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BLOW-UP CRITERION FOR TWO-DIMENSIONAL HEAT CONVECTION EQUATIONS WITH ZERO HEAT CONDUCTIVITY

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Abstract. In this article we obtain a blow-up criterion of smooth solutions to Cauchy problem for the incompressible heat convection equations with zero heat conductivity in $\mathbb{R}^2$. Our proof is based on careful Höder estimates of heat and transport equations and the standard Littlewood-Paley theory.

1. Introduction

The incompressible heat convection equations in two space dimensions take the form

$$\begin{gathered}\partial_t u + u\cdot\nabla u + \nabla\pi = \mu\Delta u + \theta e_2,\\ \partial_t\theta + u\cdot\nabla\theta - \nu\Delta\theta = \frac{\mu}{2}\sum_{i,j=1}^{2}(\partial_i u^j + \partial_j u^i)^2,\\ \nabla\cdot u = 0,\end{gathered}\tag{1.1}$$

where $u = (u^1, u^2)^t$ is the fluid velocity, π is the pressure, θ stands for the absolute temperature, μ is the coefficient of viscosity, ν is the coefficient of heat conductivity and $e_2 = (0, 1)$.

Some problems related to (1.1) have been studied in recent years (see [22], [8], [17]-[20] and [25]). Fan and Ozawa [8] obtained some regularity criteria of strong solutions to the Cauchy problem for the (1.1) in $\mathbb{R}^3$. Hiroshi [17] proved the existence of the strong solutions for the initial boundary value problems for (1.1). Kagei and Skowron [18] discussed the existence and uniqueness of solutions of the initial-boundary value problem for the heat convection equations (1.1) of incompressible asymmetric fluids in $\mathbb{R}^3$. Moreover, Kagei [19] considered global attractors for the initial-boundary value problem for (1.1) in $\mathbb{R}^2$. Lukaszewicz and Krzyzanowski [25] treated the initial-boundary value problem for (1.1) with moving boundaries in $\mathbb{R}^3$. Kakizawa [20] proved that (1.1) has uniquely a mild solution. Moreover, a mild solution of (1.1) can be a strong or classical solution under appropriate assumptions for initial data.

It is well known that the Boussinesq approximation [3] is a simplified model of heat convection of incompressible viscous fluids. There is no doubt that many

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investigations on the Boussinesq approximation have been carried out for a hundred years. For regularity criteria of weak solutions and blow up criteria of smooth solutions, we refer to [9] and so on.

Equation (1.1) is the Navier-Stokes equations coupled with the heat equation. Due to its importance in mathematics and physics, there is lots of literature devoted to the mathematical theory of the Navier-Stokes equations. Leray-Hopf weak solution were constructed by Leray [23] and Hopf [16], respectively. Later on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. Different criteria for regularity of the weak solutions have been proposed and many interesting results were established (see [7], [8]-[11], [12], [14], [29] and [33]-[34]). Serrin-type regularity criteria of Leray weak solutions in terms of pressure in Besov space were obtained in [13] and [15].

In this paper, we consider (1.1) with the zero heat conductivity; i.e., $\nu = 0$. Without loss of generality, we take $\mu = 1$. The corresponding heat convection equations thus reads

$$\begin{gathered} \partial_t u + u \cdot \nabla u + \nabla \pi = \Delta u + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta = \frac{1}{2} \sum_{i,j=1}^{2} (\partial_i u^j + \partial_j u^i)^2, \\ \nabla \cdot u = 0. \end{gathered} \tag{1.2}$$

Due to the term $\frac{1}{2} \sum_{i,j=1}^{2} (\partial_i u^j + \partial_j u^i)^2$, it is very difficult to deal with (1.2). The local well-posedness of the Cauchy problem for (1.2) is rather standard, which can be obtained by standard Galerkin method and energy estimates (for example see [8]). In the absence of global well-posedness, the development of blow-up/ non blow-up theory (see [1]) is of major importance for both theoretical and pratical purposes. In this paper, we obtain a blow-up criterion of smooth solutions to the Cauchy problem for (1.2). Our main theorem is as follows.

Theorem 1.1. *Assume that (u, θ) is a local smooth solution to the heat convection equations with zero heat conductivity* (1.2) *on* $[0, T)$ *and* $\|u(0)\|_{H^1 \cap \dot{C}^{1+\alpha}} + \|\theta(0)\|_{L^2 \cap \dot{C}^{\alpha}} < \infty$ *for some* $\alpha \in (0, 1)$. *Then*

$$\|u(t)\|_{\dot{C}^{1+\alpha}} + \|\theta(t)\|_{\dot{C}^{\alpha}} < \infty$$

for all $0 \le t \le T$ *provided that*

$$\|u\|_{L^2_T(\dot{B}^0_{\infty,\infty})} < \infty, \quad \|\theta\|_{L^1_T(\dot{B}^0_{\infty,\infty})} < \infty. \tag{1.3}$$

This article is organized as follows. We first state some preliminary on functional settings and some important inequalities in Section 2 and then prove the blow-up criterion of smooth solutions of (1.2) in Section 3.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^2)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x) dx$$

and for any given $g \in \mathcal{S}(\mathbb{R}^2)$, its inverse Fourier transform $\mathcal{F}^{-1}g = \check{g}$ is defined by

$$\check{g}(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} g(\xi) d\xi.$$

Next let us recall the Littlewood-Paley decomposition. Choose two non-negative radial functions $\chi, \phi \in \mathcal{S}(\mathbb{R}^2)$, supported respectively in $\mathbb{B} = \{\xi \in \mathbb{R}^2 : |\xi| \le \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^2 : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{k\ge 0} \phi(2^{-k}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2$$

and

$$\sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2 \backslash \{0\}.$$

The frequency localization operator is defined by

$$\Delta_k f = \int_{\mathbb{R}^2} \check{\phi}(y) f(x - 2^{-k}y) dy, \quad S_k f = \sum_{k' \le k-1} \Delta_{k'} f.$$

Let us now recall homogeneous Besov spaces (for example, see [2] and [30]). For $(p,q) \in [1,\infty]^2$ and $s \in \mathbb{R}$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined as the set of f up to polynomials such that

$$\|f\|_{\dot{B}^s_{p,q}} = \|2^{ks}\|\Delta_k f\|_{L^p}\|_{l^q(\mathbb{Z})} < \infty.$$

Finally, we recall the following space, which is defined in [6]. Let p be in $[1,\infty]$ and $r \in \mathbb{R}$; the space $\tilde{L}^p_T(C^r)$ is the space of the distributions f such that

$$\|f\|_{\tilde{L}^p_T(C^r)} = \sup_k 2^{kr} \|\Delta_k f\|_{L^p_T(L^\infty)} < \infty.$$

The open ball with radius R centered at $x_0 \in \mathbb{R}^2$ is denoted by $\mathbf{B}(x_0, R)$. The ring $\{\xi \in \mathbb{R}^2 | R_1 \le |\xi| \le R_2\}$ is denoted by $\mathbf{C}(0, R_1, R_2)$.

In what follows, we shall use Bernstein inequalities, which can be found in [4].

Lemma 2.1. *Let k a positive integer and σ any smooth homogeneous function of degree $m \in \mathbb{R}$. A constant C exists such that, for any positive real number λ and any function f in $L^p(\mathbb{R}^2)$, we have*

$$\operatorname{supp} \hat{f} \subset \lambda \boldsymbol{B} \Rightarrow \sup_{|\beta|=k} \|\partial^\beta f\|_{L^q} \le C\lambda^{k+2(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \tag{2.1}$$

$$\operatorname{supp} \hat{f} \subset \lambda \boldsymbol{C} \Rightarrow C^{-1}\lambda^k \|f\|_{L^p} \le \sup_{|\beta|=k} \|\partial^\beta f\|_{L^p} \le C\lambda^k \|f\|_{L^p}. \tag{2.2}$$

Moreover, if σ is a smooth function on $\mathbb{R}^2$ which is homogeneous of degree m outside a fixed ball, then we have

$$\operatorname{supp} \hat{f} \subset \lambda \boldsymbol{C} \Rightarrow \|\sigma(D) f\|_{L^q} \le C\lambda^{(m+2(\frac{1}{p}-\frac{1}{q}))} \|f\|_{L^p}. \tag{2.3}$$

Lemma 2.2. *For any $f \in L^p(\mathbb{R}^2)(p > 1)$ and any positive real number λ,*

$$\operatorname{supp} \hat{f} \subset \lambda \boldsymbol{C} \Rightarrow \|e^{t\Delta} f\|_{L^p} \le Ce^{-c\lambda^2 t} \|f\|_{L^p}, \tag{2.4}$$

where C and c are positive constants. See [5] *for the proof of* (2.4).

The following lemma plays an important role in the proof of Theorem 1.1 (see also [27] and [28] where similar estimate were established).

Lemma 2.3. *Assume that $\gamma > 0$, then there exists a positive constant $C > 0$ such that*

$$\|f\|_{L^\infty} \le C\left(1 + \|f\|_{L^2} + \|f\|_{\dot{B}^0_{\infty,\infty}} \ln(e + \|f\|_{\dot{C}^\gamma})\right) \tag{2.5}$$

and

$$\begin{aligned}\int_0^T \|\nabla f(\tau)\|_{L^\infty} d\tau \le C\Big(1 + \int_0^T \|f(\tau)\|_{L^2} d\tau + \sup_k \int_0^T \|\Delta_k \nabla f(\tau)\|_{L^\infty} d\tau \\ \times \ln\big(e + \int_0^T \|\nabla f(\tau)\|_{\dot{C}^\gamma} d\tau\big)\Big).\end{aligned} \tag{2.6}$$

Proof. If $f \in W^{m,p}$, $m > \frac{2}{p}$, C^γ in (2.5) is replaced by $W^{m,p}$, then (2.5) still holds. For example, see [1, 21]. It is not difficult to prove (2.5) (see [31]). For the reader convenience, we give a detail proof. It follows from Littlewood-Paley composition that

$$f = \sum_{k=-\infty}^{0} \Delta_k f + \sum_{k=1}^{A} \Delta_k f + \sum_{k=A+1}^{\infty} \Delta_k f. \tag{2.7}$$

Using (2.7) and (2.3), we obtain

$$\begin{aligned}\|f\|_{L^\infty} &\le \sum_{k=-\infty}^{0} \|\Delta_k f\|_{L^\infty} + A \max_{1\le k\le A} \|\Delta_k f\|_{L^\infty} + \sum_{k=A+1}^{\infty} \|\Delta_k f\|_{L^\infty} \\ &\le C \sum_{k=-\infty}^{0} 2^k \|\Delta_k f\|_{L^2} + A\|f\|_{\dot{B}^0_{\infty,\infty}} + \sum_{k=A+1}^{\infty} 2^{-\gamma k} 2^{\gamma k} \|\Delta_k f\|_{L^\infty} \\ &\le C\|f\|_{L^2} + A\|f\|_{\dot{B}^0_{\infty,\infty}} + \sum_{k=A+1}^{\infty} 2^{-\gamma k} \|f\|_{\dot{C}^\gamma} \\ &\le C\|f\|_{L^2} + A\|f\|_{\dot{B}^0_{\infty,\infty}} + 2^{-\gamma A} \|f\|_{\dot{C}^\gamma}.\end{aligned}$$

Equation (2.5) follows immediately by choosing

$$A = \frac{1}{\gamma} \log_2(e + \|f\|_{\dot{C}^\gamma}) \le C \ln(e + \|f\|_{\dot{C}^\gamma}).$$

Similar to the proof of (2.5), we can obtain (2.6) (see also [24]). Thus the proof is complete. □

To prove Theorem 1.1, we need the following interpolation inequalities in two space dimensions.

Lemma 2.4. *The following inequalities hold*

$$\|f\|_{L^p} \le C\|f\|_{L^q}^{1-\frac{2}{q}+\frac{2}{p}} \|\nabla f\|_{L^q}^{\frac{2}{q}-\frac{2}{p}}, \quad -\frac{2}{p} \le 1 - \frac{2}{q}, \quad p \ge q. \tag{2.8}$$

Proof. Noting $-\frac{2}{p} \le 1 - \frac{2}{q}$, $p \ge q$ and using the Sobolev embedding theorem, we obtain

$$\|f\|_{L^p} \le C(\|f\|_{L^q} + \|\nabla f\|_{L^q}). \tag{2.9}$$

Let $f_\lambda(x) = f(\lambda x)$. From (2.9), we obtain

$$\|f_\lambda\|_{L^p} \le C(\|f_\lambda\|_{L^q} + \|\nabla f_\lambda\|_{L^q}),$$

which implies

$$\|f\|_{L^p} \le C(\lambda^{\frac{2}{p}-\frac{2}{q}} \|f\|_{L^q} + \lambda^{1+\frac{2}{p}-\frac{2}{q}} \|\nabla f\|_{L^q}). \tag{2.10}$$

Taking $\lambda = \|f\|_{L^q} \|\nabla f\|_{L^q}^{-1}$, from (2.10), we immediately obtain (2.8). Thus, the proof is complete. □

3. Proof of main results

This section is devoted to the proof of Theorem 1.1, for which we need the following Lemma that is basically established in [8]. For completeness, the proof is also sketched here.

Lemma 3.1. *Assume* $\|u(0)\|_{H^1} + \|\theta(0)\|_{L^2} < \infty$ *and assume furthermore that* (u, θ) *is a smooth solution to the Cauchy problem for* (1.2) *on* $\times[0, T)$*. If*

$$u \in L^2 \left(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^2)\right), \tag{3.1}$$

then

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + \int_0^T (\|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2) dt \\ &\leq C(\|u(0)\|_{H^1}^2 + \|\theta(0)\|_{L^2}^2). \end{aligned} \tag{3.2}$$

Proof. Multiplying the first equation in (1.2) by u and using Cauchy inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \leq \frac{1}{2} \int_{\mathbb{R}^2} (|\theta|^2 + |u|^2)(x, t) dx. \tag{3.3}$$

Multiplying the first equation in (1.2) by $-\Delta u$, using integration by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2} \theta e_2 \cdot \Delta u dx + \int_{\mathbb{R}^2} u \cdot \nabla u \cdot \Delta u dx. \tag{3.4}$$

Note that (see [32])

$$-\Delta u = \nabla \times (\nabla \times u), \quad \nabla \times (u \cdot \nabla u) = u \cdot \nabla(\nabla \times u)$$

provided that $\nabla \cdot u = 0$.

Using integration by parts, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} u \cdot \nabla u \cdot \Delta u dx &= -\int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \nabla \times (\nabla \times u) dx \\ &= -\int_{\mathbb{R}^2} \nabla \times (u \cdot \nabla u) \cdot \nabla \times u dx \\ &= -\int_{\mathbb{R}^2} u \cdot \nabla(\nabla \times u) \cdot (\nabla \times u) dx = 0. \end{aligned} \tag{3.5}$$

It follows from (3.4), (3.5) and Young inequality that

$$\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2} \|\Delta u(t)\|_{L^2}^2 \leq C \|\theta(t)\|_{L^2}^2. \tag{3.6}$$

Multiplying the second equation in (1.2) by θ, using Hölder inequality and Young inequality, it holds that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2 &= -\frac{1}{2}\sum_{i,j=1}^{2}\int_{\mathbb{R}^2}\theta(\partial_i u_j+\partial_j u_i)^2 dx \\ &\le C\|\theta(t)\|_{L^2}\|\nabla u(t)\|_{L^4}^2 \\ &\le C\|\theta(t)\|_{L^2}\|u(t)\|_{\dot{B}^0_{\infty,\infty}}\|\Delta u(t)\|_{L^2} \\ &\le \frac{1}{6}\|\Delta u(t)\|_{L^2}^2 + C\|u(t)\|_{\dot{B}^0_{\infty,\infty}}^2\|\theta(t)\|_{L^2}^2, \end{aligned} \tag{3.7}$$

where we have used the interpolation inequality (see for example [26])

$$\|\nabla u(t)\|_{L^4} \le C\|u(t)\|_{\dot{B}^0_{\infty,\infty}}^{1/2}\|\Delta u(t)\|_{L^2}^{1/2}. \tag{3.8}$$

Collecting (3.3), (3.6) and (3.7) gives

$$\begin{aligned} &\frac{d}{dt}(\|u(t)\|_{L^2}^2+\|\nabla u(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2)+\|\nabla u(t)\|_{L^2}^2+\|\Delta u(t)\|_{L^2}^2 \\ &\le C\Big(\|u(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2+\|u(t)\|_{\dot{B}^0_{\infty,\infty}}^2(\|\nabla u(t)\|_{L^2}^2+\|\theta(t)\|_{L^2}^2)\Big). \end{aligned} \tag{3.9}$$

Inequality (3.2) follows immediately from (3.1), (3.9) and Gronwall's inequality. Thus, the proof complete. □

We also need the following lemma (see also [6, 24] where similar estimates were established).

Lemma 3.2. *Assume that $F \in \tilde{L}^1_T(C^{-1}) \cap L^2_T(L^2)$ and $u_0 \in L^2$. Let u be a solution of the Navier-Stokes equations*

$$\begin{gathered} \partial_t u + u\cdot\nabla u + \nabla\pi = \Delta u + F, \\ \nabla\cdot u = 0, \\ t=0: \quad u = u_0(x). \end{gathered} \tag{3.10}$$

Then it holds that

$$\begin{aligned} \|u\|_{\tilde{L}^1_T(C^1)} \le C(&\sup_k \|\Delta_k u_0\|_{L^2}(1-\exp\{-c2^{2k}T\}) + (\|u_0\|_{L^2} \\ &+ \|F\|_{L^2_T(L^2)})\|\nabla u\|_{L^2_T(L^2)}^2 + \sup_k \int_0^T 2^{-k}\|\Delta_k F(\tau)\|_{L^\infty}d\tau). \end{aligned} \tag{3.11}$$

Proof. Applying Δ_k to (3.10), we obtain

$$\Delta_k u = e^{\Delta t}\Delta_k u_0 + \int_0^t e^{\Delta(t-\tau)}\Delta_k\mathbb{P}\big(\nabla\cdot(u\otimes u)+F\big)(\tau)d\tau, \tag{3.12}$$

where operator $\mathbb{P}$ satisfies

$$(\hat{\mathbb{P}u})^i = \sum_{j=1}^{2}(\delta_{ij}-\frac{\xi^i\xi^j}{|\xi|^2})\hat{u}^j(\xi).$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}\|\Delta_k u(t)\|_{L^\infty} &\\ \le C\Big(e^{-c2^{2k}t}\|\Delta_k u_0\|_{L^\infty} &+ \int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta_k\nabla\cdot(u\otimes u)(\tau)\|_{L^\infty}d\tau\Big) \\ &+ C\int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta_k F(\tau)\|_{L^\infty}d\tau.\end{aligned} \tag{3.13}$$

This implies that

$$\begin{aligned}&\|u\|_{\tilde L^1_T(C^1)} \\ &\le C\sup_k\int_0^T 2^k e^{-c2^{2k}t}\|\Delta_k u_0\|_{L^\infty}dt \\ &\quad + C\sup_k\int_0^T\int_0^t 2^{2k}e^{-c2^{2k}(t-\tau)}\|\Delta_k u\otimes u(\tau)\|_{L^\infty}d\tau dt \\ &\quad + C\sup_k\int_0^T\int_0^t 2^{k}e^{-c2^{2k}(t-\tau)}\|\Delta_k F(\tau)\|_{L^\infty}d\tau dt \\ &\le C\sup_k\|\Delta_k u_0\|_{L^2}(1-e^{-c2^{2k}T}) \\ &\quad + C\sup_k\int_0^T\|\Delta_k(u\otimes u)(\tau)\|_{L^\infty}d\tau + \sup_k\int_0^T 2^{-k}\|\Delta_k F(\tau)\|_{L^\infty}d\tau.\end{aligned} \tag{3.14}$$

It follows from Bony decomposition that

$$\begin{aligned}&\|\Delta_k(u\otimes u)(\tau)\|_{L^\infty} \\ &= \sum_{|m-n|\le 1}\|\Delta_k(\Delta_m u\otimes\Delta_n u)(\tau)\|_{L^\infty} + \sum_{m-n\ge 2}\|\Delta_k(\Delta_m u\otimes\Delta_n u)(\tau)\|_{L^\infty} \\ &\quad + \sum_{n-m\ge 2}\|\Delta_k(\Delta_m u\otimes\Delta_n u)(\tau)\|_{L^\infty}\end{aligned}$$

By (2.1) and (2.2), a straight computation gives

$$\begin{aligned}&\int_0^T\sum_{|m-n|\le 1}\|\Delta_k(\Delta_m u\otimes\Delta_n u)(\tau)\|_{L^\infty}d\tau \\ &\le C\int_0^T\sum_{|m-n|\le 1}2^k\|\Delta_k(\Delta_m u\otimes\Delta_n u)(\tau)\|_{L^2}d\tau \\ &\le C\int_0^t\sum_{|m-n|\le 1, m\ge k-3}2^{k-\frac{m+n}{2}}\|2^m\Delta_m u(\tau)\|_{L^\infty}^{1/2}\|\Delta_n u(\tau)\|_{L^2}^{1/2}\|\Delta_m u(\tau)\|_{L^\infty}^{1/2} \\ &\quad\times\|2^n\Delta_n u(\tau)\|_{L^2}^{1/2}d\tau \\ &\le C\int_0^t\sum_{|m-n|\le 1, m\ge k-3}2^{k-\frac{m+n}{2}}\|2^m\Delta_m u(\tau)\|_{L^\infty}^{1/2}\|\Delta_n u(\tau)\|_{L^2}^{1/2}\|2^m\Delta_m u(\tau)\|_{L^2}^{1/2} \\ &\quad\times\|2^n\Delta_n u(\tau)\|_{L^2}^{1/2}d\tau \\ &\le C\|u\|_{L^\infty_T(L^2)}^{1/2}\|\nabla u\|_{L^2_T(L^2)}\|u\|_{\tilde L^1_T(C^1)}^{1/2}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^T \Big(\sum_{m-n\geq 2} \|\Delta_k(\Delta_m u \otimes \Delta_n u)(\tau)\|_{L^\infty} + \sum_{n-m\geq 2} \|\Delta_k(\Delta_m u \otimes \Delta_n u)(\tau)\|_{L^\infty} \Big) d\tau \\
&\leq C \int_0^T \sum_{m-n\geq 2, |m-k|\leq 2} \|\Delta_m u(\tau)\|_{L^\infty} \|\Delta_n u(\tau)\|_{L^\infty} d\tau \\
&\leq C \sum_{m-n\geq 2, |m-k|\leq 2} \|2^m \Delta_m u(\tau)\|_{L^\infty}^{1/2} \|2^m \Delta_m u(\tau)\|_{L^2}^{1/2} 2^{n-\frac{m}{2}} \|\Delta_n u(\tau)\|_{L^2} d\tau \\
&\leq C \|u\|_{L_T^\infty(L^2)}^{1/2} \|\nabla u\|_{L_T^2(L^2)} \|u\|_{\tilde{L}_T^1(C^1)}^{1/2}.
\end{aligned}$$

Using the above two estimates, from (3.14) and Young inequality, we obtain

$$\begin{aligned}
\|u\|_{\tilde{L}_T^1(C^1)} \leq C(&\sup_k \|\Delta_k u_0\|_{L^2}(1-\exp\{-c2^{2k}T\}) + \|u\|_{L_T^\infty(L^2)} \|\nabla u\|_{L_T^2(L^2)}^2 \\
&+ \sup_k \int_0^T 2^{-k} \|\Delta_k F(\tau)\|_{L^\infty} d\tau).
\end{aligned} \tag{3.15}$$

Combining (3.15) and the basic energy estimate

$$\|u\|_{L_T^\infty(L^2)}^2 + \|\nabla u\|_{L_T^2(L^2)}^2 \leq C(\|u_0\|_{L^2}^2 + \|F\|_{L_T^2(L^2)}^2) \tag{3.16}$$

gives (3.11). Thus, the proof is complete. □

Proof of Theorem 1.1. Set $F = u \cdot \nabla u + \theta e_2$. It follows from (1.3) and (3.2) that $F \in \tilde{L}_T^1(C^{-1}) \cap L_T^2(L^2)$. Applying Δ_k to both sides of (3.10) and using standard energy estimate, (2.2) and Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta_k u\|_{L^2}^2 + c2^{2k}\|\Delta_k u\|_{L^2}^2 \\
&\leq \frac{c}{2} 2^{2k} \|\Delta_k u\|_{L^2}^2 + C(\|\Delta_k u\|_{L^2} + \|\Delta_k F\|_{L^2}^2 + \|\Delta_k(u\otimes u)\|_{L^2}^2).
\end{aligned}$$

Integrating the above inequality with respect to t and summing over k, we obtain

$$\begin{aligned}
&\sum_k \|\Delta_k u\|_{L_T^\infty(L^2)}^2 + \sum_k \int_0^t 2^{2k} \|\Delta_k u(\tau)\|_{L^2}^2 d\tau \\
&\leq C(\|u_0\|_{L^2}^2 + \|F\|_{L_T^2(L^2)}^2 + \|u\|_{L_T^\infty(L^2)}^2 \|\nabla u\|_{L_T^2(L^2)}^2),
\end{aligned} \tag{3.17}$$

where we used the interpolation inequality (see Lemma 2.4)

$$\|u\|_{L^4} \leq C\|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2}.$$

It follows from (3.16) and (3.17) that

$$\begin{aligned}
&\sum_k \|\Delta_k u\|_{L_T^\infty(L^2)}^2 + \sum_k \int_0^t 2^{2k} \|\Delta_k u(\tau)\|_{L^2}^2 d\tau \\
&\leq C(\|u_0\|_{L^2}^2 + \|F\|_{L_T^2(L^2)}^2)(1 + \|u_0\|_{L^2}^2 + \|F\|_{L_T^2(L^2)}^2).
\end{aligned} \tag{3.18}$$

Using (3.18), for any $t_0 \in [0, T)$, we can choose $k_0 > 0$ such that

$$\sup_{k\geq k_0} \|\Delta_k u\|_{L_{[t_0,T]}^\infty(L^2)} \leq \frac{\varepsilon}{4C}.$$

By (3.16), we can choose $t_1 \in [t_0, T]$ such that

$$\begin{aligned}
&\sup_{t_1\le t\le T}\sup_{k\le k_0}\|\Delta_k u(t)\|_{L^2}(1-\exp\{-c2^{2k}(T-t)\})\\
&\le \sup_{t_1\le t\le T} 2c2^{2k_0}(T-t_1)\|u(t)\|_{L^2}\\
&\le C2^{2k_0}(\|u_0\|_{L^2}+\|F\|_{L^2_T(L^2)})(T-t_1)\le \frac{\varepsilon}{4C}.
\end{aligned}$$

Consequently,

$$\sup_{t_1\le t\le T}\sup_{k}\|\Delta_k u(t)\|_{L^2}(1-\exp\{-c2^{2k}(T-t)\})\le \frac{\varepsilon}{2C}. \tag{3.19}$$

On the other hand, we can choose $t_2 \in [t_1, T)$ such that

$$\begin{aligned}
&\Big(\sup_{t_2\le t\le T}\|u(t)\|_{L^2}+\|F\|_{L^2_{[t_2,T]}(L^2)}\Big)\|\nabla u\|^2_{L^2_{[t_2,T]}(L^2)}\\
&+\sup_k\int_{t_2}^T 2^{-k}\|\Delta_k F(\tau)\|_{L^\infty}d\tau)\\
&\le \frac{\varepsilon}{2C}.
\end{aligned} \tag{3.20}$$

It follows from (3.11) that

$$\begin{aligned}
\|u\|_{\tilde L^1_{[t_2,T]}(C^1)}\le C\Big(&\sup_k\|\Delta_k u(t_2)\|_{L^2}(1-\exp\{-c2^{2k}(T-t_2)\})\\
&+\big(\|u(t_2)\|_{L^2}+\|F\|_{L^2_{[t_2,T]}(L^2)}\big)\|\nabla u\|^2_{L^2_{[t_2,T]}(L^2)}\\
&+\sup_k\int_{t_2}^T 2^{-k}\|\Delta_k F(\tau)\|_{L^\infty}d\tau\Big).
\end{aligned} \tag{3.21}$$

Combining (3.19)-(3.21) gives

$$\|u\|_{\tilde L^1_{[t_2,T]}(C^1)}\le\varepsilon. \tag{3.22}$$

Using (3.22) and (1.3), we can choose $t^* \in [t_2, T)$ such that

$$\|u\|_{\tilde L^1_{[t^*,T]}(C^1)}\le\varepsilon,\quad \|\theta\|_{L^1_{[t^*,T]}(\dot B^0_{\infty,\infty})}\le\varepsilon. \tag{3.23}$$

For $0 \le t < T$, define

$$M(t)=\sup_{0\le\tau<t}\|u(\tau)\|_{\dot C^{1+\alpha}},\quad N(t)=\sup_{0\le\tau<t}\|\theta(\tau)\|_{\dot C^{\alpha}}.$$

In what follows, we estimate $M(t)$ and $N(t)$ for $0 \le t < T$. Applying Δ_k to the first and second equation in (1.2), we obtain

$$\begin{gathered}
\partial_t\Delta_k u-\Delta\Delta_k u+\nabla\Delta_k\pi=-\nabla\cdot\Delta_k(u\otimes u)+\Delta_k(\theta e_2),\\
\partial_t\Delta_k\theta+u\cdot\nabla\Delta_k\theta=\Delta_k\Big(\frac12\sum_{i,j=1}^2(\partial_i u^j+\partial_j u^i)^2\Big)+[u\cdot\nabla,\Delta_k]\theta.
\end{gathered} \tag{3.24}$$

Firstly, we make estimate $\|u(t)\|_{\dot C^{1+\alpha}}$. It follows from the first equation in (3.24) and (2.4) that

$$\begin{aligned}
\|\Delta_k u(t)\|_{L^\infty}\le Ce^{-c2^{2k}t}\|\Delta_k u(0)\|_{L^\infty}+C\int_0^t e^{-c2^{2k}(t-\tau)}\|\nabla\cdot\Delta_k(u\otimes u)(\tau)\|_{L^\infty}d\tau\\
+C\int_0^t e^{-c2^{2k}(t-\tau)}\|\Delta_k(\theta e_2)(\tau)\|_{L^\infty}d\tau.
\end{aligned}$$

By the above inequality, (2.1), (2.2) and Hölder inequality, we obtain

$$\begin{aligned}\|u(t)\|_{\dot C^{1+\alpha}} &\le C\|u(0)\|_{\dot C^{1+\alpha}} + C\int_0^t 2^{3k/2}e^{-c2^{2k}(t-\tau)}\|u\otimes u(\tau)\|_{\dot C^{\frac12+\alpha}}d\tau \\ &\quad + C\int_0^t 2^k e^{-c2^{2k}(t-\tau)}\|\theta(\tau)\|_{\dot C^{\alpha}}d\tau \\ &\le C(\|u(0)\|_{\dot C^{1+\alpha}} + N(t)) + C(\int_0^t \|u\otimes u(\tau)\|^4_{\dot C^{\frac12+\alpha}}d\tau)^{1/4}.\end{aligned} \tag{3.25}$$

By (2.8), we obtain

$$\|u\|_{L^4} \le C\|u\|^{1/2}_{L^2}\|\nabla u\|^{1/2}_{L^2}. \tag{3.26}$$

Using this inequality and the fact $\|u\otimes u\|_{\dot C^{\frac12+\alpha}} \le C\|u\|_{L^4}\|u\|_{\dot C^{1+\alpha}}$, we obtain

$$\begin{aligned}&\|u(t)\|^4_{\dot C^{1+\alpha}} \\ &\le C(\|u(0)\|_{\dot C^{1+\alpha}} + N(t))^4 + C\int_0^t \|u(\tau)\|^2_{L^2}\|\nabla u(\tau)\|^2_{L^2}\|\|u(\tau)\|^4_{\dot C^{1+\alpha}}d\tau \\ &\le C(\|u(0)\|_{\dot C^{1+\alpha}} + N(\tilde t))^4 + C\int_0^t \|u(\tau)\|^2_{L^2}\|\nabla u(\tau)\|^2_{L^2}\|\|u(\tau)\|^4_{\dot C^{1+\alpha}}d\tau,\end{aligned} \tag{3.27}$$

for any fixed $\tilde t : 0 \le \tilde t \le T$ and $t \le \tilde t < T$. Here we have used the fact that $N(t)$ is nondecreasing. Consequently, Gronwall's inequality gives

$$\begin{aligned}M(\tilde t)^4 &= \sup_{0\le t<\tilde t}\|u(t)\|^4_{\dot C^{1+\alpha}} \\ &\le C(\|u(0)\|_{\dot C^{1+\alpha}} + N(\tilde t))^4 \exp\{C\int_0^t \|u(\tau)\|^2_{L^2}\|\nabla u(\tau)\|^2_{L^2}\|d\tau\}.\end{aligned}$$

Since $\tilde t \in [0,T)$ is arbitrary, by (3.2), we obtain

$$M(t) \le C(\|u(0)\|_{\dot C^{1+\alpha}} + N(t)), \quad \forall t\in[0,T). \tag{3.28}$$

We next continue to estimate $N(t)$. It follows from the second equation in (3.24) that

$$\begin{aligned}\|\Delta_k\theta\|_{L^\infty} &\le C\|\Delta_k\theta(0)\|_{L^\infty} + C\int_0^t\sum_{i,j=1}^2\|\Delta_k(\partial_i u^j + \partial_j u^i)^2(\tau)\|_{L^\infty}d\tau \\ &\quad + C\int_0^t \|[u\cdot\nabla,\Delta_k]\theta(\tau)\|_{L^\infty}d\tau.\end{aligned} \tag{3.29}$$

Using (2.1), (2.2), (3.29) and Hölder inequality, we have

$$\begin{aligned}\|\theta(t)\|_{\dot C^{\alpha}} &\le C\|\theta(0)\|_{\dot C^{\alpha}} + C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\|u(\tau)\|_{\dot C^{1+\alpha}}d\tau \\ &\quad + C\int_0^t 2^{k\alpha}\|[u\cdot\nabla,\Delta_k]\theta(\tau)\|_{L^\infty}d\tau.\end{aligned} \tag{3.30}$$

It follows from Bony decomposition that

$$\begin{aligned}\theta &= \sum_{|k'-r|\le 1}[\Delta_{k'}u\cdot\nabla,\Delta_k]\Delta_r\theta + \sum_{k'\le r-2}[\Delta_{k'}u\cdot\nabla,\Delta_k]\Delta_r\theta\\ &\quad + \sum_{k'\le r-2}[\Delta_r u\cdot\nabla,\Delta_k]\Delta_{k'}\theta\\ &= \sum_{|k'-r|\le 1}[\Delta_{k'}u\cdot\nabla,\Delta_k]\Delta_r\theta + \sum_{|r-k|\le 2}[S_{r-1}u\cdot\nabla,\Delta_k]\Delta_r\theta\\ &\quad + \sum_{|r-k|\le 2}[\Delta_r u\cdot\nabla,\Delta_k]S_{r-1}\theta.\end{aligned}\tag{3.31}$$

Note that

$$[S_{r-1}u,\Delta_k]f = \int_{\mathbb{R}^2} h(y)[S_{r-1}u(x) - S_{r-1}u(x-2^{-k}y)]f(x-2^{-k}y)dy,$$

we obtain

$$\|[S_{r-1}u,\Delta_k]f\|_{L^\infty} \le C2^{-k}\|\nabla S_{r-1}u\|_{L^\infty}\|f\|_{L^\infty}.$$

Hence

$$\begin{aligned}&\sum_{|r-k|\le 2}\int_0^t 2^{k\alpha}\|[S_{r-1}u\cdot\nabla,\Delta_k]\Delta_r\theta(\tau)\|_{L^\infty}d\tau\\ &\le C\sum_{|r-k|\le 2}\int_0^t 2^{k(\alpha-1)}\|\nabla S_{r-1}u\|_{L^\infty}\|\nabla\Delta_r\theta\|_{L^\infty}(\tau)d\tau\\ &\le C\sum_{|r-k|\le 2}\int_0^t \|\nabla S_{r-1}u\|_{L^\infty}2^{r\alpha}\|\Delta_r\theta\|_{L^\infty}(\tau)d\tau\\ &\le C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\|\theta(\tau)\|_{\dot C^\alpha}d\tau.\end{aligned}\tag{3.32}$$

Note that

$$[\Delta_r u,\Delta_k]f = \int_{\mathbb{R}^2} h(y)[\Delta_r u(x) - \Delta_r u(x-2^{-k}y)]f(x-2^{-k}y)dy.$$

Then, we have

$$\|[\Delta_r u,\Delta_k]f\| \le C2^{-k}\|\nabla\Delta_r u\|_{L^\infty}\|f\|_{L^\infty}.$$

It follows from the above inequality and (2.1), (2.2) that

$$\begin{aligned}&\sum_{|r-k|\le 2}\int_0^t 2^{k\alpha}\|[\Delta_r u\cdot\nabla,\Delta_k]S_{r-1}\theta(\tau)\|_{L^\infty}\\ &\le C\sum_{|r-k|\le 2}\int_0^t 2^{k(\alpha-1)}\|\nabla\Delta_r u\|_{L^\infty}\|\nabla S_{r-1}\theta\|_{L^\infty}(\tau)d\tau\\ &\le C\int_0^t \|\theta(\tau)\|_{L^\infty}\|u(\tau)\|_{\dot C^{1+\alpha}}(\tau)d\tau.\end{aligned}\tag{3.33}$$

By a straightforward computation, we obtain

$$\begin{aligned} &\sum_{|k'-r|\le 1}\int_0^t 2^{k\alpha}\|[\Delta_{k'}u\cdot\nabla,\Delta_k]\Delta_r\theta(\tau)\|_{L^\infty}d\tau \\ &\le C\int_0^t \|\theta(\tau)\|_{L^\infty}\|u(\tau)\|_{\dot C^{1+\alpha}}d\tau. \end{aligned} \tag{3.34}$$

Collecting (3.30)-(3.34) gives

$$\begin{aligned} \|\theta(t)\|_{\dot C^\alpha} &\le C\|\theta(0)\|_{\dot C^\alpha} + C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\|u(\tau)\|_{\dot C^{1+\alpha}}d\tau \\ &\quad + C\int_0^t (\|\nabla u(\tau)\|_{L^\infty}\|\theta(\tau)\|_{\dot C^\alpha} + \|\theta(\tau)\|_{L^\infty}\|u(\tau)\|_{\dot C^{1+\alpha}})d\tau \\ &\le C\|\theta(0)\|_{\dot C^\alpha} + C\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\theta(\tau)\|_{L^\infty})(\|u(\tau)\|_{\dot C^{1+\alpha}} \\ &\quad + \|\theta(\tau)\|_{\dot C^\alpha})d\tau. \end{aligned} \tag{3.35}$$

From (3.28) and (3.35), we obtain

$$N(t) \le C\|\theta(0)\|_{\dot C^\alpha} + C\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\theta(\tau)\|_{L^\infty})(\|u(0)\|_{\dot C^{1+\alpha}} + N(\tau))d\tau. \tag{3.36}$$

With the help of Lemma 2.3 and (3.23), we obtain

$$\begin{aligned} &C\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\theta(\tau)\|_{L^\infty})d\tau \\ &\le C\int_0^{t_\star} (\|\nabla u(\tau)\|_{L^\infty} + \|\theta(\tau)\|_{L^\infty})d\tau \\ &\quad + C\int_{t_\star}^t (1 + \|u(\tau)\|_{L^2} + \|\theta(\tau)\|_{L^2})d\tau \\ &\quad + C\int_{t_\star}^t \|\theta\|_{\dot B^0_{\infty,\infty}} \ln(e + \|\theta(\tau)\|_{\dot C^\alpha})d\tau \\ &\quad + C\sup_k \int_{t_\star}^t \|\nabla\Delta_k u(\tau)\|_{L^\infty}d\tau \ln\Big(e + \int_0^t \|u(\tau)\|_{\dot C^{1+\alpha}}d\tau\Big) \\ &\le C_\star + C\varepsilon\ln(e + N(t)) + C\varepsilon\ln[e + Ct(\|u(0)\|_{\dot C^{1+\alpha}} + N(t))] \\ &\le C_\star + C\varepsilon\ln(e + \|u(0)\|_{\dot C^{1+\alpha}} + N(t)), \end{aligned} \tag{3.37}$$

where $C_\star$ is a positive constant depending on the solution (u,θ) on $[0,t_\star]$. It follows from (3.36)-(3.37) that

$$N(t) \le C_\star(1 + \|u(0)\|_{\dot C^{1+\alpha}} + \|\theta(0)\|_{\dot C^\alpha}) + C\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\theta(\tau)\|_{L^\infty})N(\tau)d\tau,$$

provided that $\varepsilon > 0$ is suitably small. By Gronwall's inequality and (3.37), we obtain

$$\begin{aligned} &e + \|u(0)\|_{\dot C^{1+\alpha}} + N(t) \\ &\le C_\star(e + \|u(0)\|_{\dot C^{1+\alpha}} + \|\theta(0)\|_{\dot C^\alpha})\exp\{C\int_0^t (\|\nabla u(\tau)\|_{L^\infty} + \|\theta(\tau)\|_{L^\infty})d\tau\} \\ &\le C_\star(e + \|u(0)\|_{\dot C^{1+\alpha}} + \|\theta(0)\|_{\dot C^\alpha})\exp\{C_\star + C\varepsilon\ln(e + \|u(0)\|_{\dot C^{1+\alpha}} + N(t))\} \end{aligned}$$

$$\leq C_\star(e + \|u(0)\|_{\dot{C}^{1+\alpha}} + \|\theta(0)\|_{\dot{C}^\alpha})(e + \|u(0)\|_{\dot{C}^{1+\alpha}} + N(t))^{C\varepsilon}.$$

Choosing $\varepsilon > 0$ suitably small, the above inequality and (3.28) yields

$$M(t) + N(t) \leq C_\star(1 + \|u(0)\|_{\dot{C}^{1+\alpha}} + \|\theta(0)\|_{\dot{C}^\alpha})^2.$$

The proof is complete. □

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