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DECAY RESULTS FOR VISCOELASTIC DIFFUSION EQUATIONS IN ABSENCE OF INSTANTANEOUS ELASTICITY

MOHAMMAD KAFINI

Abstract. We study the diffusion equation in the absence of instantaneous elasticity

$$u_t - \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau = 0, \quad (x,t) \in \Omega \times (0,+\infty),$$

where $\Omega \subset \mathbb{R}^n$, subjected to nonlinear boundary conditions. We prove that if the relaxation function g decays exponentially, then the solutions is exponential stable.

1. Introduction

A diffusion equation in the absence of instantaneous elasticity has the form

$$u_t - \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau = 0, \quad (x,t) \in \Omega \times (0,+\infty). \tag{1.1}$$

When the fluid is enclosed in a region $\Omega \subset \mathbb{R}^n$ the above equation is supplemented by conditions at $\partial\Omega$, the boundary of Ω. For instance, one can consider the nonlinear boundary condition:

$$\partial_\nu u + f(u) = 0, \quad \partial\Omega \times (0,+\infty). \tag{1.2}$$

Let us assume that

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

Denoting

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)\,ds$$

and differentiating equation (1.1), with respect to t, we arrive at the Volterra equation

$$\frac{1}{g(0)}u_{tt} = \Delta u + \frac{1}{g(0)}(g' * \Delta u). \tag{1.4}$$

Considering the Volterra inverse operator we obtain

$$u_{tt} - g(0)\Delta u + k * u_{tt} = 0, \tag{1.5}$$

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where the resolvent kernel satisfies

$$k + \frac{1}{g(0)} g' * k = -\frac{1}{g(0)} g'.$$

Thus (1.5) becomes

$$u_{tt} - g(0)\Delta u + k(0)u_t - k(t)u_t(0) + k' * u_t = 0. \tag{1.6}$$

Reciprocally, supposing in a natural way that $u_t(0) = 0$, the identity (1.6) implies (1.1). Since we are interested in relaxation functions of exponential type and (1.6) involves the resolvent kernel k, we want to know if k has the same properties. The following lemma answers this question. Let h be a relaxation function and k its resolvent kernel; that is,

$$k(t) - k * h(t) = h(t). \tag{1.7}$$

Lemma 1.1 ([5, 7, 8])**.** *If h is a positive continuous function, then k is also a positive continuous function. Moreover, if there exist positive constants $c_0 < \gamma$, such that*

$$h(t) \le c_0 e^{-\gamma t},$$

then the function k satisfies

$$k(t) \le \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$

for all $0 < \epsilon < \gamma - c_0$.

Proof. Note that $k(0) = h(0) > 0$. Now, we take

$$t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\},$$

so $k(t) > 0$ for all $t \in [0, t_0[$. If $t_0 \in \mathbb{R}^+$, from equation (1.7) we obtain that $-k * h(t_0) = h(t_0)$ but this is contradictory. Therefore, $k(t) > 0$ for all $t \in \mathbb{R}^+$. Now, let us fix ϵ, such that $0 < \epsilon < \gamma - c_0$ and denote

$$k_\epsilon(t) := e^{\epsilon t} k(t), \quad h_\epsilon(t) := e^{\epsilon t} h(t).$$

Multiplying equation (1.7) by $e^{\epsilon t}$ we obtain

$$k_\epsilon(t) = h_\epsilon(t) + k_\epsilon * h_\epsilon(t),$$

hence

$$\begin{aligned}
\sup_{s\in[0,t]} k_\epsilon(s) &\le \sup h_\epsilon(s) + \Big\{ \int_0^\infty c_0 e^{(\epsilon-\gamma)s} ds \Big\} \sup_{s\in[0,t]} k_\epsilon(s) \\
&\le c_0 + \frac{c_0}{(\gamma - \epsilon)} \sup_{s\in[0,t]} k_\epsilon(s).
\end{aligned}$$

Therefore,

$$k_\epsilon(t) \le \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0},$$

which is the desired result. □

Thanks to Lemma 1.1, we can use equation (1.6) instead of (1.1).

Usually when g is such that $\mathrm{Re}(\hat{g}) > 0$ ($\hat{g}$ is the Fourier transform of g), we can, by the Laplace transformation, reduce equation (1.1) to an elliptic problem. By the variational method, we can resolve such an equation (see Raynal [8]). In what follows we shall adopt a different procedure in order to establish the well-posedness of problem (1.1).

So, from the above comments, we can consider equation (1.6), instead of equation (1.1), supplemented by the initial data (1.3), the compatibility condition $u_t(0) = 0$, and the boundary conditions

$$\begin{gathered} \partial_\nu u + \beta u_t + |u|^\rho u = 0, \quad on \Gamma_1 \times (0, \infty) \\ u = 0, \quad \Gamma_0 \times (0, +\infty), \end{gathered} \tag{1.8}$$

assuming that $k \in W^{2,1}(0, +\infty)$, $\beta > 0$ and $0 < \rho < 2/(n-2)$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$.

We shall assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, $\Gamma_0 \neq \emptyset$; Γ_0 and Γ_1 are closed and disjoint and ν represents the unit outward normal to Γ.

The variational formulation associated with problem (1.6) is

$$\begin{aligned} &(u_{tt}(t), v)_\Omega - g(0)(\Delta u(t), v)_\Omega \\ &+ k(0)(u_t(t), v)_\Omega + \int_0^t k'(t-\tau)(u'(\tau), v)_\Omega \, d\tau + g(0)(f(u), v)_{\Gamma_1} = 0, \end{aligned}$$

for all $v \in H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\}$.

We can easily obtain the existence and uniqueness of global *regular solutions* making use, for instance, of the Faedo-Galerkin method.

Evidently the additional term given by βu_t plays an essential role by allowing us to control the nonlinear term on the boundary. This is strongly necessary because of Lopatinski condition is lost. Thus, it is clear and it has been recognized a long time ago, that well-posedness theory with semilinear boundary nonlinearity and finite energy solutions must rely on and take advantage of the boundary dissipation. See, for instance, Cavalcanti et al [3], Lasiecka and Tataru [4] and references therein.

However, from the physical point of view to have two dissipations, namely, $k(0)u_t$ (internal) and βu_t (on the boundary) is too much to establish the exponential decay. So, by considering the techniques employed in Lasiecka and Tataru [4] or in Cavalcanti, Cavalcanti, and Soriano [3], it is possible to obtain the existence of weak solutions to (1.6) subject to the boundary conditions

$$\partial_\nu u + |u|^\rho u = 0, \quad \text{on } \Gamma_1 \times (0, \infty). \tag{1.9}$$

Unfortunately, because of the nonlinear boundary condition (1.9), the uniqueness is lost.

Indeed, let A be the operator whose domain is defined by

$$D(A) = \big\{(u, v) \in H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega); u - \mathcal{N}[g_1(\gamma_0 v) + f_1(\gamma_0 u)] \in D(-\Delta)\big\}$$

and the operator by

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ \Delta(u - \mathcal{N}[g_1(\gamma_0 v) + f_1(\gamma_0 u)]) \end{pmatrix}.$$

We are assuming that

$$g_1(s) = \beta s, \quad f_1 \text{ is a Lipschitz continuous function on } \mathbb{R}, \tag{1.10}$$

and

$$\begin{aligned} D(-\Delta) &= \{v \in H^1_{\Gamma_0}(\Omega); \Delta v \in L^2(\Omega)\} \\ &= \big\{v \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega); \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_1\big\}, \end{aligned}$$

and $\mathcal{N} : H^s(\Gamma_1) \to H^{s+3/2}_{\Gamma_0}(\Omega)$, $s \in \mathbb{R}$, is the Neumann map defined by

$$\mathcal{N}p = q \Leftrightarrow \begin{cases} -\Delta q = 0 \quad \text{in } \Omega \\ q = 0 \quad \text{on } \Gamma_0 \\ \dfrac{\partial q}{\partial \nu} = p \quad \text{on } \Gamma_1 \end{cases}$$

We observe that

$$(u,v) \in D(A) \Leftrightarrow \begin{cases} (u,v) \in [H^1_{\Gamma_0}(\Omega)]^2, \\ u - \mathcal{N}[g_1(\gamma_0 v) + f_1(\gamma_0 u)] \in H^1_{\Gamma_0}(\Omega), \\ \Delta(u - \mathcal{N}[g_1(\gamma_0 v) + f_1(\gamma_0 u)]) \in L^2(\Omega). \end{cases} \tag{1.11}$$

By the nonlinear semigroup theory [4, Theorem 2.1], the operator A is ω-accretive on the space $E := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, for some ω suitably large. Moreover, $A + \omega I$ is maximal monotone and

$$D(A) \text{ is dense in } H^1_{\Gamma_0}(\Omega) \times L^2(\Omega). \tag{1.12}$$

Let us assume that $\{u^0, u^1\} \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ and consider, in view of (1.12), $\{u^0_\mu, u^1_\mu\} \subset D(A)$ such that

$$u^0_\mu \to u^0 \text{ in } H^1_{\Gamma_0}(\Omega) \quad \text{and} \quad u^1_\mu \to u^1 \text{ in } L^2(\Omega) \quad \text{as } \mu \to +\infty. \tag{1.13}$$

Thus, $\{u^0_\mu, u^1_\mu\}$ satisfies, for all $\mu \in \mathbb{N}$ the compatibility conditions

$$\frac{\partial u^0_\mu}{\partial \nu} + \frac{1}{\mu} u^1_\mu + f_{1,\mu}(u^0_\mu) = 0,$$

where β is chosen equal to $1/\mu$ and $f_{1,\mu}(s)$ is the sequance of Lipschitz continuous (truncated) functions defined by

$$f_{1,\mu}(s) := \begin{cases} |s|^\rho s, \quad |s| < \mu \\ |\mu|^\rho \mu, \quad s \geq \mu \\ |-\mu|^\rho(-\mu), \quad s \leq -\mu. \end{cases} \tag{1.14}$$

Initially, we consider regular solutions to the auxiliary problem

$$\begin{gathered} u^n_{tt} - \alpha\Delta u^n + k(0)u^n_t + \int_0^t k'(t-s)u^n_t(x,s)ds = 0 \quad \text{in } \Omega \times (0,+\infty) \\ u^n(x,t) = 0, \quad x \in \Gamma_0 \\ \frac{\partial u^n}{\partial \nu} + \frac{1}{n} u^n_t + f_{1,n}(u^n) = 0 \quad \text{on } \Gamma_1 \times (0,+\infty) \\ u^n(x,0) = u^n_0(x), \quad u^n_t(x,0) = u^n_1(x), \quad x \in \Omega. \end{gathered} \tag{1.15}$$

We obtain a sequence of regular solutions to problem (1.15) which will converge, as n approaches infinity, to a desired weak solution ($\{u^0, u^1\} \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$). The procedure described above can be followed *verbatim* as considered in [2] and therefore will be omitted. It is important to be mentioned that while problem (1.15) possesses a unique solution, the uniqueness of the limit problem, namely

$$\begin{gathered} u_{tt} - \alpha\Delta u + k(0)u_t + \int_0^t k'(t-s)u_t(x,s)ds = 0 \quad \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0, \quad x \in \Gamma_0 \\ \frac{\partial u}{\partial \nu} + |u|^\rho u = 0 \quad \text{on } \Gamma_1 \times (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{gathered} \tag{1.16}$$

is lost because of the nonlinear boundary term $|u|^\rho u$. Of course for Dirichlet or Neumann homogeneous boundary conditions the uniqueness is recovered.

The main task of this work, is to prove the exponential stability of problem (1.15). Namely, we would like to have

$$E_n(t) \le CE_n(0)e^{-\omega\, t}, \tag{1.17}$$

where $E_n(t)$ is the energy associated with (1.15) and the constants C and ω do not depend on n. So, using denseness arguments as considered in [2, 4], we can pass to the limit in (1.17) to obtain the desired exponential decay rate for those weak solutions that are limit of regular solutions of problem (1.15). Evidently the procedure is valid for any weak solution if $u = 0$ or $\partial_\nu u = 0$ on Γ.

2. Preliminaries

In this section we present some material needed in the proof of our result. We will us the following assumptions:

- (G1) $k', k''' : [0,\infty) \to \mathbb{R}^+$ with $k(0) > 0$.
- (G2) $k'' : [0,\infty) \to \mathbb{R}^-$ with $k''(0) < 0$.
- (G3) There exist two positive constants ζ_1 and ζ_2 such that

$$k'' \le -\zeta_1 k' \quad \text{and} \quad k''' \ge -\zeta_2 k''.$$

An example of function k satisfying (G1)–(G3) is $k(t) = a - e^{-bt}$, where $a > 1$, $b > 0$.

Lemma 2.1 (Poincaré)**.** *There exists a positive constant $\beta(\Omega)$ such that*

$$|u|_2^2 \le \beta |\nabla u|_2^2, \quad \forall u \in H^1_{\Gamma_0}(\Omega).$$

Our main task is concerned with the asymptotic behavior of solutions to the problem

$$\begin{gathered} u_{tt} - \alpha\Delta u + k(0)u_t + \int_0^t k'(t-s)u_t(x,s)ds = 0, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \Gamma_0, \\ \frac{\partial u}{\partial \nu} + \frac{1}{n}u_t + b|u|^\rho u = 0 \quad \text{on } \Gamma_1, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{gathered} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\alpha > 0$ and $k(t) \in C^3(\mathbb{R}^+)$ satisfying (G1)–(G3). Once exponential stability for (2.1) is established, then the same technique is used for viscoelastic diffusion problem

$$\begin{gathered} u_t - \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau = 0, \quad x \in \Omega,\ t>0, \\ u(x,t) = 0, \quad x \in \Gamma_0, \\ \frac{\partial u}{\partial \nu} + b|u|^\rho u = 0 \quad \text{on } \Gamma_1, \\ u(x,0) = u_0(x), \quad u_t(x,0) = 0, \quad x \in \Omega, \end{gathered} \tag{2.2}$$

obtained by denseness arguments, having in mind the comments established in section 1. Of course this is not true for all weak solutions of (2.2) unless we have $u = 0$ or $\partial_\nu u = 0$ on Γ.

After integrating by parts the last term in (2.1), we obtain

$$\begin{gathered} u_{tt} - \alpha\Delta u + k(0)u_t + k'(0)u(t) - k'(t)u_0 \\ + \int_0^t k''(t-s)u(x,s)ds = 0, \quad x \in \Omega,\ t>0, \\ u(x,t) = 0, \quad x \in \Gamma_0, \\ \frac{\partial u}{\partial \nu} + \frac{1}{n}u_t + b|u|^\rho u = 0 \quad \text{on } \Gamma_1, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{gathered} \tag{2.3}$$

The modified energy functional associated with (2.3) is

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega u_t^2 dx + \frac{\alpha}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}k'(t)\int_\Omega (u(x,t)-u_0(x))^2\,dx \\ &\quad - \frac{1}{2}\int_\Omega\int_0^t k''(t-s)\,(u(x,s)-u(x,t))^2\,ds\,dx + \frac{\alpha b}{\rho+2}\int_{\Gamma_1}|u|^{\rho+2}dx. \end{aligned}$$

3. Decay of solutions

In this section we state and prove our main result. For this purpose, we set

$$F(t) = E(t) + \varepsilon\varphi(t), \quad t \ge 0, \tag{3.1}$$

where ε is a positive constant and

$$\varphi(t) = \int_\Omega u_t u\,dx + \frac{k(0)}{2}\int_\Omega u^2 dx + \frac{\alpha}{2n}\int_{\Gamma_1} u^2 dx, \tag{3.2}$$

Lemma 3.1. *The modified energy satisfies, along the solution of* (2.3),

$$\begin{aligned} E'(t) &= -k(0)\int_\Omega u_t^2 dx + \frac{1}{2}k''(t)\int_\Omega (u(x,t)-u_0(x))^2\,ds\,dx \\ &\quad - \frac{1}{2}\int_\Omega\int_0^t k'''(t-s)\,(u(x,s)-u(x,t))^2\,ds\,dx - \frac{\alpha}{n}\int_{\Gamma_1}|u_t|^2 dx \le 0. \end{aligned}$$

Proof. Multiplying (2.3) by u_t and integrating over Ω, using integration by parts, hypotheses (G1) and (G2) and some manipulations as in [2] , we obtain the result for any regular solution. This result remains valid for any "limit weak" solution (not for all weak solutions) by a simple denseness argument. □

Lemma 3.2. *For $\varepsilon > 0$ small enough, we have*
$$|F(t) - E(t)| \leq \varepsilon\lambda E(t), \quad t \geq 0,$$
where λ is a constant independent of ε and n.

Proof. Using the Poincaré and the Cauchy-Schwarz inequalities, we obtain
$$\begin{aligned}
|\varepsilon\varphi(t)| &\leq \frac{\varepsilon}{2}\int_\Omega u_t^2 dx + \frac{\varepsilon}{2}(1+k(0))\int_\Omega u^2 dx + + \frac{\varepsilon\alpha}{2n}\int_{\Gamma_1} u^2 dx \\
&\leq \frac{\varepsilon}{2}\int_\Omega u_t^2 dx + \frac{\varepsilon\left(1+k(0)+\frac{\alpha}{n}\right)\beta}{2}\int_\Omega |\nabla u|^2 dx \\
&\leq \frac{\varepsilon}{2}\int_\Omega u_t^2 dx + \frac{\varepsilon\left(1+k(0)+\alpha\right)\beta}{2}\int_\Omega |\nabla u|^2 dx \leq \lambda\varepsilon E(t),
\end{aligned}$$
where β is the Poincaré constant and
$$\lambda = \max\left\{1, \frac{(1+k(0)+\alpha)\beta}{\alpha}\right\}.$$
Then from (3.1), it follows that
$$|F(t) - E(t)| \leq \varepsilon\lambda E(t), \quad t \geq 0. \tag{3.3}$$
□

Lemma 3.3. *Under assumptions*(G1)–(G2)*, the functional $\varphi(t)$ satisfies, along the solution of* (2.3) *and for any $\delta > 0$,*
$$\begin{aligned}
\varphi'(t) \leq{}& \int_\Omega u_t^2 dx - (\alpha - \delta\beta[k'(t)+1])\int_\Omega |\nabla u|^2 dx - \alpha b\int_{\Gamma_1} |u|^{\rho+2} dx \\
&- \frac{k'(0)}{4\delta}\int_\Omega\int_0^t k''(t-s)\left(u(s)-u(t)\right)^2\, ds\, dx + \frac{k'(t)}{4\delta}\int_\Omega \left(u(t)-u_0(x)\right)^2 dx.
\end{aligned}\tag{3.4}$$

Proof. Differentiation of (3.2), using (2.3), yields
$$\begin{aligned}
\varphi'(t) ={}& \int_\Omega u_t^2 dx - \alpha\int_\Omega |\nabla u|^2 dx - k'(0)\int_\Omega u^2 dx - \alpha b\int_{\Gamma_1} |u|^{\rho+2} dx \\
&+ k'(t)\int_\Omega u_0(x)u(t)dx - \int_\Omega u(t)\int_0^t k''(t-s)u(s)\, ds\, dx.
\end{aligned}\tag{3.5}$$
Using Young's, Cauchy-Schwarz's, Poincaré's and H ölder's inequalities, the last two terms in (3.5) can be estimated as follows
$$\begin{aligned}
&k'(t)\int_\Omega u_0(x)u(t)dx \\
&= k'(t)\int_\Omega \left(u_0(x) - u(t) + u(t)\right)u(t)dx \\
&= k'(t)\int_\Omega \left(u_0(x) - u(t)\right)u(t)dx + k'(t)\int_\Omega u^2(t)dx \\
&\leq \frac{k'(t)}{4\delta}\int_\Omega \left(u(t)-u_0(x)\right)^2 dx + \delta k'(t)\int_\Omega u^2(t)dx + k'(t)\int_\Omega u^2 dx \\
&\leq \frac{k'(t)}{4\delta}\int_\Omega \left(u(t)-u_0(x)\right)^2 dx + (\delta+1)\,k'(t)\int_\Omega u^2 dx
\end{aligned}\tag{3.6}$$

and

$$
\begin{aligned}
& -\int_\Omega u(t)\int_0^t k''(t-s)u(s)\,ds\,dx \\
& = -\int_\Omega u(t)\int_0^t k''(t-s)[\big(u(s)-u(t)\big)+u(t)]\,ds\,dx \\
& = -\int_\Omega u(t)\int_0^t k''(t-s)\left(u(s)-u(t)\right)\,ds\,dx - \int_\Omega u^2(t)\int_0^t k''(t-s)\,ds\,dx \\
& \leq \delta\int_\Omega u^2(t)dx + \frac{1}{4\delta}\int_\Omega\Big(\int_0^t k''(t-s)\left(u(s)-u(t)\right)ds\Big)^2 dx \\
& \quad - \int_\Omega u^2(t)\int_0^t k''(t-s)\,ds\,dx \\
& \leq \delta\int_\Omega u^2(t)dx + \frac{k'(0)}{4\delta}\int_\Omega\int_0^t -k''(t-s)\left(u(s)-u(t)\right)^2\,ds\,dx \\
& \quad + k'(0)\int_\Omega u^2dx - k'(t)\int_\Omega u^2dx \\
& \leq -\frac{k'(0)}{4\delta}\int_\Omega\int_0^t k''(t-s)\left(u(s)-u(t)\right)^2\,ds\,dx + [\delta+k'(0)-k'(t)]\int_\Omega u^2dx.
\end{aligned}
\tag{3.7}
$$

Combining (3.5)-(3.7), the result in (3.4) follows. □

At this point, we state and prove our main result.

Theorem 3.4. *Assume that* (G1)–(G3) *hold, and let* $(u_0,u_1)\in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)$. *Then, there exist two positive constants* C *and* ω, *independent of* n, *such that the limit weak solution of* (2.3) *satisfies, for all* $t\geq 0$,

$$
E(t)\leq CE(0)e^{-\omega t}.
$$

Proof. Using Lemmas 3.1 and 3.3, we have

$$
\begin{aligned}
F'(t) &= E'(t)+\varepsilon\varphi'(t) \\
&\leq -\left(k(0)-\varepsilon\right)\int_\Omega u_t^2dx - \varepsilon\left(\alpha-\delta\beta[k'(t)+1]\right)\int_\Omega|\nabla u|^2dx \\
&\quad - \varepsilon\frac{k'(0)}{4\delta}\int_\Omega\int_0^t k''(t-s)\left(u(s)-u(t)\right)^2\,ds\,dx \\
&\quad - \frac{1}{2}\int_\Omega\int_0^t k'''(t-s)\left(u(s)-u(t)\right)^2\,ds\,dx - \varepsilon\alpha b\int_{\Gamma_1}|u|^{\rho+2}dx \\
&\quad + \left(\frac{k''(t)}{2}+\varepsilon\frac{k'(t)}{4\delta}\right)\int_\Omega\left(u(t)-u_0(x)\right)^2dx - \frac{\alpha}{n}\int_{\Gamma_1}|u_t|^2dx.
\end{aligned}
\tag{3.8}
$$

Using (G3), (3.8), and $k'(t)\leq k'(0)$, and dropping the last term, we arrive at

$$
\begin{aligned}
F'(t) &\leq -\left(k(0)-\varepsilon\right)\int_\Omega u_t^2dx - \varepsilon\left(\alpha-\delta\beta[k'(0)+1]\right)\int_\Omega|\nabla u|^2dx \\
&\quad + \left(\frac{\zeta_2}{2}-\varepsilon\frac{k'(0)}{4\delta}\right)\int_\Omega\int_0^t k''(t-s)\left(u(s)-u(t)\right)^2\,ds\,dx \\
&\quad - k'(t)\left(\frac{\zeta_1}{2}-\varepsilon\frac{1}{4\delta}\right)\int_\Omega\left(u(t)-u_0(x)\right)^2dx - \varepsilon\alpha\int_{\Gamma_1}|u|^{\rho+2}dx.
\end{aligned}
\tag{3.9}
$$

Now, we choose δ such that
$$\delta < \frac{\alpha}{\beta\left(k'(0)+1\right)}.$$
Whence δ is fixed, we select ε satisfying
$$\varepsilon < \min\{k(0), \frac{2\delta\zeta_2}{k'(0)}, 2\delta\zeta_1, \frac{1}{\lambda}\},$$
hence (3.9) yields, for some $c > 0$,
$$F'(t) \le -cE(t), \quad \forall t \ge 0. \tag{3.10}$$
Also (3.3) leads to
$$(1-\lambda\varepsilon)E(t) \le F(t) \le (1+\lambda\varepsilon)E(t), \quad \forall t \ge 0.$$
Consequently, for any $0 < \gamma \le 1 - \lambda\varepsilon$, we have
$$\gamma E(t) \le F(t) \le (2-\gamma)\, E(t), \quad \forall t \ge 0. \tag{3.11}$$
Inserting (3.11) in (3.10), we obtain
$$F'(t) \le -\frac{c}{2-\gamma} F(t) = -\omega F(t), \quad \forall t \ge 0,$$
where $\omega = \frac{c}{2-\gamma}$. A direct integration yields
$$F(t) \le F(0)e^{-\omega t}, \quad \forall t \ge 0.$$
Using (3.11) again gives
$$E(t) \le \frac{1}{\gamma}F(t) \le \frac{1}{\gamma}F(0)e^{-\omega t} \le \frac{2-\gamma}{\gamma}E(0)e^{-\omega t} = CE(0)e^{-\omega t}, \quad \forall t \ge 0.$$
This completes the proof. □

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Mohammad Kafini
Department of Mathematics and Statistics, KFUPM, Dhahran 31261, Saudi Arabia
E-mail address: `mkafini@kfupm.edu.sa`