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LOCAL AND GLOBAL EXISTENCE FOR THE LAGRANGIAN AVERAGED NAVIER-STOKES EQUATIONS IN BESOV SPACES

NATHAN PENNINGTON

Abstract. Through the use of a non-standard Leibntiz rule estimate, we prove the existence of unique short time solutions to the incompressible, isotropic Lagrangian Averaged Navier-Stokes equation with initial data in the Besov space $B^r_{p,q}(\mathbb{R}^n)$, $r > 0$, for $p > n$ and $n \geq 3$. When $p = 2$, we obtain unique local solutions with initial data in the Besov space $B^{n/2-1}_{2,q}(\mathbb{R}^n)$, again with $n \geq 3$, which recovers the optimal regularity available by these methods for the Navier-Stokes equation. Also, when $p = 2$ and $n = 3$, the local solution can be extended to a global solution for all $1 \leq q \leq \infty$. For $p = 2$ and $n = 4$, the local solution can be extended to a global solution for $2 \leq q \leq \infty$. Since $B^s_{2,2}(\mathbb{R}^n)$ can be identified with the Sobolev space $H^s(\mathbb{R}^n)$, this improves previous Sobolev space results, which only held for initial data in $H^{3/4}(\mathbb{R}^3)$.

1. Introduction

The Lagrangian Averaged Navier-Stokes (LANS) equation is a recently derived approximation to the Navier-Stokes equation. The equation is obtained via an averaging process applied at the Lagrangian level, resulting in a modified energy functional. The geodesics of this energy functional satisfy the Lagrangian Averaged Euler (LAE) equation, and the LANS equation is derived from the LAE equation in an analogous fashion to the derivation of the Navier-Stokes equation from the Euler equation. For an exhaustive treatment of this process, see [12], [13], [6] and [8]. In [9] and [2], the authors discuss the numerical improvements that use of the LANS equation provides over more common approximation techniques of the Navier-Stokes equation.

On a region without boundary, the isotropic, incompressible form of the LANS equation is given by

$$\begin{gathered}\partial_t u + (u \cdot \nabla)u + \operatorname{div} \tau^\alpha u = -(1-\alpha^2\Delta)^{-1}\nabla p + \nu\Delta u \\ u = u(t,x), \quad \operatorname{div} u = 0, \quad u(0,x) = u_0(x),\end{gathered} \tag{1.1}$$

with the terms defined as follows. First, $u : I \times \mathbb{R}^n \to \mathbb{R}^n$ for some time strip $I = [0,T)$ denotes the velocity of the fluid, $\alpha > 0$ is a constant, $p : I \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the fluid pressure, $\nu > 0$ is a constant due to the viscosity of the fluid, and $u_0 : \mathbb{R}^n \to \mathbb{R}^n$, with $\operatorname{div} u_0 = 0$. Next, the differential operators ∇, Δ, and div are

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spatial differential operators with their standard definitions. The term $(v\cdot\nabla)w$, also denoted $\nabla_v w$, is the vector field with j^{th} component $\sum_{i=1}^n v_i\partial_i w_j$. The Reynolds stress τ^α is given by

$$\tau^\alpha u = \alpha^2(1-\alpha^2\Delta)^{-1}[Def(u)\cdot Rot(u)],$$

where $Rot(u) = (\nabla u - \nabla u^T)/2$ and $Def(u) = (\nabla u + \nabla u^T)/2$. We remark that setting $\alpha = 0$ in equation (1.1) recovers the Navier-Stokes equation.

There is a wide variety of local existence results for the LANS equation in various settings, including [12, 6, 7, 11]. In [7], Marsden and Shkoller proved the existence of global solutions to the LANS equation with initial data in the Sobolev space $H^{3,2}(\mathbb{R}^3)$. In [11], this result was improved, achieving global existence for data in the space $H^{3/4,2}(\mathbb{R}^3)$ and local existence for initial data in the space $H^{n/2p,p}(\mathbb{R}^n)$.

The most significant obstacle to lowering the initial data regularity necessary to obtain these results is the nonlinear terms. These terms are typically controlled by the Leibnitz rule type estimate (see [3] for the original reference or Proposition 1.1 in[16]):

$$\|fg\|_{H^{s,p}} \le \|f\|_{H^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\|g\|_{H^{s,q_2}}, \tag{1.2}$$

where $1/p = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$, $s > 0$, and $\|\cdot\|_{H^{s,p}}$ denotes the Sobolev space norm. In this article, we obtain better regularity results by changing to the Besov space $B^s_{p,q}(\mathbb{R}^n)$ setting, where we have access to the following, non-standard Leibnitz rule type result:

$$\|fg\|_{B^s_{p,q}} \le \|f\|_{B^{s_1}_{p_1,q}}\|g\|_{B^{s_2}_{p_2,q}}, \tag{1.3}$$

provided $s_1 < n/p_1$, $s_2 < n/p_2$, $s_1 + s_2 > 0$, $1/p \le 1/p_1 + 1/p_2$, and $s = s_1 + s_2 - n(1/p_1 + 1/p_2 - 1/p)$. This is Proposition 2.1 below, and can be found in [1]. This result has two advantages over equation (1.2). First, equation (1.3) allows for "spreading" the regularity s between the two terms. This is not of particular value here, since in the LANS equation the nonlinearity is of quadratic type, but it is useful when estimating products of functions with varying degrees of regularity (see, for example, [10]). The second advantage (and the the one used in this article) equation (1.3) has over (1.2) is that there is no requirement that $s > 0$ and, by allowing $s_1 + s_2 > s$, p_1, p_2 and p are no longer required to satisfy the Holder condition.

This is particularly helpful when dealing with negative regularity operators, like $\operatorname{div}(1-\alpha^2\Delta)^{-1}$. Specifically,

$$\|\operatorname{div}(\tau^\alpha(u))\|_{B^r_{p,q}} \le \|Def(u)\cdot Rot(u)\|_{B^{r-1}_{p,q}}.$$

For $r < 1$, further estimating of this term using equation (1.2) would require first embedding back to $B^s_{p,q}(\mathbb{R}^n)$, $s > 0$, and then applying the equation, which "wastes" $r-1$ derivatives. Using equation (1.3), we manage to make some (though not full) use of these $r-1$ derivatives. In the statement of our local existence results below, we will further elaborate on the benefits of equation (1.3).

The paper is organized as follows. We devote the rest of this section to defining solution spaces and stating our main theorems. In Section 2 we outline some fundamental, known Besov space results. In Sections 3 and 4, we prove Theorems 1.1 and 1.2, respectively, stated below. In Section 5 we prove Theorem 1.3, which extends some of the local solutions from Theorem 1.2 to global solutions. Section 6 contains a technical result necessary for the proof of Theorem 1.3.

As mentioned above, we denote Besov spaces by $B^s_{p,q}(\mathbb{R}^n)$, with norm denoted by $\|\cdot\|_{B^s_{p,q}} = \|\cdot\|_{s,p,q}$ (a complete definition of these spaces can be found in Section 2). We define the space

$$C^T_{a;s,p,q} = \{f \in C((0,T) : B^s_{p,q}(\mathbb{R}^n)) : \|f\|_{a;s,p,q} < \infty\},$$

where

$$\|f\|_{a;s,p,q} = \sup\{t^a \|f(t)\|_{s,p,q} : t \in (0,T)\},$$

$T > 0$, $a \geq 0$, and $C(A : B)$ is the space of continuous functions from A to B. We let $\dot{C}^T_{a;s,p,q}$ denote the subspace of $C^T_{a;s,p,q}$ consisting of f such that

$$\lim_{t\to 0^+} t^a f(t) = 0 \quad (\text{in } B^s_{p,q}(\mathbb{R}^n)).$$

Note that while the norm $\|\cdot\|_{a;s,p,q}$ lacks an explicit reference to T, there is an implicit T dependence. We also say $u \in BC(A : B)$ if $u \in C(A : B)$ and $\sup_{a\in A} \|u(a)\|_B < \infty$. Lastly, setting $\mathbb{M}((0,T) : \mathbb{E})$ to be the set of measurable functions defined on $(0,T)$ with values in the space $\mathbb{E}$, we define

$$L^a((0,T) : B^s_{p,q}(\mathbb{R}^n)) = \big\{f \in \mathbb{M}((0,T) : B^s_{p,q}(\mathbb{R}^n)) : \Big(\int_0^T \|f(t)\|^a_{s,p,q} dt\Big)^{1/a} < \infty\big\}.$$

Finally, because the Navier-Stokes equation is globally well-posed with initial data in $L^2(\mathbb{R}^2)$ (see, for example, Chapter 17 in [15]), we will restrict ourselves to the case where $n \geq 3$. We are now ready to state our two local existence theorems.

Theorem 1.1. *Let $0 < r_1 < n/p$, with $p > n$, and let $u_0 \in B^{r_1}_{p,q}(\mathbb{R}^n)$ be divergence free. Then there exists a unique local solution u to the LANS equation* (1.1)*, where*

$$u \in BC([0,T) : B^{r_1}_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{(r_2-r_1)/2;r_2,p,q}, \tag{1.4}$$

$1 < r_2 < r_1 + 1$, and T is a non-increasing function of $\|u_0\|_{B^{r_1}_{p,q}}$, with $T = \infty$ if $\|u_0\|_{s^+,2,q}$ is sufficiently small.

Similarly, with $0 < r_1 < n/p$, $p > n$, and $u_0 \in B^{r_1}_{p,q}(\mathbb{R}^n)$ divergence free, there exists a unique local solution u to the LANS equation (1.1)*, where*

$$u \in BC([0,T) : B^{r_1}_{p,q}(\mathbb{R}^n)) \cap L^a((0,T) : B^{r_2}_{p,q}(\mathbb{R}^n)), \tag{1.5}$$

$a = 2/(r_2 - r_1)$, $1 < r_2 < r_1 + 1$, and T is a non-increasing function of $\|u_0\|_{B^{r_1}_{p,q}}$, with $T = \infty$ if $\|u_0\|_{s^+,2,q}$ is sufficiently small.

Theorem 1.2. *Let $u_0 \in B^{n/2-1}_{2,q}(\mathbb{R}^n)$ be divergence free. Then there exists a unique local solution u to the LANS equation* (1.1)*, where*

$$u \in BC([0,T) : B^{n/2-1}_{2,q}(\mathbb{R}^n)) \cap \dot{C}^T_{(r-n/2+1)/2;r,2,q}, \tag{1.6}$$

$\max(1, n/2-1) < r < n/2$ and T is a non-increasing function of $\|u_0\|_{B^{n/2-1}_{2,q}}$, with $T = \infty$ if $\|u_0\|_{s^+,2,q}$ is sufficiently small.

Similarly, with $u_0 \in B^{n/2-1}_{2,q}(\mathbb{R}^n)$ divergence free, there exists a unique local solution u to the LANS equation (1.1)*, where*

$$u \in BC([0,T) : B^{n/2-1}_{2,q}(\mathbb{R}^n)) \cap L^a((0,T) : B^r_{2,q}(\mathbb{R}^n)), \tag{1.7}$$

$a = 2/(r - n/2 + 1)$, $\max(1, n/2-1) < r < n/2$, and T is a non-increasing function of $\|u_0\|_{B^{n/2-1}_{2,q}}$, with $T = \infty$ if $\|u_0\|_{s^+,2,q}$ is sufficiently small.

We pause here to address the distinction between these two results. Using techniques like those in [5], the nonlinear term from the Navier-Stokes equation $\operatorname{div}(u\otimes u)$ can be controlled provided the initial data has regularity at least $n/p-1$. Using Proposition 2.1, we are able to control the LANS specific nonlinear term, $\operatorname{div}(\tau^\alpha(u))$, provided the initial data has strictly positive regularity. Thus, when $p > n$, the limiting factor will be LANS specific term $\operatorname{div}(\tau^\alpha(u))$, and we obtain local existence provided the initial data has strictly positive regularity. For $p = 2$, the limiting factor is the Navier-Stokes nonlinear term $\operatorname{div}(u \otimes u)$, and we obtain existence provided the data has regularity $n/2-1$. We remark that this means, for $p = 2$, the additional nonlinear term in the LANS equation is no longer limiting the existence result.

Finally, we state our global existence extension.

Theorem 1.3. *When $n = 3$, the local solutions with initial data $u_0 \in B^{1/2}_{2,q}(\mathbb{R}^3)$ from Theorem 1.2 can be extended to global solutions. When $n = 4$, the local solutions with initial data $u_0 \in B^1_{2,q}(\mathbb{R}^4)$, with $2 \le q \le \infty$, can be extended to global solutions. In particular, the local solutions from Theorem 1.2 can be extended to global solutions when $u_0 \in B^{n/2-1}_{2,2}(\mathbb{R}^n) = H^{n/2-1,2}(\mathbb{R}^n)$ for $n = 3, 4$.*

We remark that this last statement improves the result from [11], which only gave global existence for initial data in $H^{3/4,2}(\mathbb{R}^3)$.

2. Besov spaces

We begin by defining the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$. Let $\psi_0 \in \mathcal{S}$ be an even, radial function with Fourier transform $\hat\psi_0$ that has the following properties:

$$\begin{gathered}
\hat\psi_0(x) \ge 0\\
\operatorname{support}\hat\psi_0 \subset A_0 := \{\xi \in \mathbb{R}^n : 2^{-1} < |\xi| < 2\}\\
\sum_{j\in\mathbb{Z}} \hat\psi_0(2^{-j}\xi) = 1, \quad \text{for all } \xi \ne 0.
\end{gathered}$$

We then define $\hat\psi_j(\xi) = \hat\psi_0(2^{-j}\xi)$ (from Fourier inversion, this also means $\psi_j(x) = 2^{jn}\psi_0(2^j x)$), and remark that $\hat\psi_j$ is supported in $A_j := \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}$. We also define Ψ by

$$\hat\Psi(\xi) = 1 - \sum_{k=0}^{\infty} \hat\psi_k(\xi). \tag{2.1}$$

We define the Littlewood Paley operators Δ_j and S_j by

$$\Delta_j f = \psi_j * f, \quad S_j f = \sum_{k=-\infty}^{j} \Delta_k f,$$

and record some properties of these operators. Applying the Fourier Transform and recalling that $\hat\psi_j$ is supported on $2^{j-1} \le |\xi| \le 2^{j+1}$, it follows that

$$\begin{gathered}
\Delta_j\Delta_k f = 0, \quad |j-k| \ge 2\\
\Delta_j(S_{k-3} f \Delta_k g) = 0 \quad |j-k| \ge 4,
\end{gathered} \tag{2.2}$$

and, if $|i-k| \le 2$, then

$$\Delta_j(\Delta_k f \Delta_i g) = 0 \quad j > k+4. \tag{2.3}$$

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ we define the space $\tilde{B}^s_{p,q}(\mathbb{R}^n)$ to be the set of distributions such that

$$\|u\|_{\tilde{B}^s_{p,q}} = \Big(\sum_{j=0}^{\infty} (2^{js} \|\Delta_j u\|_{L^p})^q \Big)^{1/q} < \infty,$$

with the usual modification when $q = \infty$. Finally, we define the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ by the norm

$$\|f\|_{B^s_{p,q}} = \|\Psi * f\|_p + \|f\|_{\tilde{B}^s_{p,q}},$$

for $s > 0$. For $s > 0$, we define $B^{-s}_{p',q'}$ to be the dual of the space $B^s_{p,q}$, where p', q' are the Holder-conjugates to p, q.

These Littlewood-Paley operators are also used to define Bony's paraproduct. We have

$$fg = \sum_k S_{k-3} f \Delta_k g + \sum_k S_{k-3} g \Delta_k f + \sum_k \Delta_k f \sum_{l=-2}^{2} \Delta_{k+l} g. \tag{2.4}$$

The estimates (2.2) and (2.3) imply that

$$\begin{aligned} \Delta_j(fg) \le& \sum_{k=-3}^{3} \Delta_j(S_{j+k-3} f \Delta_{j+k} g) + \sum_{k=-3}^{3} \Delta_j(S_{j+k-3} g \Delta_{j+k} f) \\ &+ \sum_{k>j-4} \Delta_j \Big(\Delta_k f \sum_{l=-2}^{2} \Delta_{k+l} g \Big). \end{aligned} \tag{2.5}$$

This calculation will be very useful in Section 7.

Now we turn our attention to establishing some basic Besov space estimates. First, we let $1 \le q_1 \le q_2 \le \infty$, $\beta_1 \le \beta_2$, $1 \le p_1 \le p_2 \le \infty$, $\gamma_1 = \gamma_2 + n(1/p_1 - 1/p_2)$, and $r > s > 0$. Then we have the following:

$$\begin{gathered} \|f\|_{B^{\beta_1}_{p,q_2}} \le C\|f\|_{B^{\beta_2}_{p,q_1}}, \\ \|f\|_{B^{\gamma_2}_{p_2,q}} \le C\|f\|_{B^{\gamma_1}_{p_1,q}}, \\ \|f\|_{H^{s,p}} \le \|f\|_{B^r_{p,q}}, \\ \|f\|_{H^{s,2}} = \|f\|_{B^s_{2,2}} \le \|f\|_{B^r_{2,q}}. \end{gathered} \tag{2.6}$$

These will be referred to as the Besov embedding results. Next, we record a Leibnitz-rule type estimate. This can be found in [1], and for the reader's convenience, the proof can be found in Section 7.

Proposition 2.1. *Let $f \in B^{s_1}_{p_1,q}(\mathbb{R}^n)$ and let $g \in B^{s_2}_{p_2,q}(\mathbb{R}^n)$. Then, for any p such that $1/p \le 1/p_1 + 1/p_2$ and with $s = s_1 + s_2 - n(1/p_1 + 1/p_2 - 1/p)$, we have*

$$\|fg\|_{B^s_{p,q}} \le \|f\|_{B^{s_1}_{p_1,q}} \|g\|_{B^{s_2}_{p_2,q}},$$

provided $s_1 < n/p_1$, $s_2 < n/p_2$, and $s_1 + s_2 > 0$.

Our third result is the Bernstein inequalities (see Appendix A in [14]). We let $A = (-\Delta)$, $\alpha \ge 0$, and $1 \le p \le q \le \infty$. If $\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2^j K\}$ and $\operatorname{supp} \hat{g} \subset \{\xi \in \mathbb{R}^n : 2^j K_1 \le |\xi| \le 2^j K_2\}$ for some $K, K_1, K_2 > 0$ and some integer j, then

$$\begin{gathered} \tilde{C} 2^{j\alpha + jn(1/p - 1/q)} \|g\|_p \le \|A^{\alpha/2} g\|_q \le C 2^{j\alpha + jn(1/p - 1/q)} \|g\|_p. \\ \|A^{\alpha/2} f\|_q \le C 2^{j\alpha + jn(1/p - 1/q)} \|f\|_p \end{gathered} \tag{2.7}$$

Next, we establish estimates for the heat kernel on Besov spaces.

Proposition 2.2. *Let $1 \le p_1 \le p_2 < \infty$, $-\infty < s_1 \le s_2 < \infty$, and let $0 < q < \infty$. Then*
$$\|e^{t\Delta} f\|_{B^{s_2}_{p_2,q}} \le C t^{-(s_2-s_1+n/p_1-n/p_2)/2} \|f\|_{B^{s_1}_{p_1,q}},$$
provided $0 < t < 1$.

Using the Sobolev space heat kernel estimate, we obtain, for $0 < t < 1$,
$$\begin{aligned}
\|e^{t\Delta} f\|_{B^{s_2}_{p_2,q}} &= \|\Psi * e^{t\Delta} f\|_{L^{p_2}} + \Big(\sum (2^{js_1}\|2^{j(s_2-s_1)}\Delta_j e^{t\Delta} f\|_{L^{p_2}})^q\Big)^{1/q} \\
&\le t^{(n/p_1-n/p_2)/2}\|\Psi * f\|_{L^{p_1}} + \Big(\sum (2^{js_1}\|e^{t\Delta}\Delta_j f\|_{H^{s_2-s_1,p_2}})^q\Big)^{1/q} \\
&\le t^{-(n/p_1-n/p_2)/2}\|\Psi * f\|_{L^{p_1}} + t^{\sigma}\Big(\sum (2^{js_1}\|\Delta_j * f\|_{L^{p_1}})^q\Big)^{1/q} \\
&\le t^{\sigma}\|f\|_{B^{s_1}_{p_1,q}}.
\end{aligned}$$
where $\sigma = -(s_2 - s_1 + n/p_1 - n/p_2)/2$, and we made liberal use of the fact that $e^{t\Delta}$ commutes with convolution operators. We remark that a straightforward density argument can be used to show that, for any ε,
$$\sup_{0 \le t < T} t^{(s_2-s_1+n/p_1-n/p_2)/2} \|e^{t\Delta} f\|_{B^{s_2}_{p_2,q}} < \varepsilon, \tag{2.8}$$
where T depends only on $\|f\|_{B^{s_1}_{p_1,q}}$.

2.1. **Integral-in-time results.** In this subsection we establish integral-in-time results for Besov space. The proofs are similar to those in [11] used for the analogous operators in Sobolev spaces. In this section, the operators Γ and G are defined by
$$\Gamma f = e^{t\Delta} f,$$
$$G(f)(t) = \int_0^t e^{(t-s)\Delta} f(s)ds.$$
We start with a result for Γ.

Proposition 2.3. *Let $1 < p_0 \le p_1 < \infty$, $1 \le q < \infty$, $-\infty < s_0 \le s_1 < \infty$, and assume $0 < (s_1 - s_0 + n/p_0 - n/p_1)/2 = 1/\sigma$. Then Γ maps $B^{s_0}_{p_0,q_0}$ continuously into $L^\sigma((0,\infty) : B^{s_1}_{p_1,q_1})$ with the estimate*
$$\|\Gamma f\|_{L^\sigma((0,\infty):B^{s_1}_{p_1,q_1})} \le C\|f\|_{B^{s_0}_{p_0,q_0}}.$$
Also, for any $\varepsilon > 0$,
$$\|\Gamma f\|_{L^\sigma((0,T):B^{s_1}_{p_1,q_1})} \le \varepsilon$$
provided T is sufficiently small. The necessary T depends only on $\|f\|_{B^{s_0}_{p_0,q_0}}$.

The proof is similar to [11, Prop. 4], with two main distinctions, both due to the differences in interpolation theory between Sobolev and Besov spaces. The first is that we interpolate using s_0 instead of p_0. The second difference is that we do not require $p_0 \le \sigma$, as we did in Proposition 4 of [11].

The remaining results in this section are for the operator G.

Proposition 2.4. *Given $1 \le p_0 \le p_1 < \infty$, $1 \le q < \infty$, $-\infty < s_0 \le s_1 < \infty$, $1 < \sigma_0 < \sigma_1 < \infty$ and $1/\sigma_0 - 1/\sigma_1 = 1 - (s_1 - s_0 + n/p_0 - n/p_1)/2$, for any $T \in (0, \infty]$, G sends $L^{\sigma_0}((0,T) : B^{s_0}_{p_0,q_0})$ into $L^{\sigma_1}((0,T) : B^{s_1}_{p_1,q_1})$ with the estimate*

$$\|G(f)\|_{L^{\sigma_1}((0,T):B^{s_1}_{p_1,q_1})} \le C\|f\|_{L^{\sigma_0}((0,T):B^{s_0}_{p_0,q_0})}.$$

Proposition 2.5. *$1 < p_0 \le p_1 < \infty$, $1 \le q < \infty$, $-\infty < s_0 \le s_1 < \infty$, and assume $1/p_1 \le 1/\sigma = 1 - (s_1 - s_0 + n/p_0 - n/p_1)/2 =$. Then G maps $L^{\sigma}((0,T) : B^{s_0}_{p_0,q_0})$ continuously into $BC([0,T) : B^{s_1}_{p_1,q_1})$ with the estimate*

$$\sup_{t\in[0,T)} \|G(f)(t)\|_{B^{s_1}_{p_1,q_1}} \le C\|f\|_{L^{\sigma}((0,T):B^{s_0}_{p_0,q_0})}.$$

3. Local solutions in $\dot{C}^T_{a;s,p,q}$

We begin by re-writing the LANS equation as

$$\partial_t u - Au + P^\alpha(\operatorname{div}\cdot(u \otimes u) + \operatorname{div}\tau^\alpha u) = 0, \tag{3.1}$$

where the recurring terms are as in (1.1), with the exception that we set $\nu = 1$. For the new terms, we set $A = P^\alpha\Delta$, $u \otimes u$ is the tensor with jk-component $u_j u_k$ and $\operatorname{div}\cdot(u \otimes u)$ is the vector with j-component $\sum_k \partial_k(u_j u_k)$. P^α is the Stokes Projector, defined as

$$P^\alpha(w) = w - (1 - \alpha^2\Delta)^{-1}\nabla f$$

where f is a solution of the Stokes problem: Given w, there is a unique divergence-free v and a unique (up to additive constants) function f such that

$$(1 - \alpha^2\Delta)v + \nabla f = (1 - \alpha^2\Delta)w.$$

For a more explicit treatment of the Stokes Projector, see [13, Theorem 4].

Using Duhamel's principle, we write (3.1) as the integral equation

$$u = \Gamma\varphi - G \cdot P^\alpha(\operatorname{div}(u \otimes u + \tau^\alpha(u))) \tag{3.2}$$

with

$$(\Gamma\varphi)(t) = e^{tA}\varphi,$$

where A agrees with Δ when restricted to $P^\alpha H^{r,p}$, and

$$G \cdot g(t) = \int_0^t e^{(t-s)A} \cdot g(s)ds.$$

We prove local existence using the standard contraction mapping method and heavy use of the results from Section 2. We begin by defining the nonlinear operator Φ by

$$\Phi(u) = e^{t\Delta}u_0 + \Psi(u),$$

where

$$\Psi(u) = \int_0^t e^{(t-s)\Delta}(V(u))ds$$

with V (essentially) given by

$$V(u) = \operatorname{div}(u \otimes u) + \operatorname{div}(1 - \Delta)^{-1}(\nabla u \nabla u),$$

where the full definition of V involves additional terms whose behavior is controlled by the terms shown.

The proofs of local existence in $\dot{C}_{a;r,p,q}$ for the two cases $p = 2$ and $p > n$ are sufficiently similar that we only present the $p = 2$ case here. In Section 4 we address the Integral in time case, and there we provide the details for the $p > n$ case.

Having set $p = 2$, we seek a fixed point of Φ in the space

$$E = \big\{ f \in BC([0,T) : B^{n/2-1}_{2,q}(\mathbb{R}^n)) \cap \dot{C}^T_{\frac{r-n/2+1}{2};r,2,q} : \sup_{t\in[0,T)} \|f - e^{t\Delta}u_0\|_{n/2-1,2,q} + \|f\|_{(r-n/2+1)/2;r,2,q} < M \big\},$$

for some T, M, to be determined below. First, we show that $\Phi : E \to E$, and we have

$$\|\Phi(u)\|_E = I + J + K,$$

where

$$\begin{aligned} I &= \|e^{t\Delta}u_0\|_{(r-n/2+1)/2;r,2,q} \\ J &= \sup_{t\in[0,T)} \|\Psi(u)\|_{n/2-1,2,q} \\ K &= \|\Psi(u)\|_{(r-n/2+1)/2;r,2,q}. \end{aligned} \tag{3.3}$$

For I, Proposition 2.2 and equation (2.8) give that

$$\|e^{t\Delta}u_0\|_{r-n/2+1;r,2,q} < M/3, \tag{3.4}$$

provided T is sufficiently small. Estimating J and K is significantly more work, and is the focus of the next two subsections.

3.1. **Estimating J.** We begin by writing $J \le J_1 + J_2$ where

$$J_1 = \sup_{t\in[0,T)} \| \int_0^t e^{(t-s)\Delta} \operatorname{div}(u(s)\otimes u(s))ds\|_{B^{n/2-1}_{2,q}},$$

$$J_2 = \sup_{t\in[0,T)} \| \int_0^t e^{(t-s)\Delta} \operatorname{div}(1-\Delta)^{-1}(\nabla u(s)\nabla u(s))ds\|_{B^{n/2-1}_{2,q}},$$

and for notational convenience we set $a = (r - n/2 + 1)/2$. Starting with J_1, we use Minkowski's inequality and then the heat kernel estimate to get

$$J_1 \le \sup_{t\in[0,T)} \int_0^t |t-s|^{(n/2-1-(r-1)+n/p-n/2)/2} \|u(s)\otimes u(s)\|_{B^r_{p,q}} ds, \tag{3.5}$$

where $1/p = 1 - r/n$. By Proposition 2.1, we have

$$\|u(s)\otimes u(s)\|_{B^r_{p,q}} \le \|u(s)\|_{B^r_{2,q}} \|u(s)\|_{B^0_{\tilde{p},q}} \le \|u(s)\|^2_{B^r_{2,q}},$$

where $1/p = 1/2 + 1/\tilde{p}$ (which, combined with the definition of p, implies $1/\tilde{p} = 1/2 - r/n$) and the second inequality used equation (2.6). Substituting back into equation (3.5) above, we obtain

$$\begin{aligned} J_1 &\le \sup_{t\in[0,T)} \int_0^t |t-s|^{(-r+n-r)/2} \|u(s)\otimes u(s)\|_{B^r_{p,q}} ds \\ &\le C \sup_{t\in[0,T)} \int_0^t |t-s|^{-(n/2-r)} s^{-2a} s^{2a} \|u(s)\|^2_{B^r_{2,q}} ds \\ &\le C \sup_{t\in[0,T)} \|u\|^2_{a;r,2,q} t^{-(n/2-r)-(r-n/2+1)+1} \le C\|u\|^2_{a;r,2,q}. \end{aligned} \tag{3.6}$$

We remark that this calculation required $n/2 - r < 1$ and $2a = r - n/2 + 1 < 1$, which are both satisfied for $n/2 - 1 < r < n/2$.

For J_2, with $1/p = 1 - (r-1)/n$, we have that

$$J_2 \le \sup_{t\in[0,T)} \int_0^t |t-s|^{-(n/p-n/2)/2} \| \operatorname{div}(1-\Delta)^{-1}(\nabla u(s)\nabla u(s))\|_{B^{n/2-1}_{p,q}} ds. \quad (3.7)$$

By Proposition 2.1, we have

$$\begin{aligned} \| \operatorname{div}(1-\Delta)^{-1}(\nabla u(s)\nabla u(s))\|_{B^{n/2-1}_{p,q}} &\le \|\nabla(u(s))\nabla(u(s))\|_{B^{n/2-2}_{p,q}} \\ &\le \|\nabla u(s)\|_{B^0_{2,q}} \|\nabla u(s)\|_{B^{r-1}_{2,q}} \\ &\le \|u(s)\|^2_{B^r_{2,q}}, \end{aligned}$$

provided $n/2 - 2 \le 0 + (r-1) - n/2 - n/2 + n/p$. Recalling the definition of p, this simplifies to $n/2 - 2 \le r - 1 - n + n - (r-1) = 0$, which holds for $n \le 4$. We pause here to remark that this would not follow from a more standard Leibnitz rule estimate, since $n/2 + n/2 \ne n/p$. Returning to equation (3.7), we have

$$\begin{aligned} J_2 &\le \sup_{t\in[0,T)} \int_0^t |t-s|^{-(n/2-(r-1))/2} s^{-2a} s^{2a} \|u\|^2_{B^r_{p,q}} ds \\ &\le C \sup_{t\in[0,T)} \|u\|^2_{a;r,2,q} t^{-(n/2-r+1)/2-(r-n/2+1)+1} \le C\|u\|^2_{a;r,2,q}, \end{aligned} \quad (3.8)$$

again provided $n/2 - 1 < r < n/2$. Combining equations (3.6) and (3.8), we obtain

$$J \le C\|u\|^2_{a;r,2,q} \le CM^2. \quad (3.9)$$

Now we turn to K.

3.2. **Estimating K.** As with J, we write K as $K \le K_1 + K_2$, where

$$\begin{aligned} K_1 &= \sup_{t\in[0,T)} t^a \| \int_0^t e^{(t-s)\Delta} \operatorname{div}(u(s)\otimes u(s)) ds\|_{B^r_{2,q}}, \\ K_2 &= \sup_{t\in[0,T)} t^a \| \int_0^t e^{(t-s)\Delta} \operatorname{div}(1-\Delta)^{-1}(\nabla u(s)\nabla u(s)) ds\|_{B^r_{2,q}}, \end{aligned}$$

where again $a = (r - (n/2-1))/2$. For K_1, we have

$$\begin{aligned} K_1 &\le \sup_{t\in[0,T)} t^a \int_0^t |t-s|^{-(r-(r-1)+n/p-n/2)/2} \| \operatorname{div}(u(s)\otimes u(s))\|_{B^{r-1}_{2,q}} ds \\ &\le C \sup_{t\in[0,T)} t^a \int_0^t |t-s|^{-(1+n/2-r)/2} \|u(s)\|^2_{B^r_{2,q}} ds \\ &\le C \sup_{t\in[0,T)} \|u\|_{a;r,2,q} t^a t^{-(1+n/2-r)/2-(r-n/2+1)+1} \le C\|u\|_{a;r,2,q}, \end{aligned} \quad (3.10)$$

where p is as in the estimate of J_1 and we again used Proposition 2.1.

For K_2, using an argument similar to that used for J_2, we have with $1/p = 1-(r-1)/n$,

$$\begin{aligned} K_2 &\le \sup_{t\in[0,T)} t^a \int_0^t |t-s|^{-(n/p-n/2)/2} \| \operatorname{div}(1-\Delta)^{-1}(\nabla(u(s))\nabla(u(s)))\|_{B^r_{2,q}} \\ &\le \sup_{t\in[0,T)} t^a \int_0^t |t-s|^{-(n/2-r+1)/2} \|\nabla(u(s))\nabla(u(s))\|_{B^{r-1}_{2,q}} ds \\ &\le C \sup_{t\in[0,T)} t^a \int_0^t |t-s|^{-(n/2-r+1)/2} \|u(s)\|^2_{B^r_{2,q}} ds \\ &\le C \sup_{t\in[0,T)} \|u\|^2_{a;r,2,q} t^a t^{-(1+n/2-r)/2-(r-n/2+1)+1} \le C\|u\|^2_{a;r,2,q}, \end{aligned} \tag{3.11}$$

where this time the use of Proposition 2.1 required $n/2-1 \le r$. Combining equations (3.10) and (3.11), we obtain

$$K \le C\|u\|^2_{a;r,2,q} \le CM^2. \tag{3.12}$$

3.3. **Finishing Theorem 1.2.** From equations (3.4), (3.9) and (3.12), we have that

$$\|\Phi(u)\|_E \le I + J + K < M/3 + CM^2 < M,$$

provided T and M are sufficiently small, and thus $\Phi : E \to E$. To show that Φ is a contraction, we observe that

$$u \otimes u - v \otimes v = (u-v) \otimes u + v \otimes (u-v),$$
$$\nabla u \nabla u - \nabla v \nabla v = \nabla(u-v)\nabla u + v\nabla(u-v),$$

and so, using a slight modification of equations (3.9) and (3.12), we have

$$\|\Phi(u) - \Phi(v)\|_E \le CM\|u-v\|_E,$$

which proves that Φ is a contraction for a sufficiently small choice of M. This completes the proof of the first part of Theorem 1.2. Adapting the above argument to the $p > n$ case proves the first part of Theorem 1.1. The details necessary for this adaptation are similar to those found in the next section.

4. Local solutions in $L^a((0,T) : B^s_{p,q}(\mathbb{R}^n))$

As in Section 3, we seek a fixed point of the map

$$\Phi(u) = e^{t\Delta}u_0 + \Psi(u),$$

where

$$\Psi(u) = \int_0^t e^{(t-s)\Delta}(V(u))ds$$

with V (essentially) given by

$$V(u) = \operatorname{div}(u \otimes u) + \operatorname{div}(1-\Delta)^{-1}(\nabla u \nabla u).$$

We present the details for the $p > n$ case. The $p = 2$ case is handled by a combination of the arguments presented here and the arguments used in Section 3.

We begin by defining F, for a T and M to be chosen later, as

$$\begin{aligned} F = \{f \in BC([0,T) : B^{r_1}_{p,q}(\mathbb{R}^n)) \cap L^a((0,T) : B^{r_2}_{p,q}) : \\ \sup_{t\in[0,T)} \|f - e^{t\Delta}u_0\|_{B^{r_1}_{p,q}} + \|f\|_{L^a(B^{r_2}_{p,q})} < M\}, \end{aligned}$$

where $a = 2/(r_2 - r_1)$, r_1 is an arbitrarily small positive number, and $1 < r_2 < 1 + r_1$. As in the previous section, we first show that $\Phi : F \to F$, and we have

$$\|\Phi(u)\|_F = I + J + K,$$

where

$$\begin{gathered} I = \|e^{t\Delta}u_0\|_{L^a(B^{r_2}_{p,q})} \\ J = \sup_{t\in[0,T)} \|\Psi(u)\|_{B^{r_1}_{p,q}} \\ K = \|\Psi(u)\|_{L^a(B^{r_2}_{p,q})}. \end{gathered}$$

For I, Proposition 2.4 gives that

$$\|e^{t\Delta}u_0\|_{L^a(B^{r_2}_{p,q})} < M/3, \tag{4.1}$$

provided T is sufficiently small. As in the previous section, estimating J and K is the focus of the next two subsections.

4.1. **Estimating J.** We write $J \le J_1 + J_2$, where

$$\begin{gathered} J_1 = \sup_{t\in[0,T)} \|G(\operatorname{div}(u\otimes u))(t))\|_{B^{r_1-1}_{p,q}}, \\ J_2 = \sup_{t\in[0,T)} \|G(\operatorname{div}(1-\Delta)^{-1}(\nabla u \nabla u))(t)\|_{B^{r_1}_{p,q}}. \end{gathered}$$

For J_1, we use Proposition 2.5 and get

$$J_1 \le \|\operatorname{div}(u\otimes u)\|_{L^\sigma(B^{r-1}_{p/2,q})} \le \|u\otimes u\|_{L^\sigma(B^r_{\tilde{p},q})}, \tag{4.2}$$

where $1/\sigma = 1 - (r_1 - r_2 + 1 + 2n/p - n/p)/2 = (r_2 - r_1 + 1 - n/p)/2$. Using Proposition 2.1, we have

$$\|u(s)\otimes u(s)\|_{B^{r_2}_{p/2,q}} \le \|u(s)\|_{B^{r_2}_{p,q}} \|u(s)\|_{B^0_{p,q}}.$$

Plugging back into equation (4.2), we obtain

$$\begin{aligned} J_1 \le \|u\otimes u\|_{L^\sigma(B^r_{\tilde{p},q})} &\le \Big(\int_0^T (\|u(s)\|_{B^{r_2}_{p,q}} \|u(s)\|_{B^{r_1}_{p,q}})^\sigma ds\Big)^{1/\sigma} \\ &\le C \sup_{t\in[0,T)} \|u(t)\|_{B^{r_1}_{p,q}} \|u\|_{L^a(B^{r_2}_{p,q})} \le CM^2, \end{aligned} \tag{4.3}$$

where we used that $\|\cdot\|_{L^\sigma} \le \|\cdot\|_{L^a}$, since $\sigma \le a$.

For J_2, again using Proposition 2.5, we have

$$J_2 \le \|\operatorname{div}(1-\Delta)^{-1}(\nabla u\nabla u)\|_{L^1(B^{r_1}_{p,q})} \le \|\nabla u \nabla u\|_{L^1(B^{r_1-1}_{p,q})}. \tag{4.4}$$

Using Proposition 2.1, we have

$$\|\nabla u(s)\nabla u(s)\|_{B^{r_1-1}_{p,q}} \le \|\nabla u\|^2_{B^{r_2-1}_{p,q}} \le \|u(s)\|^2_{B^{r_2}_{p,q}}, \tag{4.5}$$

provided $r_1 - 1 \le 2(r_2 - 1) - n/p$ (recall $r_2 > 1$, so $r_2 - 1 > 0$). This condition is equivalent to $n/p \le 2r_2 - 1 - r_1$, and since $r_1 < n/p - 1$, equation (4.5) holds. Using equation (4.5) in equation (4.4), we have

$$J_2 \le \|\nabla u\nabla u\|_{L^1(B^{r_1-1}_{p,q})} \le C\|u\|^2_{L^2(B^{r_2}_{p,q})} \le C\|u\|^2_{L^a(B^{r_2}_{p,q})} \le CM^2, \tag{4.6}$$

since $2 < a = 2/(r_2 - r_1)$. So using equations (4.3) and (4.6), we have

$$J \le CM^2. \tag{4.7}$$

4.2. **Estimating K.** We have that $K \le K_1 + K_2$ with

$$K_1 = \|G(\operatorname{div}(u \otimes u))\|_{L^a(B^{r_2}_{p,q})},$$
$$K_2 = \|G(\operatorname{div}(1-\Delta)^{-1}(\nabla u \nabla u))\|_{L^a(B^{r_2}_{p,q})}.$$

Using Proposition 2.4, for K_1, we have

$$K_1 = \|G(\operatorname{div}(u \otimes u))\|_{L^a(B^{r_2}_{p,q})} \le \|\operatorname{div}(u \otimes u)\|_{L^\sigma(B^{r_2-1}_{p/2,q})} \le \|u \otimes u\|_{L^\sigma(B^{r_2}_{p/2,q})}, \tag{4.8}$$

where $1/\sigma - 1/a = 1 - (r_2 - (r_2 - 1) + 2p/n - n/p)/2$, which can be rewritten as $1/\sigma = (r_2 - r_1)/2 + (1 - n/p)/2$. Using Proposition 2.1, we have

$$\|u(s) \otimes u(s)\|_{B^{r_2}_{p/2,q}} \le \|u(s)\|_{B^{r_2}_{p,q}} \|u(s)\|_{B^0_{p,q}}.$$

Applying this to equation (4.8), we have

$$\begin{aligned} K_1 &\le C\Big(\int_0^T (\|u(s)\|_{B^0_{p,q}} \|u(s)\|_{B^{r_2}_{p,q}})^\sigma ds\Big)^{1/\sigma} \\ &\le C \sup_{t\in[0,T)} \|u(t)\|_{B^{r_1}_{p,q}} \|u\|_{L^\sigma(B^{r_2}_{p,q})} \le CM\|u\|_{L^a(B^{r_2}_{p,q})} \le CM^2, \end{aligned} \tag{4.9}$$

which required $1/\sigma > 1/a$, which holds since $p > n$. Now we turn to K_2, where we have

$$\begin{aligned} K_2 &= \|G(\operatorname{div}(1-\Delta)^{-1}(\nabla u \nabla u))\|_{L^a(B^{r_2}_{p,q})} \\ &\le \|\operatorname{div}(1-\Delta)^{-1} \nabla u \nabla u\|_{L^\sigma(B^{r_1}_{p,q})} \le \|\nabla u \nabla u\|_{L^\sigma(B^{r_1-1}_{p,q})}, \end{aligned} \tag{4.10}$$

provided $1/\sigma - 1/a = 1 - (r_2 - r_1)/2$, which implies $\sigma = 1$. Then, by equation (4.10) above, we have

$$K_2 \le \|\nabla u \nabla u\|_{L^\sigma(B^{r_1-1}_{p,q})} \le CM^2. \tag{4.11}$$

Combining equations (4.3) and (4.6), we obtain

$$K \le CM^2. \tag{4.12}$$

Given equations (4.1), (4.7), and (4.12), we have that

$$\Phi(u) \le M/3 + CM^2 < M,$$

provided M is sufficiently small. From here, local existence follows from the standard method.

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3, and we start by proving the following *a priori* estimate.

Lemma 5.1. *Let f be a solution to the LANS equation such that $f(t) \in H^{2,2}(\mathbb{R}^n)$ for all $t \in [a,T)$ for some $a \ge 0$. Then*

$$\sup_{t\in[a,T)} \|f(t)\|_{H^{1,2}} \le \|f(a)\|_{H^{1,2}}.$$

We begin the proof of the Lemma by stating the following equivalent form of the LANS equation (see [7, Section 3]):

$$\begin{aligned} &\partial_t(1-\alpha^2\Delta)f(t) - (1-\alpha^2\Delta)\Delta f(t) \\ &= -\nabla p - \alpha^2(\nabla f(t))^T \cdot (-\Delta)f(t) - \nabla_{f(t)}[(1-\alpha^2\Delta)f(t)] \end{aligned} \tag{5.1}$$

Taking the L^2 product of the equation with $f(t)$, we obtain

$$\partial_t(\|f(t)\|_{L^2}^2 + \alpha^2\|\nabla f(t)\|_{L^2}^2) + \|\nabla f(t)\|_{L^2}^2 + \alpha^2\|\Delta f(t)\|_{L^2}^2 \le I_1 + I_2 + I_3, \quad (5.2)$$

where

$$\begin{gathered} I_1 = (\nabla_{f(t)} f(t), f(t)), \\ I_2 = \alpha^2\Big((\nabla_{f(t)}\Delta f(t), f(t)) + ((\nabla f(t))^T \cdot (-\Delta) f(t), f(t))\Big), \\ I_3 = (\nabla p, f(t)). \end{gathered}$$

An application of integration by parts and recalling that $\operatorname{div} f(t) = 0$ gives that $I_3 = 0$. For I_1, writing the expression in its coordinate form gives

$$\begin{aligned} I_1 = (\nabla_{f(t)} f(t), f(t)) &= \sum_{i,j=1}^n \int f_i(t)(\partial_{x_i} f_j(t)) f_j(t) \\ &= \sum_{i,j=1}^n \frac{1}{2}\int f_i(t)(\partial_{x_i}(f_j(t))^2) = -\frac{1}{2}\int \operatorname{div}(f(t))|f(t)|^2 = 0. \end{aligned}$$

For I_2, writing it in coordinates (and temporarily suppressing the time dependence), we see that

$$\begin{aligned} I_2 &= \sum_{i,j=1}^n \alpha^2 \int f_i(\partial_{x_i}\Delta f_j) f_j + (\Delta f_i)(\partial_{x_j} f_i) f_j \\ &= \sum_{i,j=1}^3 \alpha^2 \int -(f_i(\Delta f_j)(\partial_{x_i} f_j)) + (\Delta f_i)(\partial_{x_j} f_i) f_j = 0, \end{aligned}$$

where we again used integration by parts and exploited the divergence free condition. We remark here that it is these cancellations which make it easier to control the long time behavior of the LANS equations. Returning to equation (5.2), we have

$$\partial_t(\|f(t)\|_{L^2}^2 + \alpha^2\|\nabla f(t)\|_{L^2}^2) \le -(\|\nabla f(t)\|_{L^2}^2 + \alpha^2\|\Delta f(t)\|_{L^2}^2),$$

which, combined with Gronwall's inequality, completes the Lemma. Note that, if $\alpha = 0$, this reduces to the well-known L^2 control of the solution.

Now we are ready to prove Theorem 1.3. The extension arguments for the two different local existence results from Theorem 1.2 are similar, and we present here the argument for the local solution u given in equation (1.6). First, because the time interval of the local solution given by Theorem 1.2 depends only on $\|u_0\|_{B^{n/2-1}_{2,q}}$, global existence will follow from a standard bootstrapping argument once we have a uniform in time bound on $\|u(t)\|_{B^{n/2-1}_{p,q}}$.

Because $u \in BC([0,T)^{n/2-1}_{2,q}(\mathbb{R}^n))$, there exists an $a < T$ such that

$$\sup_{t\in[0,T)} \|u(t)\|_{B^{n/2-1}_{2,q}} \le 2\|u_0\|_{B^{n/2-1}_{2,q}} + \sup_{t\in[a,T)} \|u(t)\|_{B^{n/2-1}_{2,q}}. \quad (5.3)$$

So our remaining task is to bound the second term, and this will follow from Lemma 5.1. First, from Lemma 6.1 in the next section, we have that $u(t) \in B^{n/2+1}_{2,q}(\mathbb{R}^n)$ for all $t > 0$. From the Besov embedding results in equation (2.6), this means

$u(t) \in H^{2,2}(\mathbb{R}^n)$ for all $t > 0$, and thus Lemma 5.1 can be applied to our solution u. Using Lemma 5.1, when $n = 3$, we have

$$\sup_{t\in[a,T)} \|u(t)\|_{B^{n/2-1}_{2,q}} \leq \sup_{t\in[a,T)} \|u(t)\|_{H^{1,2}} \leq \|u(a)\|_{H^{1,2}}.$$

Plugging this back into (5.3) gives the desired uniform bound on $\|u(t)\|_{B^{3/2-1}_{2,q}}$. For $n = 4$, $n/2 - 1 = 1$, and Lemma 5.1 provides the desired bound when $\|u(t)\|_{B^1_{2,q}} \leq \|u(t)\|_{H^{1,2}} = \|u(t)\|_{B^1_{2,2}}$, which holds for $2 \leq q \leq \infty$.

For the integrable in time spaces, the only distinction in the argument is that Lemma 5.1 only provides a bound almost everywhere, since Lemma 6.1 gives that $u(t) \in B^2_{2,q}(\mathbb{R}^n)$ for almost every $t > 0$. So, in this case, Lemma 5.1 and the Besov embedding results only give that $\|u(t)\|_{B^{n/2-1}_{2,q}}$ is uniformly bounded for almost all t. However, since $u \in BC([0,T) : B^{n/2-1}_{2,q}(\mathbb{R}^n))$, continuity extends the bound to all time.

6. Higher regularity for the local existence result

In this section we quantify the smoothing effect of the heat kernel on our local solutions. The proof is an induction argument, similar to the one in [11] applied to the LANS equation (which was in turn inspired by the argument in [4] for the Navier-Stokes equation).

Lemma 6.1. *Let $u_0 \in B^s_{p,q}(\mathbb{R}^n)$ and let u be an associated solution to the LANS equation with initial data u_0 such that*

$$u \in BC([0,T) : B^r_{p,q}(\mathbb{R}^n)) \cap \dot{C}^T_{(s-r)/2;s,p,q},$$

where $0 < s - r < 1$ and $s > 1$. Then for all $k \geq s$, we have that $u \in \dot{C}^T_{(k-s)/2;k,p,q}$.

We have an analogous result for the integral in time case.

Lemma 6.2. *Let $k > s_2 > s_1$, with $s_2 \geq 1$, and let ε be a small positive number. Then, for $k - s_2 = s_2 - s_1 = \varepsilon$, for any solution u to the LANS equation* (1.1) *where*

$$u \in BC([0,T) : B^{s_1}_{p,q}(\mathbb{R}^n) \cap L^{2/(s_2-s_1)}((0,T) : B^{s_2}_{p,q}(\mathbb{R}^n)),$$

we have that $u \in L^1((0,T) : B^k_{p,q}(\mathbb{R}^n))$.

The proofs of the two Lemmas are similar. The rest of the section is devoted to the proof of Lemma 6.1.

Proof. We start with a solution to the LANS equation u. Then let $\delta > 0$ be arbitrary, and let $w = t^\delta u$. We note that $w(0) = 0$. Then

$$\begin{aligned}
\partial_t w &= \delta t^{\delta-1} u + t^\delta \partial_t u \\
&= \delta t^{-1} w + t^\delta (\Delta u - \operatorname{div}(u \otimes u + \tau^\alpha(u,u))) \\
&= \delta t^{-1} w + \Delta w - t^{-\delta} \operatorname{div}(w \otimes w + \tau^\alpha(w,w)).
\end{aligned}$$

Applying Duhamel's principle, we obtain

$$\begin{aligned}
w = e^{t\Delta} w_0 &+ \int_0^t e^{(t-s)\Delta} s^{-1} w(s) ds \\
&+ \int_0^t e^{(t-s)\Delta} s^{-\delta} (\operatorname{div}(w(s) \otimes w(s) + \tau^\alpha(w(s), w(s))))ds.
\end{aligned}$$

Recalling that $w(0) = w_0 = 0$, and substituting $w = t^\delta u$, we obtain

$$u = t^{-\delta}\int_0^t e^{(t-s)\Delta}s^{\delta-1}u(s)ds+t^{-\delta}\int_0^t e^{(t-s)\Delta}s^{\delta}(\operatorname{div}(u(s)\otimes u(s)+\tau^{\alpha}(u(s),u(s))))ds.$$

Now we are ready to apply the induction. We have by assumption that u is in $\dot{C}^T_{(r-s)/2;r,p,q}$, where $r > 1$. For induction, we assume this solution u is also in $\dot{C}^T_{(k-r)/2;k,p,q}$, and seek to show that u is in $\dot{C}^T_{(k+h-r)/2;k+h,p,q}$, where $0 < h < 1$ is fixed and will be chosen later. We have

$$\|u\|_{B^{k+h}_{p,q}} \le I + J_1 + J_2,$$

with I, J_1, and J_2 defined by

$$I = t^{-\delta}\int_0^t \|e^{(t-s)\Delta}s^{\delta-1}u(s)\|_{B^{k+h}_{p,q}}ds$$

$$J_1 = t^{-\delta}\int_0^t \|e^{(t-s)\delta}s^{\delta}(\operatorname{div}(1-\alpha^2\Delta)^{-1}(\nabla u(s)\nabla u(s)))\|_{B^{k+h}_{p,q}}ds$$

$$J_2 = t^{-\delta}\int_0^t \|e^{(t-s)\delta}s^{\delta}(\operatorname{div}(u(s)\otimes u(s)))\|_{B^{k+h}_{p,q}}ds$$

where, as usual, we have suppressed terms from τ^α that are controlled by the terms we included.

6.1. **Bounding I, J_1, and J_2.** Starting with I, we have

$$\begin{aligned} I &\le t^{-\delta}\int_0^t |t-s|^{-h/2}s^{\delta-1}\|u(s)\|_{B^k_{p,q}} \\ &\le t^{-\delta}\|u\|_{(k-r)/2;k,p,q}\int_0^t |t-s|^{-h/2}s^{\delta-1-(k-r)/2}ds \\ &\le C\|u\|_{(k-r)/2;k,p,q}t^{-\delta}t^{-h/2}t^{\delta-1-(k-n/2)/2+1} \\ &\le Ct^{-(k+h-r)/2}\|u\|_{(k-r)/2;k,2,q}, \end{aligned} \tag{6.1}$$

provided

$$1 > h/2, \quad -1 < \delta - 1 - (k-r)/2,$$

which clearly holds for sufficiently large δ. We observe that, without modifying the PDE to include these t^δ terms, we would need $(k-r)/2$ to be less than 1, which does not hold for large k.

For J_1, we have

$$\begin{aligned} J_1 &\le t^{-\delta}\int_0^t |t-s|^{-(h+2n/p-n/p)/2}s^{\delta}\|\operatorname{div}(1-\Delta)^{-1}(\nabla u\nabla u)\|_{B^k_{p/2,q}}ds \\ &\le t^{-\delta}\int_0^t |t-s|^{-(h+n/p)/2}s^{\delta}\|(\nabla u\nabla u)\|_{B^{k-1}_{p/2,q}}ds \\ &\le t^{-\delta}\int_0^t |t-s|^{-(h+n/p)/2}s^{\delta}\|u\|_{B^k_{p,q}}\|\nabla u\|_{B^1_{p,q}}ds \\ &\le t^{-\delta}\|u\|_{(k-r)/2;k,p,q}\|u\|_{(1-r)/2;1,p,q}\int_0^t |t-s|^{-(h+n/p)/2}s^{\delta-(k-r)/2-(1-r)/2}ds \\ &\le t^{-\delta-(h+n/p)/2-(k-r)/2-(1-r)/2+1+\delta}\|u\|^2_{(k-r)/2;k,p,q} \\ &\le t^{-(k+h-r))/2-(n/p-1-r)/2}\|u\|^2_{(k-r)/2;k,p,q} \le t^{-(k+h-r)/2\|u\|^2_{(k-r)/2;k,p,q}}, \end{aligned} \tag{6.2}$$

provided

$$\delta > (k-r)/2 + (1-r)/2, \quad 2 > h + n/p, \quad r \geq n/p - 1,$$

and we again see that this is easily satisfied by choosing δ large and h small. For J_2, we handle the cases $p = 2$ and $p > n$ separately. For $p > n$, we have

$$\begin{aligned}
J_2 &\leq t^{-\delta} \int_0^t |t-s|^{-(h+1+2n/p-n/p)/2} s^{\delta} \|u \otimes u\|_{B^k_{p/2,q}} ds \\
&\leq t^{-\delta} \int_0^t |t-s|^{-(h+1+n/p)/2} s^{\delta} \|u\|_{B^k_{p,q}} \|u\|_{B^s_{p,q}} ds \\
&\leq t^{-\delta} \|u\|_{(k-r)/2;k,p,q} \|u\|_{0;s,p,q} \int_0^t |t-s|^{-(h+1+n/p)/2} s^{\delta-(k-r)/2} ds \\
&\leq t^{-(h+k-r)/2-(1+n/p-2)/2} \|u\|_{(k-n/2)/2;k,2,q} \|u\|_{0;n/2,2,q} \\
&\leq t^{-(h+k-r)/2} \|u\|_{(k-n/2)/2;k,2,q} \|u\|_{0;n/2,2,q},
\end{aligned} \tag{6.3}$$

provided

$$1 > h + n/p, \quad -1 < \delta - (k-r)/2.$$

For the $p = 2$ case, we specialize to the case $r = n/2 - 1$, which is the minimal s allowed by our local existence theorem. The argument for larger s is a straightforward generalization of the one presented here. Defining $1/\tilde{p} = 1 - 1/n$, we have

$$\begin{aligned}
J_2 &\leq t^{-\delta} \int_0^t |t-s|^{-(h+1+n/\tilde{p}-n/2)/2} s^{\delta} \|u \otimes u\|_{B^k_{\tilde{p},q}} ds \\
&\leq t^{-\delta} \int_0^t |t-s|^{-(h+1+n/2-1)/2} s^{\delta} \|u\|_{B^k_{2,q}} L^{2n/(n-2)} ds \\
&\leq t^{-\delta} \|u\|_{(k-r)/2;k,2,q} \|u\|_{(1-r)/2;1,2,q} \int_0^t |t-s|^{-(h+n/2)/2} s^{\delta-(k-r)/2-(1-r)/2} ds \\
&\leq t^{-(h+k-r)/2-(n/p-r-1)/2} \|u\|_{(k-n/2)/2;k,2,q} \|u\|_{(1-r)/2;1,2,q} \\
&\leq t^{-(h+k-r)/2} \|u\|_{(k-n/2)/2;k,2,q} \|u\|_{(1-r)/2;1,2,q},
\end{aligned} \tag{6.4}$$

provided

$$2 > h + n/2, \quad -1 < \delta - (k-r)/2 - (1-r)/2, \quad r \geq n/2 - 1,$$

which, again, are easily satisfied.

Combining equations (6.1), (6.2) and (6.3) for $p > n$ (or (6.4) if $p = 2$), we have that, for h small enough and δ large enough,

$$I + J_1 + J_2 \leq C t^{-(h+k-n/2)/2} \|u\|^2_{(k-n/2)/2;k,2,q}$$

This in turn gives

$$\|u\|_{B^{k+h}_{p,q}} \leq C t^{(k+h-r)/2} \|u\|^2_{(k-n/2)/2;k,2,q}$$

which proves the desired result. We remark that δ is chosen after beginning the induction step, while the appropriate value of h is fixed by the choices of the parameters. □

7. Appendix: A Modified Product Estimate

In this appendix we prove Proposition 2.1, which can be found in Corollary 1.3.1 in [1]. Before beginning, we establish another result for the Littlewood-Paley operators and make a slight notational change. First, we observe that, by changing variables,

$$\|\psi_j\|_{L^p} \le 2^{jn/p'}\|\psi_0\|_{L^p} \le C2^{jn/p'}, \tag{7.1}$$

where p' is the Holder' conjugate to p; i.e., $1 = 1/p + 1/p'$. Next, we make a slight notational change. For $j > 0$, we leave ψ_j as defined in Section 2. For $j = 0$, we set $\psi_0 = \Psi$, so $\hat{\psi}_0$ is now supported on the ball centered at the origin of radius 1/2 and $\Delta_0 f = \psi_0 * f = \Psi * f$. Then the Besov norm can be defined by

$$\|f\|_{B^r_{p,q}} = \Big(\sum_{j=0}^{\infty} 2^{rjq}\|\Delta_j u\|^q_{L^p}\Big)^{1/q}.$$

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. We start by taking the L^p norm of equation (2.5), and get:

$$\begin{aligned}\|\Delta_j(fg)\|_{L^p} \le \sum_{k=-3}^{3}\|\Delta_j(S_{j+k-3}f\Delta_{j+k}g)\|_{L^p} + \sum_{k=-3}^{3}\|\Delta_j(S_{j+k-3}g\Delta_{j+k}f)\|_{L^p}\\ + \sum_{k>j-4}\|\Delta_j\Big(\Delta_k f\sum_{l=-2}^{2}\Delta_{k+l}g\Big)\|_{L^p}.\end{aligned}$$

We first observe that, without loss of generality, we can set $k = l = 0$ in the finite sums and replace $k > j-4$ with $k > j$. Doing so, we obtain

$$\|\Delta_j(fg)\|_{L^p} \le \|\Delta_j(S_{j-3}f\Delta_j g)\|_{L^p} + \|\Delta_j(S_{j-3}g\Delta_j f)\|_{L^p} + \sum_{k>j}\|\Delta_j\Big(\Delta_k f\Delta_k g\Big)\|_{L^p}.$$

Starting with the first term, and defining $\tilde{p}$ by $1 + 1/p = 1/\tilde{p} + 1/p_2$, we have

$$\begin{aligned}\|\Delta_j(S_{j-3}f\Delta_j g)\|_{L^p} &\le \|\psi_j\|_{L^{\tilde{p}}}\|\Delta_j f S_{j-3}g\|_{L^{p_2}}\\ &\le C2^{jn/\tilde{p}'}\|\Delta_j g\|_{L^{p_2}}\|S_{j-3}f\|_{L^\infty}\\ &\le C2^{jn/\tilde{p}'}\|\Delta_j g\|_{L^{p_2}}\sum_{m<j-3}\|\Delta_m f\|_{L^\infty}\\ &\le C2^{jn(1/p_2-1/p)/\tilde{p}'}\|\Delta_j g\|_{L^{p_2}}\sum_{m<j-3}2^{mn/p_1}\|\Delta_m f\|_{L^{p_1}},\end{aligned}$$

where we used Young's inequality, equation (7.1), Holder's inequality, and finally Bernstein's inequality.

A similar calculation for the second term yields

$$\|\Delta_j(S_{j-3}g\Delta_j f)\|_{L^p} \le C2^{jn(1/p_1-1/p)}\|\Delta_j f\|_{L^{p_2}}\sum_{m<j-3}2^{mn/p_2}\|\Delta_m g\|_{L^{p_1}}.$$

For the third term, we have

$$\sum_{k>j}\|\Delta_j(\Delta_k f\Delta_k g\|_p) \le \|\psi_j\|_{\tilde{q}}\sum_{k>j}\|\Delta_k u\Delta_k v\|_{L^q}$$

$$
\begin{aligned}
&\le 2^{jn/\tilde{p}'} \sum_{k>j} \|\Delta_k f\|_{p_1} \|\Delta_k g\|_{p_2} \\
&\le 2^{jn(1/p-1/p_1-1/p_2)} \sum_{k>j} \|\Delta_k f\|_{p_1} \|\Delta_k g\|_{p_2},
\end{aligned}
$$

where $1+1/p = 1/\tilde{q} + 1/q$ and $1/q = 1/p_1 + 1/p_2$.

So we have that

$$
\begin{aligned}
\|\Delta_j(fg)\|_{L^p} \le\ & 2^{jn(1/p_2-1/p)} \|\Delta_j g\|_{L^{p_2}} \sum_{m<j-3} 2^{jn/p_1} \|\Delta_m f\|_{L^{p_1}} \\
&+ 2^{jn(1/p_1-1/p)} \|\Delta_j f\|_{L^{p_1}} \sum_{m<j-3} 2^{jn/p_2} \|\Delta_m g\|_{L^{p_2}} \\
&+ 2^{jn(1/p-1/p_1-1/p_2)} \sum_{k>j} \|\Delta_k f\|_{p_1} \|\Delta_k g\|_{p_2}
\end{aligned}
\tag{7.2}
$$

Multiplying (7.2) by $2^{j(s_1+s_2-n(1/p_2+1/p_1-1/p))}$ and taking the l^q norm in j, we obtain

$$
\|fg\|_{B^s_{p,q}} \le I + J + K,
$$

where

$$
\begin{gathered}
I = \Big(\sum_j 2^{(s_1+s_2-n/p_1)jq} \|\Delta_j g\|^q_{L^{p_2}} \Big(\sum_{m<j-3} 2^{mn/p_1} \|\Delta_m f\|_{L^{p_1}} \Big)^q \Big)^{1/q}, \\
J = \Big(\sum_j 2^{(s_1+s_2-n/p_2)jq} \|\Delta_j f\|^q_{L^{p_1}} \Big(\sum_{m<j-3} 2^{mn/p_2} \|\Delta_m g\|_{L^{p_2}} \Big)^q \Big)^{1/q}, \\
K = \Big(\sum_j (2^{j(s_1+s_2)} \sum_{k>j} \|\Delta_k f\|_{p_1} \|\Delta_k g\|_{p_2})^q \Big)^{1/q}.
\end{gathered}
$$

For I, we have

$$
\begin{aligned}
I &\le \Big(\sum_j 2^{(s_1+s_2-n/p_1)jq} \|\Delta_j g\|^q_{L^{p_2}} (\sum_{m<j-3} 2^{jn/p_1} \|\Delta_m f\|_{L^{p_1}})^q \Big)^{1/q} \\
&\le \Big(\sum_j (2^{js_2} \|\Delta_j g\|_{L^{p_2}})^q (\sum_{m<j-3} 2^{m(n/p_1+s_1-n/p_1)} 2^{(j-m)(s_1-n/p_1)} \|\Delta_m f\|_{L^{p_1}})^q \Big)^{1/q} \\
&\le \|f\|_{B^{s_1}_{p_1,\infty}} \sum_k 2^{-(s_1-n/p_2)} \Big(\sum_j (2^{js_2} \|\Delta_j g\|_{L^{p_2}}) \Big)^{1/q} \\
&\le \|f\|_{B^{s_1}_{p,q}} \|g\|_{B^{s_2}_{s_2,q}},
\end{aligned}
$$

provided $s_1 < n/p_1$. A similar calculation for J yields

$$
J \le \|f\|_{B^{s_1}_{p,q}} \|g\|_{B^{s_2}_{s_2,q}},
$$

provided $s_2 < n/p_2$. For K, we have, using Young's inequality for sums,

$$
\begin{aligned}
K &= \Big(\sum_j (\sum_{k>j} 2^{(j-k)(s_1+s_2)} 2^{ks_1} \|\Delta_k f\|_{p_1} 2^{ks_2} \|\Delta_k g\|_{p_2})^q \Big)^{1/q} \\
&\le \|g\|_{B^{s_2}_{p_2,\infty}} \Big(\sum_j (\sum_{k>j} 2^{(j-k)(s_1+s_2)} 2^{ks_1} \|\Delta_k f\|_{p_1})^q \Big)^{1/q}
\end{aligned}
$$

$$\leq \|g\|_{B^{s_2}_{p_2,\infty}} \sum_k 2^{-k(s_1+s_2)} \Big(\sum_k (2^{ks_1} \|\Delta_k f\|_{p_1})^q \Big)^{1/q}$$
$$\leq C \|f\|_{B^{s_1}_{p_1,q}} \|g\| + B^{s_2}_{p_2,q},$$

provided $s_1 + s_2 > 0$. This completes the proof. □

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Nathan Pennington

Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, USA

E-mail address: `npenning@math.ksu.edu`