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POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this article, we study the existence of the positive solutions for a class of differential equations of fractional order with variable coefficients. The equation of this type plays an important role in the description and modeling of control systems, such as PD^{μ}-controller. The differential operator is taken in the Riemann-Liouville sense. Our analysis relies on the Leggett-Williams fixed point theorem.

1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration. It plays an important role in science, engineering, economy, and other fields, see [6, 7, 8, 10, 13, 14, 16, 17]. For example, the book [14] details the use of fractional calculus in the description and modeling of systems, and in a range of control design and practical applications. And today there are many papers dealing with the fractional differential equations due to its various applications, see [1, 2, 3, 4, 5, 12, 15, 19, 20, 21, 22, 23].

In [14], the authors considered the dynamic model of an immersed plate, which is modeled by

$$\begin{gathered} A_B D_{0+}^{2} y(t) + B_B D_{0+}^{1.5} y(t) + C_B y(t) = f(t), \\ y(0) = y'(0) = 0. \end{gathered}$$

As indicated in [17], a fractional order PD^{μ}-controller can be more suitable for the control of "reality" than integer order. For example, the fractional-order PD^{μ}-controller can be characterized by (see [17, equation (9.33)])

$$a_2 D_{0+}^{\beta} y(t) + T_d D_{0+}^{\mu} y(t) + a_1 D_{0+}^{\alpha} y(t) + (a_0 + K) y(t) = K w(t) + T_d D_{0+}^{\mu} w(t), \tag{1.1}$$

where $\alpha < \mu < \beta$. And, (1.1) and (1.1) are the particular case of the equation of type (1.2) in our paper. And for the system of this type, we can find its many other real applications in [16, 17, 18] and in [14, Chapter 14-18].

Problems of this type, with constant coefficients, have provoked some interest in recent literature, such as [3, 19, 20, 21] and references therein. In [19], the author

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indicated that: "Some of the earlier results of this type contains errors in the proof of equivalence of the initial value problems and the corresponding Volterra integral equations (see survey paper by Kilbas and Trujillo [9])".

Motivated by these papers, in this paper, we consider the following initial value problems of fractional differential equations with variable coefficients

$$\begin{gathered} D_{0+}^{\alpha_n}u(t) - \sum_{j=1}^{n-1} a_j(t) D_{0+}^{\alpha_j}u(t) = f(t, u(t)), \quad 0 \le t \le 1, \\ u(0) = u'(0) = 0, \end{gathered} \tag{1.2}$$

where $0 < \alpha_1 < \alpha_2 < \dots < \alpha_{n-1} < \alpha_n - 1 < 1 < \alpha_n < 2$, $n \ge 2$, $n \in \mathbb{Z}$, $a_n \in \mathbb{R}$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $a_j : [0,1] \to (0,+\infty)$ $(j = 1,2,\dots,n-1)$ are continuously differentiable. We will study the problem (1.2) in the Banach space $C[0,1]$ equipped with the maximum norm $\|\cdot\|$.

To the best of our knowledge, the results on the existence of solutions for the fractional differential equations with variable coefficients are relatively scare. The variable coefficients cause the problem more complex. The main difficulty in dealing with such issues is that the classical integration by parts formula is no longer applicable for the fractional integration. And how to get the equivalent integral equation of the problem (1.2) differs from the equations with constant coefficients. In the paper we solve these problems.

This article is organized as follows. In Section 2, we present some results of fractional calculus theory and auxiliary technical lemmas, which are used in the next section. Section 3, applying the results of Section 2, we obtain the existence and multiplicity results of the positive solutions for the problem (1.2) by the Leggett-Williams fixed point theorem in a cone. Then an example is given in Section 4 to demonstrate the application of our results.

2. Preliminaries

First of all, we present the necessary definitions and fundamental facts on the fractional calculus theory. These can be found in [8, 13, 17].

Definition 2.1 ([8, 16, 17])**.** The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $h : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\nu}h(t) = D_{0+}^{-\nu}h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s)ds \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([8, 16, 17])**.** The Riemann-Liouville fractional derivative of order $\nu > 0$ of a continuous function $h : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\nu}h(t) = \frac{1}{\Gamma(n-\nu)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\nu-1} h(s)ds, \tag{2.2}$$

where $n = [\nu] + 1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.3 ([5])**.** *Assume that $h(t) \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\nu > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\nu} D_{0+}^{\nu} h(t) = h(t) + C_1 t^{\nu-1} + C_2 t^{\nu-2} + \dots + C_N t^{\nu-N}, \tag{2.3}$$

for some $C_i \in \mathbb{R}$, $i = 1,2,\dots,N$, where N is the smallest integer such that $N \ge \nu$.

Lemma 2.4 ([8, 16, 17]). *If $\nu_1, \nu_2, \nu > 0$, $t \in [0,1]$ and $h(t) \in L[0,1]$, then*

$$I_{0+}^{\nu_1} I_{0+}^{\nu_2} h(t) = I_{0+}^{\nu_1+\nu_2} h(t), \quad D_{0+}^{\nu} I_{0+}^{\nu} h(t) = h(t). \tag{2.4}$$

Lemma 2.5 ([12, 17]). *If $h(t) \in C[0,1]$ and $\nu > 0$, then we have*

$$\big[I_{0+}^{\nu} h(t)\big]_{t=0} = 0, \quad or \quad \lim_{t\to 0} \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds = 0. \tag{2.5}$$

Let

$$g_j(t,s) = (\alpha_n - 1) a_j(s) - (t-s) a_j'(s), \quad (t,s) \in [0,1] \times [0,1], \tag{2.6}$$

$$h_j(t,\tau) = \int_0^1 \xi^{-\alpha_j} (1-\xi)^{\alpha_n - 2} g_j(t, \tau + \xi(t-\tau)) d\xi, \quad (t,\tau) \in [0,1] \times [0,1], \tag{2.7}$$

where $j = 1, 2, \dots, n-1$. It is obvious that $g_j(t,s), h_j(t,\tau)$ are continuous and that for $0 < s < t$,

$$\begin{aligned} \frac{d}{ds}\big((t-s)^{\alpha_n - 1} a_j(s)\big) &= -(\alpha_n - 1)(t-s)^{\alpha_n - 2} a_j(s) + (t-s)^{\alpha_n - 1} a_j'(s) \\ &= -(t-s)^{\alpha_n - 2} g_j(t,s), \quad j = 1, 2, \dots, n-1. \end{aligned}$$

Set

$$b_j(t) = \ln a_j(t), \quad j = 1, 2, \dots, n-1, \tag{2.8}$$

then it is clear that $b_j(t)$ $(j = 1, 2, \dots, n-1)$ is continuously differentiable.

Lemma 2.6. *Let $a_j : [0,1] \to (0, +\infty)$ $(j = 1, 2, \dots, n-1)$ are continuously differentiable. Assume that the condition*

(H1) $|{b_j}'(t)| < \alpha_n - 1$, $j = 1, 2, \dots, n-1$.

Then $g_j(t,s) > 0$, for $j = 1, 2, \dots, n-1$.

Proof. In view of (2.8), we have

$$a_j(t) = e^{b_j(t)}, \quad {a_j}'(t) = {b_j}'(t) e^{b_j(t)}, \quad j = 1, 2, \dots, n-1.$$

Then, by (H_1), we deduce that

$$\begin{aligned} g_j(t,s) &= (\alpha_n - 1) a_j(s) - (t-s) a_j'(s) \\ &= (\alpha_n - 1) e^{b_j(t)} - (t-s) b_j'(t) e^{b_j(t)} \\ &= e^{b_j(t)}\big((\alpha_n - 1) - (t-s) b_j'(t)\big) > 0. \end{aligned}$$

The proof is complete. □

For convenience, denote

$$M_j = \max_{0\le t\le 1,\, 0\le s\le 1} g_j(t,s), \quad m_j = \min_{0\le t\le 1,\, 0\le s\le 1} g_j(t,s), \quad j = 1, 2, \dots, n-1;$$

$$P_1 = \sum_{j=1}^{n-1} \frac{M_j B(1-\alpha_j, \alpha_n - 1)}{(\alpha_n - \alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)} = \sum_{j=1}^{n-1} \frac{M_j}{(\alpha_n - 1)\Gamma(\alpha_n - \alpha_j + 1)};$$

$$P_2 = \sum_{j=1}^{n-1} \frac{m_j B(1-\alpha_j, \alpha_n - 1)}{(\alpha_n - \alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)} = \sum_{j=1}^{n-1} \frac{m_j}{(\alpha_n - 1)\Gamma(\alpha_n - \alpha_j + 1)};$$

We can easily show that $M_j \ge m_j > 0$ and $P_1 \ge P_2 > 0$. Then we have the following lemma.

Lemma 2.7. *Let $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $a_j : [0,1] \to (0,+\infty)$ $(j = 1,2,\dots,n-1)$ are continuously differentiable, then*

$$m_j B(1-\alpha_j, \alpha_n - 1) \le h_j(t,\tau) \le M_j B(1-\alpha_j, \alpha_n - 1), \quad j = 1,2,\dots,n-1, \tag{2.9}$$

where $B(\cdot,\cdot)$ is the Beta function.

Proof. According to (2.7), for each $j = 1,2,\dots,n-1$, we have

$$h_j(t,\tau) \le M_j \int_0^1 \xi^{-\alpha_j}(1-\xi)^{\alpha_n-2} d\xi = M_j B(1-\alpha_j, \alpha_n-1), \quad (t,\tau) \in [0,1]\times[0,1].$$

Analogously,

$$h_j(t,\tau) \ge m_j B(1-\alpha_j, \alpha_n - 1),\ (t,\tau) \in [0,1]\times[0,1].$$

Then we obtain (2.9). □

Definition 2.8. $u(t) \in C[0,1]$ is called a solution of the problem (1.2) if $u'(t)$ exists in $[0,1]$ and $u(t)$ satisfied the equation and the initial conditions in (1.2).

Lemma 2.9. *Let $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $a_j : [0,1] \to (0,+\infty)$ $(j = 1,2,\dots,n-1)$ are continuously differentiable, then $u(t)$ is a solution of the equation* (1.2) *if and only if $u(t) \in C[0,1]$ is the solution of the integral equation*

$$\begin{aligned} u(t) = &\sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t,\tau) d\tau \\ &+ \frac{1}{\Gamma(\alpha_n)} \int_0^t (t-\tau)^{\alpha_n-1} f(\tau, u(\tau)) d\tau \end{aligned} \tag{2.10}$$

Proof. "Necessity". Applying Lemma 2.3 and the initial conditions, we have

$$u(t) = \lambda_1 \sum_{j=1}^{n-1} I_{0+}^{\alpha_n}\big(a_j(t) D_{0+}^{\alpha_j} u(t)\big) + \lambda_2 I_{0+}^{\alpha_n} f(t,u(t)). \tag{2.11}$$

Combining Definition 2.2 and Lemma 2.5, we have

$$\begin{aligned} &I_{0+}^{\alpha_n}\big(a_j(t) D_{0+}^{\alpha_j} u(t)\big) \\ &= \frac{1}{\Gamma(\alpha_n)} \int_0^t (t-s)^{\alpha_n-1} a_j(s) \Big(\frac{d}{ds} \frac{1}{\Gamma(1-\alpha_j)} \int_0^s (s-\tau)^{-\alpha_j} u(\tau) d\tau\Big) ds \\ &= \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \Big((t-s)^{\alpha_n-1} a_j(s) \int_0^s (s-\tau)^{-\alpha_j} u(\tau) d\tau\Big)\Big|_{s=0}^{s=t} \\ &\quad - \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t \int_0^s \frac{d}{ds}\Big((t-s)^{\alpha_n-1} a_j(s)\Big)(s-\tau)^{-\alpha_j} u(\tau) d\tau ds \\ &= \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t \int_0^s (t-s)^{\alpha_n-2} g_j(t,s)(s-\tau)^{-\alpha_j} u(\tau) d\tau ds \\ &= \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t \int_\tau^t (t-s)^{\alpha_n-2} g_j(t,s)(s-\tau)^{-\alpha_j} u(\tau) ds\, d\tau \\ &= \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) \int_0^1 \xi^{-\alpha_j}(1-\xi)^{\alpha_n-2} \\ &\quad \times g_j(t, \tau + \xi(t-\tau)) d\xi d\tau \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t,\tau) d\tau,$$

where $j = 1, 2, \dots, n-1$. Thus, in view of (2.11), we derive (2.10).

"Sufficiency". Suppose that $u(t) \in C[0,1]$ is the solution of (2.10). Then we canshow that $u'(t)$ exists in $[0,1]$ by (2.10). Also by exploiting Lemma 2.4 and Lemma 2.5, we can deduce the equation (1.2) easily and prove that the initial conditions are satisfied. This completes the proof. □

Theorem 2.10 ([11]). *Let $(E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone of E and $c > 0$ be a constant. Suppose that there exists a concave nonnegative continuous functional ω on P with $\omega(x) \le \|x\|$ for all $x \in \overline{P}_c$. Let $B : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator. Assume there are numbers a, b and d with $0 < d < a < b \le c$ such that*

1. *$\{x \in P(\omega, a, b) : \omega(x) > a\} \neq \emptyset$ and $\omega(Bx) > a$ for all $x \in P(\omega, a, b)$;*
2. *$\|Bx\| < d$ for all $x \in \overline{P}_d$;*
3. *$\omega(Bx) > a$ for all $x \in P(\omega, a, c)$ with $\|Bx\| > b$.*

Then B has at least three fixed points x_1, x_2 and x_3 in $\overline{P}_c$. Furthermore, $x_1 \in P_d$, $x_2 \in \{x \in P(\omega, a, c) : \omega(x) > a\}$; $x_3 \in \overline{P}_c \backslash (P(\omega, a, c) \cup \overline{P}_d)$.

3. Existence and multiplicity of positive solutions

Let

$$K = \{x \in C[0,1] : x(t) \ge 0,\ t \in [0,1],\ \min_{t \in [0, l_1]} x(t) \ge L\|x\|\},$$

where $0 < l_1 < 1$ and $0 < L < 1$. Evidently, K is a cone of the Banach space $C[0,1]$. In the following we will assume that $d/a < L < 1$ (the constant d and a are defined in Theorem 3.4). Define $\omega : K \to [0, +\infty)$ by

$$\omega(u) = \min_{t \in [l_1,\, l_2]} u(t), \quad 0 < l_1 < l_2 \le 1.$$

It is easy to check that $\omega(u)$ is a concave nonnegative continuous functional on K, and satisfies $\omega(u) \le \|u\|$ for all $u \in K$.

Denote $C^+[0,1] = \{x \in C[0,1] : x(t) \ge 0,\ t \in [0,1]\}$. Then let us define three operators $A, B, T : C^+[0,1] \to C^+[0,1]$ as follows

$$(Au)(t) = \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t,\tau) d\tau,$$

$$(Bv)(t) = \frac{1}{\Gamma(\alpha_n)} \int_0^t (t-\tau)^{\alpha_n-1} f(\tau, v(\tau)) d\tau,$$

$$(T\varphi)(t) = (A\varphi)(t) + (B\varphi)(t),$$

where $u, v, \varphi \in C^+[0,1]$.

Lemma 3.1. *The operator $A : C^+[0,1] \to C^+[0,1]$ is continuous and compact.*

Proof. Obviously, A is continuous. So we only need to prove that A is compact. Let $U \subset C^+[0,1]$ be bounded; i.e., there exists a positive constant r such that $\|u\| \le r$, $\forall u \in U$, for each $u \in U$, via Lemma 2.7, we have

$$|(Au)(t)| = \Big| \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t,\tau) d\tau \Big|$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n-1} \frac{M_j B(1-\alpha_j, \alpha_n - 1)}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) d\tau \\
&\le \sum_{j=1}^{n-1} \frac{r M_j B(1-\alpha_j, \alpha_n - 1)}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \frac{t^{\alpha_n-\alpha_j}}{\alpha_n - \alpha_j} \\
&\le \sum_{j=1}^{n-1} \frac{r M_j B(1-\alpha_j, \alpha_n - 1)}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)} = P_1 r.
\end{aligned}
$$

Thus $\|Au\| \le P_1 r$. Hence $A(U)$ is bounded.

Next, let

$$
\gamma_1 = 2P_1 r, \quad \gamma_2 = \sum_{j=1}^{n-1} \frac{r}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)}.
$$

Since $h_j(t,\tau)$ $(j = 1, 2, \dots, n-1)$ is uniformly continuous on $[0,1] \times [0,1]$, $\forall \varepsilon > 0$, there exists a $\delta_j > 0$ $(\delta_j < 1)$ such that

$$
|h_j(t_1, \tau_1) - h_j(t_2, \tau_2)| \le \frac{\varepsilon}{3\gamma_2}, \tag{3.1}
$$

for all $(t_1, \tau_1), (t_2, \tau_2) \in [0,1] \times [0,1]$ with $|t_1 - t_2| \le \delta_j$ and $|\tau_1 - \tau_2| \le \delta_j$, $j = 1, 2, \dots, n-1$.

Next prove that $A(U)$ is equicontinuous. For the given $\varepsilon > 0$, there exists $\rho_j > 0 (1 \le j \le n-1)$ such that $|{t_2}^{\alpha_n-\alpha_j} - {t_1}^{\alpha_n-\alpha_j}| < \varepsilon/(3\gamma_1)$, where $|t_2 - t_1| < \rho_j$. Let

$$
\delta = \min\big\{\delta_1,\ \delta_2,\ \dots,\ \delta_{n-1},\ \rho_1,\ \rho_2, \dots, \rho_{n-1}, (\frac{\varepsilon}{3})^{1/(\alpha_n-\alpha_{n-1})}\big\}.
$$

For each $u \in U$, $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \le \delta (t_1 < t_2)$, we have

$$
\begin{aligned}
&|(Au)(t_1) - (Au)(t_2)| \\
&= \Big| \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^{t_1} (t_1-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t_1,\tau) d\tau \\
&\quad - \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^{t_2} (t_2-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t_2,\tau) d\tau \Big| \\
&\le \Big| \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^{t_1} [(t_1-\tau)^{\alpha_n-\alpha_j-1} - (t_2-\tau)^{\alpha_n-\alpha_j-1}] u(\tau) h_j(t_1,\tau) d\tau \Big| \\
&\quad + \Big| \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^{t_1} (t_2-\tau)^{\alpha_n-\alpha_j-1} u(\tau) [h_j(t_1,\tau) - h_j(t_2,\tau)] d\tau \Big| \\
&\quad + \Big| \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t_2,\tau) d\tau \Big| \\
&\le \sum_{j=1}^{n-1} \frac{r M_j B(1-\alpha_j, \alpha_n - 1)}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^{t_1} [(t_2-\tau)^{\alpha_n-\alpha_j-1} - (t_1-\tau)^{\alpha_n-\alpha_j-1}] d\tau \\
&\quad + (\frac{\varepsilon}{3\gamma_2}) \sum_{j=1}^{n-1} \frac{r}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^{t_1} (t_2-\tau)^{\alpha_n-\alpha_j-1} d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n-1}\frac{rM_jB(1-\alpha_j,\alpha_n-1)}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha_n-\alpha_j-1}d\tau\\
&=\sum_{j=1}^{n-1}\frac{rM_jB(1-\alpha_j,\alpha_n-1)}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)}[t_2^{\alpha_n-\alpha_j}-t_1^{\alpha_n-\alpha_j}-(t_2-t_1)^{\alpha_n-\alpha_j}]\\
&\quad+\big(\frac{\varepsilon}{3\gamma_2}\big)\sum_{j=1}^{n-1}\frac{r}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)}[t_2^{\alpha_n-\alpha_j}-(t_2-t_1)^{\alpha_n-\alpha_j}]\\
&\quad+\sum_{j=1}^{n-1}\frac{rM_jB(1-\alpha_j,\alpha_n-1)}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)}(t_2-t_1)^{\alpha_n-\alpha_j}\\
&\le r\sum_{j=1}^{n-1}\frac{M_jB(1-\alpha_j,\alpha_n-1)}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)}[t_2^{\alpha_n-\alpha_j}-t_1^{\alpha_n-\alpha_j}]+\big(\frac{\varepsilon}{3\gamma_2}\big)\gamma_2\\
&\quad+r\sum_{j=1}^{n-1}\frac{M_jB(1-\alpha_j,\alpha_n-1)}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)}(t_2-t_1)^{\alpha_n-\alpha_j}\\
&\le rP_1\big(\frac{\varepsilon}{3\gamma_1}\big)+(rP_1)\big(\frac{\varepsilon}{3\gamma_1}\big)+\frac{\varepsilon}{3}\le\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{3}<\varepsilon.
\end{aligned}
$$

Therefore, $A(U)$ is equicontinuous. And the Arzela-Ascoli theorem implies that $A(U)$ is relatively compact. Thus, the operator $A : C^+[0,1] \to C^+[0,1]$ is compact. □

Lemma 3.2. *The operator $B : C^+[0,1] \to C^+[0,1]$ is continuous and compact.*

Proof. It is obvious that the operator $B : C^+[0,1] \to C^+[0,1]$ is continuous. Similar to the proof of Lemma 3.1, by the Arzela-Ascoli theorem, we can conclude that the operator $B : C^+[0,1] \to C^+[0,1]$ is compact. Here we omit the proof. □

Lemma 3.3. *The operator $T : C^+[0,1] \to C^+[0,1]$ is continuous and compact.*

The above lemma is obtained from Lemmas 3.1 and 3.2. Now we present the main result of this article.

Theorem 3.4. *Let $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $a_j : [0,1] \to (0,+\infty)$ $(j = 1,2,\dots,n-1)$ are continuously differentiable. Assume that (H_1) holds and there exist three positive constants $0 < d < a < b$ such that the following conditions are satisfied.*

- (H2) *$P_1 < 1$ and $f(t,u) \le \Gamma(\alpha_n+1)(1-P_1)b$, for all $(t,u) \in [0,1]\times[0,b]$;*
- (H3) *$C_1P_2L < 1$, and $f(t,u) \ge C_2\, a$, for all $(t,u) \in [0,l_1]\times[La,b]$, where*

$$
C_1=\min\big\{\min_{t\in[l_1,\,l_2]}[t^{\alpha_n-\alpha_j}-(t-l_1)^{\alpha_n-\alpha_j}],\ j=1,2,\dots,n-1\big\},
$$

$$
C_2>\frac{\Gamma(\alpha_n+1)(1-C_1P_2L)}{l_1^{\alpha_n}};
$$

- (H4) *$P_1 < 1$ and $f(t,u) < \Gamma(\alpha_n+1)(1-P_1)d$, for all $(t,u) \in [0,1]\times[0,d]$.*

Then problem (1.2) has at least three positive solutions u_1, u_2 and u_3 in $\overline{K}_b$. Furthermore, $u_1 \in K_d$; $u_2 \in \{u \in K(\omega,a,b) : \omega(u) > a\}$; $u_3 \in \overline{K}_b\backslash(K(\omega,a,b)\cup\overline{K}_d)$.

Proof. From Section 2, we know that $K_d = \{u \in K : \|u\| < d\}$, $\overline{K}_d = \{u \in K : \|u\| \leq d\}$, $\overline{K}_b = \{u \in K : \|u\| \leq b\}$ and $K(\omega, a, b) = \{u \in K : \omega(u) \geq a, \|u\| \leq b\}$. We prove the results by three steps.

Step 1: $T : \overline{K}_b \to \overline{K}_b$ is a completely continuous operator. For any $u \in \overline{K}_b$, from (H_2) and Lemma 2.7, we have

$$
\begin{aligned}
(Tu)(t) &= \sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t,\tau)d\tau \\
&\quad + \frac{1}{\Gamma(\alpha_n)} \int_0^t (t-\tau)^{\alpha_n-1} f(\tau, u(\tau))d\tau \\
&\leq \sum_{j=1}^{n-1} \frac{M_j B(1-\alpha_j, \alpha_n-1)}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} b \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} d\tau \\
&\quad + \frac{\Gamma(\alpha_n+1)(1-P_1)b}{\Gamma(\alpha_n)} \int_0^t (t-\tau)^{\alpha_n-1} d\tau \\
&= \sum_{j=1}^{n-1} \frac{M_j B(1-\alpha_j, \alpha_n-1)}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)} bt^{\alpha_n-\alpha_j} + (1-P_1)bt^{\alpha_n} \\
&\leq P_1 b + (1-P_1)b = b.
\end{aligned}
$$

Thus, $\|Tu\| \leq b$, that is, $T : \overline{K}_b \to \overline{K}_b$. Also, T is a completely continuous operator via Lemma 3.3.

Step 2: $\{u \in K(\omega, a, b) : \omega(u) > a\} \neq \emptyset$ and $\omega(Tu) > a$ for all $u \in K(\omega, a, b)$. Take $u_0(t) = (a+b)/2$, then $\omega(u_0) = \min_{t\in[l_1, l_2]} u_0(t) = (a+b)/2 > a$ and $\|u_0\| = (a+b)/2 < b$. Thus, $u_0(t) \in \{u \in K(\omega, a, b) : \omega(u) > a\} \neq \emptyset$. For each $u \in K(\omega, a, b)$, applying condition (H_3), the definition of K and Lemma 2.7, we can show

$$
\begin{aligned}
\omega(Tu) &= \min_{t\in[l_1, l_2]} \Big(\sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t,\tau)d\tau \\
&\quad + \frac{1}{\Gamma(\alpha_n)} \int_0^t (t-\tau)^{\alpha_n-1} f(\tau, u(\tau))d\tau \Big) \\
&\geq \min_{t\in[l_1, l_2]} \Big(\sum_{j=1}^{n-1} \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^t (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau) h_j(t,\tau)d\tau \Big) \\
&\quad + \min_{t\in[l_1, l_2]} \Big(\frac{1}{\Gamma(\alpha_n)} \int_0^t (t-\tau)^{\alpha_n-1} f(\tau, u(\tau))d\tau \Big) \\
&\geq \min_{t\in[l_1, l_2]} \Big(\sum_{j=1}^{n-1} \frac{m_j B(1-\alpha_j, \alpha_n-1)}{\Gamma(\alpha_n)\Gamma(1-\alpha_j)} \int_0^{l_1} (t-\tau)^{\alpha_n-\alpha_j-1} u(\tau)d\tau \Big) \\
&\quad + \frac{1}{\Gamma(\alpha_n)} \int_0^{l_1} (l_1-\tau)^{\alpha_n-1} f(\tau, u(\tau))d\tau \\
&\geq \min_{t\in[l_1, l_2]} \Big(\sum_{j=1}^{n-1} \frac{m_j B(1-\alpha_j, \alpha_n-1)}{(\alpha_n-\alpha_j)\Gamma(\alpha_n)\Gamma(1-\alpha_j)} La[t^{\alpha_n-\alpha_j} - (t-l_1)^{\alpha_n-\alpha_j}] \Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_n)}\int_0^{l_1}(l_1-\tau)^{\alpha_n-1}C_2 a d\tau\\
&\geq C_1P_2La+\frac{C_2a}{\Gamma(\alpha_n+1)}l_1^{\alpha_n}\\
&> C_1P_2La+(1-C_1P_2L)a=a,
\end{aligned}
$$

which implies $\omega(Tu)>a$.

Step 3: $\|Tx\|<d$ for all $u\in\overline{K}_d$. Proceeding as step 1, we can obtain the result easily by making use of the condition (H_4) and Lemma 2.7.

Then we obtain the conclusion of the theorem by employing Theorem 2.10. □

Corollary 3.5. *If the conditions* (H2) *and* (H3) *in Theorem 3.4 are replaced by*

(H2') $P_1<1$ *and* $\limsup_{u\to+\infty}\max_{t\in[0,1]}\frac{f(t,u)}{u}<\Gamma(\alpha_n+1)(1-P_1)$;
(H3') $C_1P_2L<1$, *and* $f(t,u)\geq C_2a$, *for all* $(t,u)\in[0,1]\times[La,+\infty)$.

Then the conclusion of Theorem 3.4 holds.

Proof. Since (H2') holds, there exists $0<\sigma<\Gamma(\alpha_n+1)(1-P_1)$ and $r_1>0$, such that $f(t,u)\leq\sigma u$, for all $u\geq r_1$. Let $\beta=\max_{0\leq t\leq r_1}u(t)$, then

$$0\leq f(t,u)\leq\sigma u+\beta,\quad 0\leq u<+\infty.$$

Let $b>\max\{\beta/\big(\Gamma(\alpha_n+1)(1-P_1)-\sigma\big),\ a\}$. Combining this with condition (H3'), we obtain the conditions (H2) and (H3). Therefore, the conclusion of Theorem 3.4 holds. □

Corollary 3.6. *If the condition* (H4) *in Theorem 3.4 is replaced by*

(H4') $P_1<1$ *and* $\limsup_{u\to0^+}\max_{t\in[0,1]}\frac{f(t,u)}{u}<\Gamma(\alpha_n+1)(1-P_1)$.

Then the conclusion of Theorem 3.4 holds.

4. Examples

To illustrate our main results, we present an example:

$$
\begin{gathered}
D_{0+}^{1.5}u(t)-a_3(t)D_{0+}^{0.3}u(t)-a_2(t)D_{0+}^{0.2}u(t)-a_1(t)D_{0+}^{0.1}u(t)=f(t,u(t)),\\
u(0)=u'(0)=0,\quad 0\leq t\leq 1,
\end{gathered}
\tag{4.1}
$$

where

$$
\begin{gathered}
a_1(t)=\ln(t^2+4),\quad a_2(t)=\frac{1}{8}(\sin t+1),\quad a_3(t)=\frac{1}{12}\big(\frac{t^2}{t^2+1}+1\big),\\
f(t,u)=(t^2+1)u+\beta,\quad \beta>0.
\end{gathered}
$$

Note that

$$0\leq {b_1}'(t)<\frac{1}{2\ln 4}<0.5,\quad 0\leq {b_2}'(t)<0.5,\quad 0\leq {b_3}'(t)\leq\frac{1}{6}<0.5.$$

A simple computation shows that $P_1\approx0.5146$. Let $l_1=0.98$, $L=0.95$, $d=2\beta$, $a=20\beta/9$, $b=4\beta$, $C_1<1$, and $C_2=1.4$. It is easy to check that all the hypotheses in Theorem 3.4 are satisfied. Thus, we conclude that problem (4.1) has at least three positive solutions.

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