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POLYNOMIAL DIFFERENTIAL SYSTEMS WITH EXPLICIT NON-ALGEBRAIC LIMIT CYCLES

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Abstract. Up to now all the examples of polynomial differential systems for which non-algebraic limit cycles are known explicitly have degree at most 5. Here we show that already there are polynomial differential systems of degree at least exhibiting explicit non-algebraic limit cycles. It is well known that polynomial differential systems of degree 1 (i.e. linear differential systems) has no limit cycles. It remains the open question to determine if the polynomial differential systems of degree 2 can exhibit explicit non-algebraic limit cycles.

1. Introduction and statement of the main results

Probably the existence of limit cycles is one of the more difficult objects to study in the qualitative theory of differential equations in the plane. There is a huge literature dedicated to this topic, see for instance the book of Ye et al [12], or the famous Hilbert 16th problem [6] and [7]. Publications more closely related to the problem in this article are [4, 5, 1, 2, 8, 9].

A *polynomial differential system* is a system of the form

$$\begin{aligned}\dot{x} &= P(x, y),\\ \dot{y} &= Q(x, y),\end{aligned} \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials in the variables x and y. The *degree* of the system is the maximum of the degrees of the polynomials P and Q. As usual the dot denotes derivative with respect to the independent variable t.

A *limit cycle* of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system (1.1). If a limit cycle is contained in an algebraic curve of the plane, then we say that it is *algebraic*, otherwise it is called *non-algebraic*. In other words a limit cycle is algebraic if there exists a real polynomial $f(x, y)$ such that the algebraic curve $f(x, y) = 0$ contains the limit cycle. In general, the orbits of a polynomial differential system (1.1) are contained in analytic curves which are not algebraic.

To distinguish when a limit cycle is algebraic or not, usually, it is not easy. Thus, the well-known limit cycle of the van der Pol differential system exhibited in 1926 (see [11]) was not proved until 1995 by Odani [10] that it was non-algebraic.

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The van der Pol system can be written as a polynomial differential system (1.1) of degree 3, but its limit cycle is not known explicitly.

These previous years (from 2006 up to now) several papers have been published exhibiting polynomial differential systems for which non-algebraic limit cycles are known explicitly. This means that in some coordinates we have an explicit analytic expression of the curve containing the non-algebraic limit cycle. The first explicit non-algebraic limit cycle, due to Gasull, Giacomini and Torregrosa [4], was for a polynomial differential system of degree 5. Of course, multiplying the right hand part of this polynomial differential system of degree 5 by $(ax + by + c)^n$ with n an arbitrary positive integer, where the straight line $ax+by+c = 0$ must be chosen in such a way that it does not intersect the explicit limit cycle of the system, we get a polynomial differential system of degree $5 + n$ exhibiting an explicit non-algebraic limit cycle.

Immediately after this first paper appeared the paper of Al-Dosary [1] inspired by [4] (note that this reference is quoted in [1]), providing a similar polynomial differential system of degree 5 exhibiting an explicit non-algebraic limit cycle.

Giné and Grau [5] provide a polynomial differential system of degree 9 exhibiting simultaneously two explicit limit cycles one algebraic and another non-algebraic. Note that the paper [4] is also quoted in [5].

The aim of this paper is to show that there exist polynomial differential systems of degree 3 exhibiting explicit non-algebraic limit cycles. Thus, our main result is the following one.

Theorem 1.1. *The differential polynomial system of degree* 3*,*

$$\begin{aligned} \dot{x} &= x + (y - x)(x^2 - xy + y^2), \\ \dot{y} &= y - (y + x)(x^2 - xy + y^2), \end{aligned} \tag{1.2}$$

has a unique non-algebraic limit cycle whose expression in polar coordinates (r, θ)*, defined by* $x = r\cos\theta$ *and* $y = r\sin\theta$*, is*

$$r(\theta) = e^{\theta}\sqrt{r_*^2 - f(\theta)}, \tag{1.3}$$

where

$$\begin{gathered} r_* = e^{2\pi}\sqrt{\frac{f(2\pi)}{e^{4\pi} - 1}} \approx 1.1911644871948721\dots, \\ f(\theta) = 4\int_0^{\theta} \frac{e^{-2s}}{2 - \sin(2s)}\, ds. \end{gathered}$$

Moreover, this limit cycle is a stable hyperbolic limit cycle.

The above theorem is proved in section 2. In short, since it is well known that the linear differential systems (or polynomial differential systems of degree 1) have no limit cycles, it remains the following open question:

Open question. *Are there or not polynomial differential systems of degree* 2 *exhibiting explicit non-algebraic limit cycles.*

2. Proof of Theorem 1.1

The polynomial differential system (1.2) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= r + \frac{1}{2}(\sin(2\theta) - 2)r^3, \\ \dot{\theta} &= \frac{1}{2}r^2(\sin(2\theta) - 2). \end{aligned} \tag{2.1}$$

Taking as independent variable the coordinate θ, this differential system writes

$$\frac{dr}{d\theta} = r + \frac{2}{r(\sin(2\theta) - 2)}. \tag{2.2}$$

Note that since $\dot{\theta} < 0$, the orbits $r(\theta)$ of the differential equation (2.2) has reversed their orientation with respect to the orbits $(r(t), \theta(t))$ or $(x(t), y(t))$ of the differential systems (2.1) and (1.2), respectively.

It is easy to check that the solution $r(\theta; r_0)$ of the differential equation (2.2) such that $r(0; r_0) = r_0$ is

$$r(\theta; r_0) = e^{\theta}\sqrt{r_0^2 - f(\theta)}, \tag{2.3}$$

where $f(\theta)$ is the function defined in the statement of Theorem 1.1.

Clearly the unique equilibrium point of the differential system (1.2) is the origin of coordinates, which is an unstable node because its eigenvalues are 1 with multiplicity two, for more details see for instance [3, Theorem 2.15]. This equilibrium point in polar coordinates become $r = 0$. This is the unique point of the plane where the differential equation (2.2) is not defined. But we can extend the flow of this differential equation to $r = 0$, assuming that at the origin of the plane in polar coordinates we have an unstable node.

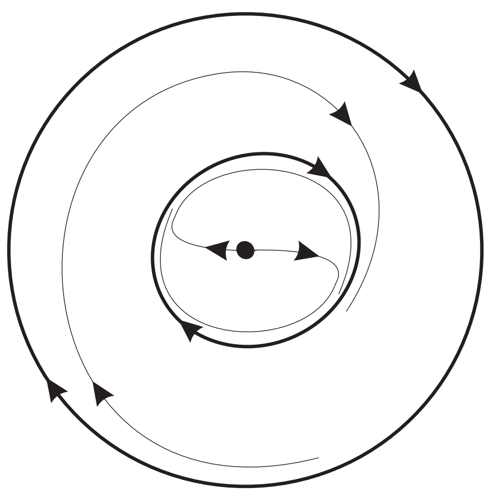

Figure 1. The phase portrait in the Poincaré disc of the polynomial differential system (1.2)

The periodic orbits $r(\theta; r_0)$ of (2.2) must satisfy $r(2\pi; r_0) = r_0$. A solution of this equation is $r_0 = r_*$, where r_* is defined in the statement of Theorem 1.1. So, if $r(\theta; r_*) > 0$ for all $\theta \in \mathbb{R}$, we shall have $r(\theta; r_*) > 0$ would be a periodic orbit, and consequently a limit cycle. In what follows it is proved that $r(\theta; r_*) > 0$ for all $\theta \in \mathbb{R}$. Indeed

$$\begin{aligned} r(\theta; r_*) &= e^{\theta} \sqrt{\frac{e^{4\pi}}{e^{4\pi}-1} f(2\pi) - f(\theta)} \\ &\geq e^{\theta} \sqrt{f(2\pi) - f(\theta)} \\ &= 2e^{\theta} \sqrt{\int_{\theta}^{2\pi} \frac{e^{-2s}}{2-\sin(2s)}\, ds} > 0, \end{aligned}$$

because $e^{-2s}/(2-\sin(2s)) > 0$ for all $s \in \mathbb{R}$.

An easy computation shows that

$$\frac{d\, r(2\pi; r_0)}{d\, r_0}\Big|_{r_0=r_*} = e^{4\pi} > 1.$$

Therefore the limit cycle of the differential equation (2.2) is unstable and hyperbolic, for more details see [3, section 1.6]. Consequently, this is a stable and hyperbolic limit cycle for the differential system (1.2).

Clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x, y) plane with

$$r(\theta)^2 = e^{2\theta}(r_*^2 - f(\theta)),$$

is not algebraic, due to the expression $e^{2\theta} r_*^2$. More precisely, in cartesian coordinates the curve defined by this limit cycle is

$$f(x,y) = x^2 + y^2 - e^{2\arctan(y/x)} \Big(r_*^2 - 4 \int_0^{\arctan(y/x)} \frac{e^{-2s}}{2-\sin(2s)}\, ds \Big) = 0.$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $f(x, y)$ in the variables x and y satisfies that there is a positive integer n such that $\partial^n f/(\partial x)^n = 0$, and this is not the case because in the derivative

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + \frac{2y e^{2\arctan(y/x)}}{x^2+y^2} \Big(r_*^2 - 4 \int_0^{\arctan(y/x)} \frac{e^{-2s}}{2-\sin(2s)}\, ds \Big) \\ &\quad - \frac{4y}{(x^2+y^2)\big(2-\sin\big(2\arctan(y/x)\big)\big)} \end{aligned}$$

it appears again the expression

$$e^{2\arctan(y/x)} \Big(r_*^2 - 4 \int_0^{\arctan(y/x)} \frac{e^{-2s}}{2-\sin(2s)}\, ds \Big),$$

which already appears in $f(x, y)$, and this expression will appear in the partial derivative at any order.

Now we shall prove that the limit cycle given by $r(\theta; r_*)$ is the unique periodic orbit of the differential system, and consequently the unique limit cycle. We recall the so called Generalized Dulac's Theorem, for a proof of it see [3, Theorem 7.12].

Theorem 2.1. *Let R be an n-multiply connected region of $\mathbb{R}^2$ (i.e. R has one outer boundary curve, and $n-1$ inner boundary curves). Assume that the divergence function $\partial P/\partial x + \partial Q/\partial y$ of the C^1 differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ has constant sign in the region R, and is not identically zero on any subregion of R. Then this differential system has at most $n-1$ periodic orbits which lie entirely in R.*

We take as new independent variable the variable τ defined by $d\tau = (x^2 + y^2)(x^2 - xy + y^2)dt$. Since $(x^2+y^2)(x^2-xy+y^2)$ only vanishes at the origin of coordinates the differential system (1.2) and the differential system

$$\begin{aligned} x' &= \frac{x + (y-x)(x^2 - xy + y^2)}{(x^2+y^2)(x^2-xy+y^2)}, \\ y' &= \frac{y - (y+x)(x^2 - xy + y^2)}{(x^2+y^2)(x^2-xy+y^2)}, \end{aligned} \tag{2.4}$$

where the prime denotes derivative with respect to the variable τ, have the same phase portrait in $R = \mathbb{R}^2 \setminus \{(0,0)\}$. An easy computation shows that the divergence of the differential system (2.4) is

$$-\frac{2}{(x^2+y^2)(x^2-xy+y^2)} < 0 \quad \text{in } R.$$

So, by Theorem 2.1, and since R is 2-multiply connected region of $\mathbb{R}^2$ it follows that the differential system (2.4) and consequently the differential system (1.2) has at most one periodic solution. In short, the unique periodic solution of system (1.2) is $r(\theta; r_*)$. This completes the proof of Theorem 1.1.

Now we shall present the phase portrait of the differential system (1.2) in the Poincaré disc, see the Poincaré compactification in [3, Chapter 5].

Since the polynomial $\dot{x}y - \dot{y}x = (x^2+y^2)(x^2-xy+y^2)$ has no real linear factors, the compactification of Poincaré of the differential system (1.2) has no equilibrium points at infinity, i.e. the infinity is a periodic orbit. Doing the change of variables $r = 1/\rho$, the infinity of the differential equation (2.2) passes at the origin, and equation (2.2) becomes

$$\frac{d\rho}{d\theta} = -\rho - \frac{2\rho^3}{\sin(2\theta) - 2}.$$

Hence, clearly $\rho = 0$ is an stable equilibrium point of this differential equation, consequently the periodic orbit at infinity of the differential equation (1.3) is an unstable limit cycle. Then the phase portrait in the Poincaré disc of the polynomial differential system (1.2) is given in Figure 1.

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