Theorem 3.1, we obtain

$$\begin{aligned}\varphi(r_k)&=\inf_{u\in\Phi^{-1}(]-\infty,r_k[)}\frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_k])}\Psi(v))-\Psi(u)}{r_k-\Phi(u)}\\&\le\frac{\sup_{v\in\Phi^{-1}(]-\infty,r_k])}\Psi(v)-\int_a^bF(x,w_{1k}(x),\dots,w_{nk}(x))dx}{r_k-\sum_{i=1}^n\frac{\tilde{K}_i(\|w_{ik}\|_*^{p_i})}{p_i}}\\&\le\frac{\int_\Omega\sup_{(t_1,\dots,t_n)\in Q(a_k)}F(x,t_1,\dots,t_n)dx-\int_{B(x_0,\underline{\mu}\tau)}F(x,b_{1k},\dots,b_{nk})dx}{\frac{a_k^{\underline{p}}}{\left(\sum_{i=1}^n(p_i\frac{C}{\underline{m}})^{\frac{1}{p_i}}\right)^{\underline{p}}}-\sum_{i=1}^n\frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|b_{ik}|^{p_i})}{p_i}}\end{aligned}$$

where $w_k=(w_{1k},\dots,w_{nk})$, with w_{ik} for $1\le i\le n$, as given in (3.6) with $b_{i,k}$ instead of $d_{i,k}$. So, we have the desired conclusion. □

Now we point out a consequence of Theorem 3.1, under the assumptions

(B1)
$$\liminf_{\xi\to+\infty}\frac{\int_\Omega\sup_{(t_1,\dots,t_n)\in Q(\xi)}F(x,t_1,\dots,t_n)dx}{\xi^{\underline{p}}}<\frac{1}{\left(\sum_{i=1}^n(p_i\frac{C}{\underline{m}})^{\frac{1}{p_i}}\right)^{\underline{p}}};$$

(B2)
$$\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)_{(t_1,\dots,t_n)\in\mathbb{R}_+^n}}\frac{\int_{B(x_0,\underline{\mu}\tau)}F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n\frac{\tilde{K}_i(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})}{p_i}}>1.$$

Corllary 3.4. *Assume* (A1), (B1), (B2). *Then the system*

$$\begin{aligned}&-\Big[K_i(\int_\Omega(|\nabla u_i(x)|^{p_i}+a_i(x)|u_i(x)|^{p_i})dx)\Big]^{p_i-1}\\&\times\Big(\operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)+a_i(x)|u_i|^{p_i-2}u\Big)\\&=F_{u_i}(x,u_1,\dots,u_n)\quad\text{in }\Omega,\end{aligned}$$

$$u_i = 0 \quad on\ \partial\Omega,$$

for $1 \leq i \leq n$, has an unbounded sequence of weak solutions in X.

As an example, we state a special case of our main result.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^2$ be a non-empty bounded open set with a smooth boundary $\partial\Omega$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two positive $C^0(\mathbb{R}^2)$-functions such that the differential 1-form $w := f(\xi,\eta)d\xi + g(\xi,\eta)d\eta$ is integrable and let F be a primitive of w such that $F(0,0) = 0$. Fix $p, q > 2$, with $p \leq q$, and assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi,\xi)}{\xi^p} = 0, \quad \limsup_{\xi\to+\infty} \frac{F(\xi,\xi)}{\frac{\tilde{K_1}(\overline{\mu}^2\tau^2\pi\upsilon|t_1|^p)}{p} + \frac{\tilde{K_2}(\overline{\mu}^2\tau^2\pi\upsilon|t_2|^q)}{q}} = +\infty$$

where

$$\upsilon := \max\Big\{\frac{\sigma(p,2)}{\tau^p} + \|a_1\|_\infty \frac{g_{\mu_1}(p,2)}{\overline{\mu_1}^2},\ \frac{\sigma(q,2)}{\tau^q} + \|a_2\|_\infty \frac{g_{\mu_2}(q,2)}{\overline{\mu_2}^2}\Big\}.$$

Then, the system

$$\begin{gathered} -\Big[K_1\Big(\int_\Omega (|\nabla u(x)|^p + a_1(x)|u(x)|^p)dx\Big)\Big]^{p-1}\Big(\operatorname{div}(|\nabla u|^{p-2}\nabla u) + a_1(x)|u|^{p-2}u\Big) \\ = f(u,v) \quad in\ \Omega, \\ -\Big[K_2\Big(\int_\Omega (|\nabla v(x)|^q + a_2(x)|v(x)|^q)dx\Big)\Big]^{q-1}\Big(\operatorname{div}(|\nabla v|^{q-2}\nabla v) + a_2(x)|v|^{q-2}v\Big) \\ = g(u,v) \quad in\ \Omega, \\ u = v = 0 \quad on\ \partial\Omega \end{gathered}$$

admits a sequence of pairwise distinct positive weak solutions in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Proof. Take $n = 2$ and set $F(x,t_1,t_2) = F(t_1,t_2)$ for all $x \in \Omega$ and $t_1, t_2 \in \mathbb{R}$. From the conditions

$$\liminf_{\xi\to+\infty} \frac{F(\xi,\xi)}{\xi^p} = 0, \quad \limsup_{\xi\to+\infty} \frac{F(\xi,\xi)}{\frac{\tilde{K_1}(\overline{\mu}^2\tau^2\pi\upsilon|t_1|^p)}{p} + \frac{\tilde{K_2}(\overline{\mu}^2\tau^2\pi\upsilon|t_2|^q)}{q}} = +\infty,$$

we see that the assumptions (B1) and (B2), respectively, are satisfied. So, taking into account that $F_{t_1}(t_1,t_2) = f(t_1,t_2)$, $F_{t_2}(t_1,t_2) = g(t_1,t_2)$ for all $(t_1,t_2) \in \mathbb{R}^2$, and $f, g : \mathbb{R}^2 \to \mathbb{R}$ are positive, the conclusion follows from Corollary 3.4. $\square$

Remark 3.6. We observe in Theorem 3.1 we can replace $\xi \to +\infty$ with $\xi \to 0^+$, that by the same way as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.1 instead of (b), the system (1.1) has a sequence of weak solutions, which strongly converges to 0 in X.

Now, we want to point out a remarkable particular situation of Theorem 3.1, using the assumption

(C1)

$$\begin{aligned} &\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi)} F(x,t_1,\dots,t_n)dx}{\xi^{\underline{p}}} \\ &< \frac{1}{\Big(\sum_{i=1}^n (p_i\frac{C}{\underline{\alpha}})^{\frac{1}{p_i}}\Big)^{\underline{p}}} \end{aligned}$$

$$\times \limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)_{(t_1,\dots,t_n)\in\mathbb{R}^n_+}} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\alpha_i\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i}+\frac{\beta_i}{2}(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})^2}{p_i}}.$$

Corllary 3.7. *Fix $\alpha_i,\beta_i>0$ for $1\le i\le n$, and denote $\underline{\alpha}=\min\{\alpha_i;\ 1\le i\le n\}$. Suppose that Assumptions* (A1), (C1) *hold, and let λ belong to the interval*

$$\Big]\frac{1}{\limsup_{(t_1,\dots,t_n)\to(+\infty,\dots,+\infty)} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\alpha_i\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i}+\frac{\beta_i}{2}(\overline{\mu}^N\omega_\tau\upsilon|t_i|^{p_i})^2}{p_i}}},$$
$$\frac{\frac{1}{\left(\sum_{i=1}^n (p_i\frac{C}{\underline{\alpha}})^{\frac{1}{p_i}}\right)^{\underline{p}}}}{\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in Q(\xi)} F(x,t_1,\dots,t_n)dx}{\xi^{\underline{p}}}}\Big[.$$

Then the system

$$\begin{gathered}
-\Big[\alpha_i+\beta_i\int_\Omega(|\nabla u_i(x)|^{p_i}+a_i(x)|u_i(x)|^{p_i})dx\Big]^{p_i-1}\\
\times\Big(\operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)+a_i(x)|u_i|^{p_i-2}u\Big)\\
=\lambda F_{u_i}(x,u_1,\dots,u_n)\quad \text{in } \Omega,\\
u_i=0\quad \text{on } \partial\Omega
\end{gathered}$$

has an unbounded sequence of weak solutions in X.

Proof. For fixed $\alpha_i,\beta_i>0$ and $1\le i\le n$, set $K_i(t)=\alpha_i+\beta_i t$ for all $t\ge 0$. Bearing in mind that $m_i=\alpha_i$ for $1\le i\le n$, the conclusion follows immediately from Theorem 3.1. □

We illustrate our results by giving the following example whose construction is motivated by [6, Example 3.1].

Example 3.8. Let $\Omega\subset\mathbb{R}^2$ be a non-empty open set with a smooth boundary $\partial\Omega$ and consider the increasing sequence of positive real numbers given by

$$a_1=2,\quad a_{n+1}=n!(a_n)^{7/3}+2\quad \text{for } n\ge 1.$$

Define the function $F:\Omega\times\mathbb{R}^2\to\mathbb{R}$ by

$$F(x,y,t_1,t_2)=\begin{cases}(a_{n+1})^7 e^{x^2+y^2-\frac{1}{1-(t_1-a_{n+1})^2-(t_2-a_{n+1})^2}+1}\\ \quad \text{if } (x,y,t_1,t_2)\in\Omega\times\cup_{n\ge1}S((a_{n+1},a_{n+1}),1),\\ 0 \qquad \text{otherwise,}\end{cases}$$

where $S((a_{n+1},a_{n+1}),1)$ denotes the open unit ball with center at (a_{n+1},a_{n+1}). It is clear that $F:\Omega\times\mathbb{R}^2\to\mathbb{R}$ is a non-negative function such that the mapping $(t_1,t_2)\to F(x,t_1,t_2)$ is in C^1 in $\mathbb{R}^2$ for all $x\in\Omega$, F_{t_i} is continuous in $\Omega\times\mathbb{R}^2$, for $i=1,2$, and $F(x,y,0,0)=0$ for all $(x,y)\in\Omega$. Now, for every $n\in\mathbb{N}$, one has

$$\begin{aligned}
&\int_{B(x_0,\underline{\mu}\tau)}\sup_{(t_1,t_2)\in S((a_{n+1},a_{n+1}),1)} F(x,y,t_1,t_2)\,dx\,dy\\
&=\int_{B(x_0,\underline{\mu}\tau)} F(x,y,a_{n+1},a_{n+1})\,dx\,dy
\end{aligned}$$

$$= (a_{n+1})^7 \int_{B(x_0,\underline{\mu}\tau)} e^{x^2+y^2}\, dx\, dy.$$

We will denote by f and g the partial derivative of F respect to t_1 and t_2, respectively. Since

$$\lim_{n\to+\infty} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,y,a_{n+1},a_{n+1})\, dx\, dy}{\sum_{i=1}^{2} \frac{\overline{\mu}^2\tau^2\pi\upsilon a_{n+1}^3+\frac{1}{2}(\overline{\mu}^2\tau^2\pi\upsilon a_{n+1}^3)^2}{3}} = +\infty,$$

where

$$\upsilon := \max\Big\{\frac{\sigma(3,2)}{\tau^3} + \|a_1\|_\infty \frac{g_{\mu_1}(3,2)}{\overline{\mu_1}^2},\ \frac{\sigma(3,2)}{\tau^3} + \|a_2\|_\infty \frac{g_{\mu_2}(3,2)}{\overline{\mu_2}^2}\Big\},$$

we see that

$$\limsup_{(t_1,t_2)\to(+\infty,+\infty)_{(t_1,t_2)\in\mathbb{R}_+^2}} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,y,t_1,t_2)dx}{\sum_{i=1}^{2} \frac{\overline{\mu}^2\tau^2\pi\upsilon|t_i|^3+\frac{1}{2}(\overline{\mu}^2\tau^2\pi\upsilon|t_i|^3)^2}{3}} = +\infty.$$

Moreover, by choosing $\xi_n = a_{n+1} - 1$, for every $n \in \mathbb{N}$, one has

$$\int_{B(x_0,\underline{\mu}\tau)} \sup_{(t_1,t_2)\in K(\xi)} F(x,y,t_1,t_2)\, dx\, dy = (a_n)^7 \int_{B(x_0,\underline{\mu}\tau)} e^{x^2+y^2}\, dx\, dy,$$

Then

$$\lim_{n\to+\infty} \frac{\int_{B(x_0,\underline{\mu}\tau)} \sup_{(t_1,t_2)\in K(\xi)} F(x,y,t_1,t_2)\, dx\, dy}{(a_{n+1}-1)^3} = 0,$$

and so

$$\liminf_{\xi\to+\infty} \frac{\int_{B(x_0,\underline{\mu}\tau)} \sup_{(t_1,t_2)\in K(\xi)} F(x,y,t_1,t_2)\, dx\, dy}{\xi^3} = 0.$$

Therefore,

$$\begin{aligned} 0 &= \liminf_{\xi\to+\infty} \frac{\int_{B(x_0,\underline{\mu}\tau)} \sup_{(t_1,t_2)\in K(\xi)} F(x,y,t_1,t_2)\, dx\, dy}{\xi^3} \\ &< \frac{1}{24C} \limsup_{(t_1,t_2)\to(+\infty,+\infty)_{(t_1,t_2)\in\mathbb{R}_+^2}} \frac{\int_{B(x_0,\underline{\mu}\tau)} F(x,y,t_1,t_2)dx}{\sum_{i=1}^{2} \frac{\overline{\mu}^2\tau^2\pi\upsilon|t_i|^3+\frac{1}{2}(\overline{\mu}^2\tau^2\pi\upsilon|t_i|^3)^2}{3}} = +\infty. \end{aligned}$$

Hence, all the assumptions of Corollary 3.7 are satisfied, and it is applicable to the system

$$\begin{gathered} -\Big[1 + \int_\Omega (|\nabla u(x)|^3 + a_1(x)|u(x)|^3)dx\Big]^2 \Big(\operatorname{div}(|\nabla u|\nabla u) + a_1(x)|u|u\Big) \\ = \lambda f(x,y,u,v) \quad \text{in } \Omega, \\ -\Big[1 + \int_\Omega (|\nabla v(x)|^3 + a_2(x)|v(x)|^3)dx\Big]^2 \Big(\operatorname{div}(|\nabla v|\nabla v) + a_2(x)|v|v\Big) \\ = \lambda g(x,y,u,v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega \end{gathered}$$

for every $\lambda \in]0,+\infty[$.

As an application of our results, we consider the problem

$$
\begin{gathered}
-\Big[\alpha+\beta\int_\Omega(|\nabla u(x)|^p+a(x)|u(x)|^p)dx\Big]^{p-1}\Big(\operatorname{div}(|\nabla u|^{p-2}\nabla u)+a(x)|u|^{p-2}u\Big) \\
=\lambda f(x,u)\quad \text{in } \Omega, \\
u=0\quad \text{on } \partial\Omega
\end{gathered}
\tag{3.9}
$$

where $p>N$, $\lambda>0$, $\alpha,\beta>0$, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is an L^1-Caratéodory function and $a\in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a(x)\geq 0$. Put

$$
F(x,t)=\int_0^t f(x,\xi)d\xi\quad \text{for all } (x,t)\in\Omega\times\mathbb{R}.
$$

The following existence result is an immediate consequence of Theorem 3.1.

Theorem 3.9. *Assume that*

(D1) $F(x,t)\geq 0$ *for each* $(x,t)\in\Omega\times\mathbb{R}_+$*;*
(D2)

$$
\liminf_{\xi\to+\infty}\frac{\int_\Omega\sup_{|t|\leq\xi}F(x,t)dx}{\xi^p}<\frac{\alpha}{C^p}\limsup_{t\to+\infty_{t\in\mathbb{R}_+}}\frac{\int_{B(x_0,\underline{\mu}\tau)}F(x,t)dx}{\alpha\overline{\mu}^N\omega_\tau\upsilon|t|^p+\frac{\beta}{2}(\overline{\mu}^N\omega_\tau\upsilon|t|^p)^2},
$$

where

$$
C:=\sup_{u\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u(x)|}{\Big(\int_\Omega|\nabla u(x)|^pdx\Big)^{1/p}}.
$$

Then, for each λ *in the interval*

$$
\Big]\frac{\frac{1}{p}}{\limsup_{t\to+\infty\ t\in\mathbb{R}_+}\frac{\int_{B(x_0,\underline{\mu}\tau)}F(x,t)dx}{\alpha\overline{\mu}^N\omega_\tau\upsilon|t|^p+\frac{\beta}{2}(\overline{\mu}^N\omega_\tau\upsilon|t|^p)^2}},\ \frac{\frac{\alpha}{pC^p}}{\liminf_{\xi\to+\infty}\frac{\int_\Omega\sup_{|t|\leq\xi}F(x,t)dx}{\xi^p}}\Big[
$$

the problem (3.9) *has an unbounded sequence of weak solutions in* $W_0^{1,p}(\Omega)$.

Acknowledgments. The authors express their sincere gratitude to the anonymous referee for the valuable suggestions concerning improvement of the manuscript. Shapour Heidarkhani was supported by grant 90470020 from IPM.

References

[1] C. O. Alves, F. S. J. A. Corrêa, T. F. Ma; *Positive solutions for a quasilinear elliptic equations of Kirchhoff type,* Comput. Math. Appl. 49 (2005) 85-93.
[2] G. Bonanno, G. D'Aguì; *On the Neumann problem for elliptic equations involving the p-Laplacian,* J. Math. Anal. Appl. 358 (2009) 223-228.
[3] G. Bonanno, B. Di Bella; *Infinitely many solutions for a fourth-order elastic beam equation,* Nonlinear Differential Equations and Applications NoDEA, 18(3) (2011) 357-368.
[4] G. Bonanno, G. Molica Bisci; *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities,* Bound. Value Probl. 2009 (2009) 1-20.
[5] G. Bonanno, G. Molica Bisci; *Infinitely many solutions for a Dirichlet problem involving the p-Laplacian,* Proceedings of the Royal Society of Edinburgh 140A (2010) 737-752.
[6] G. Bonanno, G. Molica Bisci, D. O'Regan; *Infinitely many weak solutions for a class of quasilinear elliptic systems,* Math. Comput. Modelling 52 (2010) 152-160.
[7] G. Bonanno, G. Molica Bisci, V. Rădulescu; *Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz-Sobolev spaces,* Monatsh Math. 165 (2012) 305-318.
[8] G. Bonanno, G. Molica Bisci, V. Rădulescu; *Variational analysis for a nonlinear elliptic problem on the Sierpiński gasket,* ESAIM: Control, Optimisation and Calculus of Variations, DOI: 10.1051/cocv/2011199.

[9] G. Bonanno, G. Molica Bisci, V. Rădulescu; *Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz-Sobolev spaces,* Nonlinear Anal. (2012); doi: 10.1016 /j.na.2011.12.016.
[10] P. Candito; *Infinitely many solutions to the Neumann problem for elliptic equations involving the p-Laplacian and with discontinuous nonlinearities,* Proc. Edin. Math. Soc. 45 (2002) 397-409.
[11] P. Candito, R. Livrea; *Infinitely many solutions for a nonlinear Navier boundary value problem involving the p-biharmonic,* Studia Univ. "Babeş-Bolyai", Mathematica, Volume **LV**, Number 4, December 2010.
[12] B. Cheng, X. Wu, J. Liu; *Multiplicity of solutions for nonlocal elliptic system of (p, q)-Kirchhoff type,* Abstract and Applied Analysis 2011 (2011), doi:10.1155/2011/526026.
[13] M. Chipot, B. Lovat; *Some remarks on non local elliptic and parabolic problems,* Nonlinear Anal. 30 (1997) 4619-4627.
[14] G. Dai; *Infinitely many solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian,* Nonlinear Anal. 70 (2009) 2297-2305.
[15] X. Fan, C. Ji; *Existence of infinitely many solutions for a Neumann problem involving the p(x)-Laplacian,* J. Math. Anal. Appl. 334 (2007) 248-260.
[16] J. R. Graef, S. Heidarkhani, L. Kong; *Infinitely many solutions for systems of multi-point boundary value equations,* Topological Methods in Nonlinear Analysis, to appear.
[17] J. R. Graef, S. Heidarkhani, L. Kong; *A variational approach to a Kirchhoff-type problem involving two parameters,* Results. Math., (2012) DOI:10.1007/s00025-012-0238-x.
[18] X. He, W. Zou; *Infinitely many positive solutions for Kirchhoff-type problems,* Nonlinear Anal. 70 (2009) 1407-1414.
[19] S. Heidarkhani, G. A. Afrouzi, D. O'Regan; *Existence of three solutions for a Kirchhoff-type boundary value problem,* Electronic Journal of Differential Equations Vol. 2011 (2011) No. 91, pp. 1-11.
[20] S. Heidarkhani, Y. Tian; *Three solutions for a class of gradient Kirchhoff-type systems depending on two parameters,* Dynamic Systems and Applications 20 (2011) 551-562.
[21] A. Kristály; *Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$,* J. Differential Equations 220 (2006) 511-530.
[22] T. F. Ma; *Remarks on an elliptic equation of Kirchhoff type,* Nonlinear Anal. 63 (2005) e1957-e1977.
[23] S. A. Marano, D. Motreanu; *Infinitely many critical points of non-differentiable functions and applications to a Neumann type problem involving the p-Laplacian,* J. Differential Equations 182 (2002) 108-120.
[24] A. Mao, Z. Zhang; *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition,* Nonlinear Anal. 70 (2009) 1275-1287.
[25] K. Perera, Z. T. Zhang; *Nontrivial solutions of Kirchhoff-type problems via the Yang index,* J. Differential Equations 221 (2006) 246-255.
[26] B. Ricceri; *A general variational principle and some of its applications,* J. Comput. Appl. Math. 113 (2000) 401-410.
[27] B. Ricceri; *Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian,* Bull. London Math. Soc. 33 (3) (2001) 331-340.
[28] B. Ricceri; *On an elliptic Kirchhoff-type problem depending on two parameters,* J. Global Optimization 46 (2010) 543-549.
[29] J. Simon; *Regularitè de la solution d'une equation non lineaire dans R^N,* in: Journées d'Analyse Non Linéaire (Proc. Conf., Besanon, 1977), (P. Bénilan, J. Robert, eds.), Lecture Notes in Math., 665, pp. 205-227, Springer, Berlin-Heidelberg-New York, 1978.
[30] G. Talenti; *Some inequalities of Sobolev type on two-dimensional spheres,* in: W. Walter (Ed.), General Inequalities, Vol. 5, Internat. Ser Numer. Math. 8 (1987) 401-408. J. Math. Anal. Appl. 282 (2003) 531-552.
[31] Z. T. Zhang, K. Perera; *Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow,* J. Math. Anal. Appl. 317
[32] E. Zeidler; *Nonlinear Functional Analysis and Its Applications,* Vol. II. Berlin-Heidelberg-New York 1985.

Shapour Heidarkhani
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran
E-mail address: s.heidarkhani@razi.ac.ir

Johnny Henderson
Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA
E-mail address: Johnny_Henderson@baylor.edu