Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 83, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLUTIONS TO OVER-DETERMINED SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS RELATED TO HAMILTONIAN STATIONARY LAGRANGIAN SURFACES

BANG-YEN CHEN

Abstract. This article concerns the over-determined system of partial differential equations

$$\Big(\frac{k}{f}\Big)_x + \Big(\frac{f}{k}\Big)_y = 0, \quad \frac{f_y}{k} = \frac{k_x}{f}, \quad \Big(\frac{f_y}{k}\Big)_y + \Big(\frac{k_x}{f}\Big)_x = -\varepsilon f k\,.$$

It was shown in [6, Theorem 8.1] that this system with $\varepsilon = 0$ admits traveling wave solutions as well as non-traveling wave solutions. In this article we solve completely this system when $\varepsilon \neq 0$. Our main result states that this system admits only traveling wave solutions, whenever $\varepsilon \neq 0$.

1. Introduction

A submanifold M of a Kähler manifold $\tilde{M}$ is called Lagrangian if the complex structure J of $\tilde{M}$ interchanges each tangent space T_pM with the corresponding normal space $T_p^{\perp}M$, $p \in M$ (cf. [1]).

A vector field X on a Kähler manifold $\tilde{M}$ is called Hamiltonian if $\mathcal{L}_X\omega = f\omega$ for some function $f \in C^{\infty}(\tilde{M})$, where $\mathcal{L}$ is the Lie derivative. Thus, there is a smooth real-valued function φ on $\tilde{M}$ such that $X = J\tilde{\nabla}\varphi$, where $\tilde{\nabla}$ is the gradient in $\tilde{M}$. The diffeomorphisms of the flux ψ_t of X satisfy $\psi_t\omega = e^{h_t}\omega$. Thus they transform Lagrangian submanifolds of $\tilde{M}$ into Lagrangian submanifolds. A normal vector field ξ to a Lagrangian immersion $\psi : M \to \tilde{M}$ is called Hamiltonian if $\xi = J\nabla f$, for some $f \in C^{\infty}(M)$, where ∇f is the gradient of f. A Lagrangian submanifold of a Kähler manifold is called Hamiltonian stationary if it is a critical point of the volume under Hamiltonian deformations.

Related to the classification of Hamiltonian stationary Lagrangian surfaces of constant curvature ε in a Kähler surface of constant holomorphic sectional curvature 4ε via a construction method introduced by Chen, Dillen, Verstraelen and Vrancken in [4] (see also [2, 3, 5]), one has to determine the exact solutions of the following overdetermined system of PDEs (see [6, 7] for details):

$$\Big(\frac{k}{f}\Big)_x + \Big(\frac{f}{k}\Big)_y = 0, \quad \frac{f_y}{k} = \frac{k_x}{f}, \quad \Big(\frac{f_y}{k}\Big)_y + \Big(\frac{k_x}{f}\Big)_x = -\varepsilon f k. \tag{1.1}$$

2000 *Mathematics Subject Classification.* 35N05, 35C07, 35C99.
Key words and phrases. Over-determined PDE system; traveling wave solution; exact solution; Hamiltonian stationary Lagrangian surfaces.

Submitted December 1, 2011. Published May 23, 2012.

This over-determined system was solved completely in [6] for the case $\varepsilon = 0$. In particular, it was shown that system (1.1) with $\varepsilon = 0$ admits traveling wave solutions as well as non-traveling wave solutions. More precisely, we have the following result from [6, Theorem 8.1].

Theorem 1.1. *The solutions $\{f, k\}$ of the over-determined PDE system*

$$\Big(\frac{k}{f}\Big)_x + \Big(\frac{f}{k}\Big)_y = 0, \quad \frac{f_y}{k} = \frac{k_x}{f}, \quad \Big(\frac{f_y}{k}\Big)_y + \Big(\frac{k_x}{f}\Big)_x = 0,$$

are the following:

$$f(x,y) = \pm k(x,y) = ae^{b(x+y)}; \tag{1.2}$$

$$f(x,y) = ame^{b(m^2x+y)}, \quad k(x,y) = \pm ae^{b(m^2x+y)}; \tag{1.3}$$

$$f(x,y) = \frac{a}{\sqrt{x}} e^{c \arctan\sqrt{-y/x}}, \quad k(x,y) = \pm\frac{a}{\sqrt{-y}} e^{c\arctan\sqrt{-y/x}}, \tag{1.4}$$

where a, b, c, m are real numbers with $a, c, m \neq 0$ and $m \neq \pm 1$.

The main purpose of this article is to solve the over-determined system (1.1) completely. Our main result states that the over-determined PDE system (1.1) with $\varepsilon \neq 0$ admits only traveling wave solutions.

2. Exact solutions of the over-determined system with $\varepsilon = 1$

Theorem 2.1. *The solutions $\{f, k\}$ of the over-determined PDE system*

$$\Big(\frac{k}{f}\Big)_x + \Big(\frac{f}{k}\Big)_y = 0, \quad \frac{f_y}{k} = \frac{k_x}{f}, \quad \Big(\frac{f_y}{k}\Big)_y + \Big(\frac{k_x}{f}\Big)_x = -fk, \tag{2.1}$$

are the traveling wave solutions given by

$$f = cm \operatorname{sech}\Big(\frac{c(m^2x+y)}{\sqrt{1+m^2}}\Big), \quad k = c \operatorname{sech}\Big(\frac{c(m^2x+y)}{\sqrt{1+m^2}}\Big), \tag{2.2}$$

where c and m are nonzero real numbers.

Proof. First, let us assume that $f = mk$ for some nonzero real number m. Then the first equation of system (2.1) holds identically.

If $\{f, k\}$ satisfies the second equation of system (2.1), then we have $k_x = m^2 k_y$, which implies that

$$f = mK(s), \quad k = K(s), \quad s = m^2x + y, \tag{2.3}$$

for some function K. By substituting (2.3) into the third equation in system (2.1), we find

$$(1+m^2)(K(s)K''(s) - (K')^2(s)) + K^4(s) = 0. \tag{2.4}$$

Since $K \neq 0$, (2.4) implies that K is non-constant. Thus (2.4) gives

$$(1+m^2)\frac{K'^2}{K^2} + K^2 = c^2 \tag{2.5}$$

for some positive real number c_1. After solving (2.5) we conclude that, up to translations and sign, K is given by

$$K = c \operatorname{sech}\Big(\frac{cs}{\sqrt{1+m^2}}\Big). \tag{2.6}$$

Now, after combining (2.3) and (2.6) we obtain the traveling wave solutions of the over-determined PDE system given by (2.2).

Next, let us assume that $v = f(x,y)/k(x,y)$ is a non-constant function. It follows from the first equation of system (2.1) that $\frac{\partial v}{\partial y} \neq 0$. Therefore, after solving the first equation of system (2.1), we obtain

$$y = -q(v) - xv^2, \quad f = vk \tag{2.7}$$

for some function q. Let us consider the new variables (u,v) with $u = x$ and v being defined by (2.7). Then we have

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 0, \quad \frac{\partial y}{\partial u} = -v^2, \quad \frac{\partial y}{\partial v} = -q'(v) - 2uv, \tag{2.8}$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = \frac{-v^2}{q'(v)+2uv}, \quad \frac{\partial v}{\partial y} = \frac{-1}{q'(v)+2uv}, \tag{2.9}$$

It follows from (2.7), (2.8) and (2.9) that

$$f_y = -\frac{k + vk_v}{q'(v)+2uv}, \quad k_x = k_u - \frac{v^2 k_v}{q'(v)+2uv}. \tag{2.10}$$

By substituting (2.7), and (2.10) into the second equation of (2.1) we obtain

$$k_u + \Big(\frac{v}{q'(v)+2uv}\Big)k = 0. \tag{2.11}$$

After solving this equation we obtain

$$f = \frac{vA(v)}{\sqrt{2uv+q'(v)}}, \quad k = \frac{A(v)}{\sqrt{2uv+q'(v)}}. \tag{2.12}$$

Now, by applying (2.9) and (2.12), we find

$$\begin{gathered}
f_x = \frac{v^2A(v)(vq''(v) - 6uv - 4q'(v)) - 2v^3A'(v)(2uv+q'(v))}{2(2uv+q'(v))^{5/2}}, \\
f_y = \frac{A(v)(vq''(v) - 2uv - 2q'(v)) - 2vA'(v)(2uv+q'(v))}{2(2uv+q'(v))^{5/2}}, \\
k_x = \frac{vA(v)(vq''(v) - 2uv - 2q'(v)) - 2v^2A'(v)(2uv+q'(v))}{2(2uv+q'(v))^{5/2}}, \\
k_y = \frac{A(v)(2u+q''(v)) - 2A'(v)(2uv+q'(v))}{2(2uv+q'(v))^{5/2}}.
\end{gathered} \tag{2.13}$$

After substituting (2.13) into the last equation in (2.1) and by applying (2.8) and (2.9), we obtain a polynomial equation of degree 3 in u:

$$A^4(v)u^3 + B(v)u^2 + C(v)u + D(v) = 0, \tag{2.14}$$

where B, C and D are functions in v. Consequently, we must have $A(v) = 0$ which is a contradiction according to (2.14). Therefore this case cannot happen. □

3. Exact solutions of the over-determined system with $\varepsilon = -1$

Theorem 3.1. *The solutions $\{f,k\}$ of the over-determined PDE system*

$$\Big(\frac{k}{f}\Big)_x + \Big(\frac{f}{k}\Big)_y = 0, \quad \frac{f_y}{k} = \frac{k_x}{f}, \quad \Big(\frac{f_y}{k}\Big)_y + \Big(\frac{k_x}{f}\Big)_x = fk, \tag{3.1}$$

are the following traveling wave solutions:

$$f = cm\operatorname{csch}\Big(\frac{c(m^2x+y)}{\sqrt{1+m^2}}\Big), \quad k = c\operatorname{csch}\Big(\frac{c(m^2x+y)}{\sqrt{1+m^2}}\Big); \tag{3.2}$$

$$f = cm \sec\Big(\frac{c(m^2x+y)}{\sqrt{1+m^2}}\Big), \quad k = c\sec\Big(\frac{c(m^2x+y)}{\sqrt{1+m^2}}\Big); \tag{3.3}$$

$$f = \frac{m\sqrt{1+m^2}}{m^2x+y}, \quad k = \frac{\sqrt{1+m^2}}{m^2x+y}, \tag{3.4}$$

where c and m are nonzero real numbers.

Proof. First, let us assume that $f = mk$ for some nonzero real number m. Then the first equation of system (3.1) holds identically. As in the previous section, we obtain from the second equation of system (3.1) that

$$f = mK(s), \quad k = K(s), \quad s = m^2x + y, \tag{3.5}$$

for some function K. By substituting (2.3) into the third equation in system (3.1), we find

$$(1+m^2)(K(s)K''(s) - K'^2(s)) = K^4(s). \tag{3.6}$$

Since $K \neq 0$, (3.6) implies that K is non-constant. Thus (2.4) gives

$$(1+m^2)\frac{K'^2}{K^2} - K^2 = c_1 \tag{3.7}$$

for some real number c_1.

If $c_1 > 0$, we put $c_1 = c^2$ with $c \neq 0$. Then (3.7) becomes

$$(1+m^2)\frac{K'^2}{K^2} - K^2 = c^2. \tag{3.8}$$

After solving (3.8) we conclude that, up to translations and sign, K is given by

$$K = c\,\mathrm{csch}\Big(\frac{cs}{\sqrt{1+m^2}}\Big). \tag{3.9}$$

Now, after combining (3.5) and (3.9) we obtain the traveling wave solutions (3.2).

If $c_1 < 0$, we put $c_1 = -c^2$ with $c \neq 0$. Then (3.7) becomes

$$(1+m^2)\frac{(K')^2}{K^2} - K^2 = -c^2. \tag{3.10}$$

After solving (3.10) we conclude that, up to translations and sign, K is given by

$$K = c\sec\big(\frac{cs}{\sqrt{1+m^2}}\big). \tag{3.11}$$

By combining (3.5) and (3.11) we obtain the traveling wave solutions of the over-determined PDE system given by (3.3).

If $c_1 = 0$, (3.7) becomes

$$(1+m^2)K'^2 = K^4. \tag{3.12}$$

After solving (3.12) we conclude that, up to translations and sign, K is given by

$$K = \frac{\sqrt{1+m^3}}{m^2x+y}, \tag{3.13}$$

which yields solutions (3.4).

Finally, by applying a argument similar to the one given in section 2, we conclude that the remaining case is impossible. □

4. Applications to Hamiltonian-stationary Lagrangian surfaces

Let (M_j, g_j), $j = 1, \dots, m$, be Riemannian manifolds, f_i a positive function on $M_1 \times \cdots \times M_m$ and $\pi_i : M_1 \times \cdots \times M_m \to M_i$ the i-th canonical projection for $i = 1, \dots, m$. The *twisted product*

$$_{f_1}M_1 \times \cdots \times_{f_m} M_m$$

is the product manifold $M_1 \times \cdots \times M_m$ equipped with the twisted product metric g defined by

$$g(X,Y) = f_1^2 \cdot g_1(\pi_{1*}X, \pi_{1*}Y) + \cdots + f_m^2 \cdot g_m(\pi_{m*}X, \pi_{m*}Y). \tag{4.1}$$

Let $N^{n-\ell}(\varepsilon)$ be an $(n-\ell)$-dimensional real space form of constant curvature ε. For $\ell < n-1$ we consider the following twisted product:

$$_{f_1}I_1 \times \cdots \times_{f_\ell} I_\ell \times_1 N^{n-\ell}(\varepsilon) \tag{4.2}$$

with twisted product metric given by

$$g = f_1^2 dx_1^2 + \cdots + f_\ell^2 dx_\ell^2 + g_0, \tag{4.3}$$

where g_0 is the canonical metric of $N^{n-\ell}(\varepsilon)$ and $I_1, \dots, I_\ell$ are open intervals. When $\ell = n-1$, we shall replace $N^{n-1}(\varepsilon)$ by an open interval. If the twisted product is a real-space-form $M^n(\varepsilon)$, it is called a *twisted product decomposition* of $M^n(\varepsilon)$ (cf. [4]). We denote such a decomposition by $\mathcal{T}P^n_{f_1 \dots f_\ell}(\varepsilon)$.

We recall the following result from [6, Theorem 3.2] (see also [7]).

Theorem 4.1. *Let f, k be a pair of positive functions satisfying PDE system* (2.1). *Then, up to rigid motions of $\tilde{M}^2(4\varepsilon)$, there is a unique H-stationary Lagrangian isometric immersion:*

$$L_{f,k} : \mathcal{T}P^2_{f^2k^2}(\varepsilon) \to \tilde{M}^2(4\varepsilon) \tag{4.4}$$

whose second fundamental form satisfies

$$h\Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\Big) = J\frac{\partial}{\partial x_1}, \quad h\Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big) = 0, \quad h\Big(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\Big) = J\frac{\partial}{\partial x_2}. \tag{4.5}$$

If the two twistor functions f^2 and k^2 are equal and if they satisfy PDE system (1.1), then the corresponding Hamiltonian-stationary adapted Lagrangian immersion of $\mathcal{T}P^2_{f^2k^2}(\varepsilon)$ is said to be *of type I*. If the two twistor functions f^2 and k^2 are unequal, then the corresponding Hamiltonian-stationary adapted Lagrangian immersion is said to be *of type II*.

By applying Theorem 2.1 and results of [6, Section 5], we can determine all type II adapted Hamiltonian stationary Lagrangian surfaces in the complex projective plane $CP^2(4)$ of constant holomorphic sectional curvature 4. In fact, by combining Theorem 2.1 and [6, Section 5] we have the following.

Corollary 4.2. *A type II adapted Hamiltonian-stationary Lagrangian surface in $CP^2(4)$ is congruent to $\pi \circ L$, where $\pi : S^5(1) \to CP^2(4)$ is the Hopf fibration and L is given by*

$$L(x,y) = \frac{\operatorname{sech}\big(\frac{m^2x+y}{\sqrt{1+m^2}}\big)}{\sqrt{2+m^2}}\Big(\frac{2m\sqrt{2+m^2}}{\sqrt{1+5m^2}}e^{i(x+y)/2}\sin\Big(\frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}}(x-y)\Big),$$
$$e^{i(x+y)/2}\Big[\sqrt{1+m^2}\cos\Big(\frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}}(x-y)\Big)$$

$$- i(1-m^2)\sqrt{1+5m^2}\sin\Big(\frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}}(x-y)\Big)\Big],$$
$$\frac{1}{\sqrt{2}}\sqrt{1+\cosh\big(\frac{2m^2x+2y}{\sqrt{1+m^2}}\big)}\Big(1-i\sqrt{1+m^2}\tanh\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big)\Big)\Big)$$

for some positive number $m \neq 1$.

Similarly, by applying Theorem 3.1 and results of [6, Section 7] we can determine all type II adapted Hamiltonian-stationary Lagrangian surfaces in the complex hyperbolic plane $CH^2(-4)$ of constant holomorphic sectional curvature -4. More precisely, we have the following result.

Corollary 4.3. *A type II adapted Hamiltonian-stationary Lagrangian surface in $CH^2(-4)$ is congruent to $\pi \circ L$, where $\pi : H_1^5(-1) \to CH^2(-4)$ denotes the Hopf fibration and $L(x,y)$ is given by one of the following five immersions:*
(a)
$$L = \Big(1 - \frac{i(1+m^2)}{m^2x+y},\ \frac{m\sqrt{1+m^2}}{m^2x+y}e^{ix},\ \frac{\sqrt{1+m^2}}{m^2x+y}e^{iy}\Big);$$
(b)
$$L = \operatorname{sech}\Big(\frac{x+3y}{2\sqrt{3}}\Big)\Big(\frac{x-y+4i}{2}e^{i(x+y)/2},\ \frac{x-y}{2}e^{i(x+y)/2},\ \sqrt{3}+2i\tan\Big(\frac{x+3y}{2\sqrt{3}}\Big)\Big);$$
(c)
$$L = \Big(\frac{\sqrt{3m^4+2m^2-1}\cosh(\alpha(x-y))+i(m^2-1)\sinh(\alpha(x-y))}{m\sqrt{3m^2-1}e^{-i(x+y)/2}}\sec\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big),$$
$$\frac{2me^{i(x+y)/2}}{\sqrt{3m^2-1}}\sec\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big)\sinh(\alpha(x-y)),\ \frac{1}{m}+\frac{i\sqrt{1+m^2}}{m}\tan\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big)\Big);$$
(d)
$$L = \Big(\frac{\sqrt{1-2m^2-3m^4}\cos(\beta(x-y))+i(1-m^2)\sin(\beta(x-y))}{m\sqrt{1-3m^2}e^{-i(x+y)/2}}\sec\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big),$$
$$\frac{2me^{i(x+y)/2}}{\sqrt{1-3m^2}}\sec\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big)\sinh(\beta(x-y)),\ \frac{1}{m}+\frac{i\sqrt{1+m^2}}{m}\tan\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big)\Big);$$
(e)
$$L = \frac{1}{\sqrt{2+m^2}}\operatorname{csch}\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big)\Big(\sinh\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big)-i\sqrt{1+m^2}\cosh\Big(\frac{m^2x+y}{\sqrt{1+m^2}}\Big),$$
$$e^{i(x+y)/2}\Big\{\sqrt{1+m^2}\cos\Big(\frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}}(x-y)\Big)$$
$$+\frac{i(m^2-1)}{\sqrt{1+5m^2}}\sin\Big(\frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}}(x-y)\Big)\Big\},$$
$$\frac{2m\sqrt{2+m^2}}{\sqrt{1+5m^2}}e^{i(x+y)/2}\sin\Big(\frac{\sqrt{1+5m^2}}{2\sqrt{1+m^2}}(x-y)\Big)\Big),$$
where α and β are constants given by
$$\alpha = \frac{\sqrt{3m^2-1}}{2\sqrt{1+m^2}},\quad \beta = \frac{\sqrt{1-3m^2}}{2\sqrt{1+m^2}}.$$

References

[1] Chen, B.-Y.: *Pseudo-Riemannian geometry, δ-invariants and Applications*, World Scientific, Hackensack, NJ, 2011.

[2] Chen, B.-Y.: *Classification of a family of Hamiltonian-stationary Lagrangian submanifolds in complex hyperbolic 3-space*, Taiwanese J. Math. 12 (2008), 1261–1284.

[3] Chen, B.-Y.; Dillen, F.: *Warped product decompositions of real space forms and Hamiltonian stationary Lagrangian submanifolds*, Nonlinear Anal. 69 (2008), 3462–3494.

[4] Chen, B.-Y.; Dillen, F.; Verstraelen, L.; Vrancken, L.: *Lagrangian isometric immersions of a real-space-form $M^n(c)$ into a complex-space-form $\tilde{M}^n(4c)$*, Math. Proc. Cambridge Philo. Soc. **124** (1998), 107–125.

[5] Chen, B.-Y.; Garay, O. J.: *Classification of Hamiltonian-stationary Lagrangian submanifolds of constant curvature in CP^3 with positive relative nullity*, Nonlinear Anal. 69 (2008), 747–762.

[6] Chen, B.-Y.; Garay, O. J.; Zhou, Z.: *Hamiltonian stationary Lagrangian surfaces of constant curvature ε in complex space form $\tilde{M}^2(4\varepsilon)$*, Nonlinear Anal. **71** (2009), 2640–2659.

[7] Dong, Y.; Han, Y.: *Some explicit examples of Hamiltonian minimal Lagrangian submanifolds in complex space forms*, Nonlinear Anal. **66** (2007), 1091–1099.

Bang-Yen Chen

Department of Mathematics, Michigan State University, 619 Red Cedar Road, East Lansing, Michigan 48824-1027, USA

E-mail address: bychen@math.msu.edu