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POSITIVE ALMOST PERIODIC SOLUTIONS FOR STATE-DEPENDENT DELAY LOTKA-VOLTERRA COMPETITION SYSTEMS

YONGKUN LI, CHAO WANG

Abstract. In this article, using Mawhin's continuation theorem of coincidence degree theory, we obtain sufficient conditions for the existence of positive almost periodic solutions for the system of equations

$$\dot{u}_i(t) = u_i(t)\Big[r_i(t) - a_{ii}(t)u_i(t) - \sum_{j=1,j\neq i}^{n} a_{ij}(t)u_j\big(t - \tau_j(t, u_1(t), \dots, u_n(t))\big)\Big],$$

where $r_i, a_{ii} > 0$, $a_{ij} \geq 0 (j \neq i,\ i, j = 1, 2, \dots, n)$ are almost periodic functions, $\tau_i \in C(\mathbb{R}^{n+1}, \mathbb{R})$, and $\tau_i (i = 1, 2, \dots, n)$ are almost periodic in t uniformly for $(u_1, \dots, u_n)^T \in \mathbb{R}^n$. An example and its simulation figure illustrate our results.

1. Introduction

Proposed by Lotka [14] and Volterra [18], the well-known Lotka-Volterra models concerning ecological population modeling have been extensively investigated in the literature. When two or more species live in proximity and share the same basic requirements, they usually compete for resources, food, habitat, or territory. In recent years, it has also been found with successful and interesting applications in epidemiology, physics, chemistry, economics, biological science and other areas (see [3, 5, 6]). Owing to their theoretical and practical significance, the Lotka-Volterra systems have been studied extensively [7, 8, 9, 10, 15]. To consider periodic environmental factors, it is reasonable to study the Lotka-Volterra system with both the periodically changing environment and the effects of time delays. Li [11] studied the state dependent delay Lotka-Volterra competition system by using coincidence degree theory:

$$\dot{u}_i(t) = u_i(t)\Big[r_i(t) - a_{ii}(t)u_i(t) - \sum_{j=1,j\neq i}^{n} a_{ij}(t)u_j\big(t - \tau_j(t, u_1(t), \dots, u_n(t))\big)\Big], \tag{1.1}$$

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where $i = 1, 2, \dots, n$, $u_i(t)$ stands for the ith species population density at time t, $r_i(t)$ is the natural reproduction rate for the ith species, a_{ij} represents the effect of interspecific (if $i \neq j$) or intraspecific (if $i = j$) interaction.

Virtually all biological systems exist in environments which vary with time, frequently in a periodic way. Ecosystem effects and environmental variability are very important factors and mathematical models cannot ignore, for example, year-to-year changes in weather, habitat destruction and exploitation, the expanding food surplus, and other factors that affect the population growth.

Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically varying environment are considered as important selective forces on systems in a fluctuating environment. Therefore, on the one hand, models should take into account the seasonality of the periodically changing environment. However, on the other hand, in fact, it is more realistic to consider almost periodic system than periodic system. Recently, there are two main approaches to obtain sufficient conditions for the existence and stability of the almost periodic solutions of biological models: One is by using the fixed point theorem, Lyapunov functional method and differential inequality techniques (see [2, 12]); the other is by using functional hull theory and Lyapunov functional method (see [16, 17]). To the best of our knowledge, there are few papers published on the existence of almost periodic solutions to almost periodic differential equations done by the method of coincidence degree theory [16-18] and no published papers considering the almost periodic solutions for non-autonomous Lotka-Volterra competitive system with time delay by applying the method of coincidence degree theory.

Motivated above, we apply the coincidence degree theory to study the existence of positive almost periodic solutions for the state dependent delay Lotka-Volterra competition system (1.1) under the following assumptions:

- (H1) $r_i, a_{ii} > 0$, $a_{ij} \geq 0 (j \neq i,\ i, j = 1, 2, \dots, n)$ are almost periodic functions, $\tau_i \in C(\mathbb{R}^{n+1}, \mathbb{R})$, and $\tau_i(t, u_1, \dots, u_n)$ $(i = 1, 2, \dots, n)$ are bounded and almost periodic in t uniformly for $(u_1, \dots, u_n)^T \in \mathbb{R}^n$.

The result obtained in this paper is new, and our method can be used to study other population models.

2. Preliminaries

Let X, Y be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exists continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} L = \ker L, \ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, it follows that the mapping $L_{\operatorname{Dom} L \cap \ker P} : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, then the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

For convenience, we introduce the Mawhin's continuation theorem [4] as follows.

Lemma 2.1 ([4])**.** *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\bar{\Omega}$. Assume that*

(1) *$Ly \neq \lambda Ny$ for every $y \in \partial\Omega \cap \operatorname{Dom} L$ and $\lambda \in (0, 1)$;*

(2) $QNy \neq 0$ *for every* $y \in \partial\Omega \cap \ker L$;
(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then $Ly = Ny$ *has at least one solution in* $\operatorname{Dom} L \cap \bar{\Omega}$.

For $f \in AP(\mathbb{R}, \mathbb{R}^n)$ we denote by

$$\Lambda(f) = \Big\{\lambda \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)e^{-i\lambda s}\,\mathrm{d}s \neq 0\Big\}$$

and

$$\operatorname{mod}(f) = \Big\{\sum_{j=1}^{m} n_j\lambda_j : n_j \in \mathbb{Z},\, m \in \mathbb{N},\, \lambda_j \in \Lambda(f),\, j = 1, 2, \dots, m\Big\}$$

the set of Fourier exponents and the module of f, respectively.

Suppose that $f(t,\phi)$ is almost periodic in t, uniformly with respect to $\phi \in S$. $E\{f, \varepsilon, S\}$ denotes the set of ε-almost periods for f with respect to $S \subset C([-\sigma, 0], \mathbb{R}^n)$, $l(\varepsilon, S)$ denotes the length of the inclusion interval and $m[f] = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s$ denote the mean value of f. Set

$$\mathbb{X} = \mathbb{Y} = V_1 \oplus V_2,$$

where

$$V_1 = \big\{y = (x_1, x_2, \dots, x_n)^T \in AP(\mathbb{R}, \mathbb{R}^n) : \operatorname{mod}(y) \subset \operatorname{mod}(F)\, \forall \mu_0 \in \Lambda(y) \text{ satisfies } |\mu_0| > \alpha\big\}$$

and

$$V_2 = \big\{y = (x_1(t), \dots, x_n(t))^T \equiv (k_1, \dots, k_n)^T,\, (k_1, \dots, k_n)^T \in \mathbb{R}^n\big\},$$

where $F = (F_1, F_2, \dots, F_n)^T$. For $i = 1, 2, \dots, n$,

$$\begin{aligned} F_i(t,\varphi) &= r_i(t) - a_{ii}(t)\exp\{\varphi_i(0)\} \\ &\quad - \sum_{j=1, j\neq i}^{n} a_{ij}(t)\exp\big\{\varphi_j\big(-\tau_j(t, \varphi_1(0), \dots, \varphi_n(0))\big)\big\}, \end{aligned}$$

$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T \in C([-\sigma, 0], \mathbb{R}^n)$, $\sigma = \max_{1\le j\le n}\sup_{(t,u)\in\mathbb{R}\times\mathbb{R}^n}\{\tau_j(t,u)\}$ and α is a given positive constant. Define the norm

$$\|y\| = \sup_{t\in\mathbb{R}} |y(t)| = \sup_{t\in\mathbb{R}} \max_{1\le i\le n}\{|x_i(t)|\}, \quad y \in \mathbb{X}(\text{or } \mathbb{Y}).$$

The following lemma will play an important role in the proof of our main result.

Lemma 2.2. *If* $f \in C(\mathbb{R}, \mathbb{R})$ *is almost periodic,* $t_0 \in \mathbb{R}$. *For any* $\varepsilon > 0$ *and inclusion length* $l(\varepsilon)$, *for all* $t_1, t_2 \in [t_0, t_0 + l(\varepsilon)] := I_{l(\varepsilon)}$. *Then for all* $t \in \mathbb{R}$, *the following two inequalities hold*

$$f(t) \le f(t_1) + \int_{t_0}^{t_0+l(\varepsilon)} |f'(s)|\,ds + \varepsilon \tag{2.1}$$

$$f(t) \ge f(t_2) - \int_{t_0}^{t_0+l(\varepsilon)} |f'(s)|\,ds - \varepsilon. \tag{2.2}$$

Proof. For any $t \in \mathbb{R}$, there exists $\tau \in E\{f, \varepsilon\}$ such that $t \in [t_0 - \tau, t_0 - \tau + l(\varepsilon)]$. Thus, $t + \tau \in [t_0, t_0 + l(\varepsilon)]$. So we can obtain

$$\begin{aligned} f(t) - f(t_1) &= \int_{t_1}^{t} f'(s)\,\mathrm{d}s = \int_{t_1}^{t+\tau} f'(s)\mathrm{d}s + \int_{t+\tau}^{t} f'(s)\,\mathrm{d}s \\ &\le \int_{t_1}^{t+\tau} |f'(s)|\,\mathrm{d}s + |f(t+\tau) - f(t)| \\ &\le \int_{t_0}^{t_0 + l(\varepsilon)} |f'(s)|\,\mathrm{d}s + \varepsilon. \end{aligned}$$

Hence, (2.1) holds.

Similarly, we have

$$\begin{aligned} f(t) - f(t_2) &= \int_{t_2}^{t} f'(s)\,\mathrm{d}s = \int_{t_2}^{t+\tau} f'(s)\mathrm{d}s + \int_{t+\tau}^{t} f'(s)\,\mathrm{d}s \\ &\ge -\int_{t_2}^{t+\tau} |f'(s)|\,\mathrm{d}s - |f(t+\tau) - f(t)| \\ &\ge -\int_{t_0}^{t_0 + l(\varepsilon)} |f'(s)|\,\mathrm{d}s - \varepsilon. \end{aligned}$$

Thus, (2.2) holds. The proof is complete. $\square$

3. Main results

By making the substitution

$$u_i(t) = \exp\{x_i(t)\}, \quad i = 1, 2, \dots, n,$$

Equation (1.1) is reformulated as

$$\begin{aligned} \dot{x}_i(t) &= r_i(t) - a_{ii}(t)\exp\{x_i(t)\} \\ &\quad - \sum_{j=1, j\neq i}^{n} a_{ij}(t)\exp\big\{x_j\big(t - \tau_j(t, \exp\{x_1(t)\}, \dots, \exp\{x_n(t)\})\big)\big\}. \end{aligned} \tag{3.1}$$

Lemma 3.1. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with the norm $\|\cdot\|$.*

Proof. If $\{y_n\} \subset V_1$ and y_n converges to y_0, then it is easy to show that $y_0 \in AP(\mathbb{R}, \mathbb{R}^n)$ with $\mathrm{mod}(y_0) \subset \mathrm{mod}(F)$. Indeed, for all $|\lambda| \le \alpha$ we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T y_n(s)e^{-i\lambda s}\,\mathrm{d}s = 0.$$

Thus

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T y_0 e^{-i\lambda s}\,\mathrm{d}s = 0,$$

which implies that $y_0 \in V_1$. One can easily see that V_1 is a Banach space endowed with the norm $\|\cdot\|$. The same can be concluded for the spaces $\mathbb{X}$ and $\mathbb{Y}$. The proof is complete. $\square$

Lemma 3.2. *Let $L : \mathbb{X} \to \mathbb{Y}$ such that $Ly = \dot{y}$. Then L is a Fredholm mapping of index zero.*

Proof. Clearly, $\ker L = V_2$. It remains to prove that $\operatorname{Im} L = V_1$. Suppose that $\phi \in \operatorname{Im} L \subset \mathbb{Y}$. Then, there exist $\phi_{V_1} = (\phi_1^{(1)}, \phi_1^{(2)}, \dots, \phi_1^{(n)})^T \in V_1$ and $\phi_{V_2} = (\phi_2^{(1)}, \phi_2^{(2)}, \dots, \phi_2^{(n)})^T \in V_2$ such that

$$\phi = \phi_{V_1} + \phi_{V_2}.$$

From the definitions of $\phi(t)$ and $\phi_{V_1}(t)$, we deduce that $\int^t \phi(s)\,\mathrm{d}s$ and $\int^t \phi_{V_1}(s)\,\mathrm{d}s$ are almost periodic functions and thus $\phi_{V_2}(t) \equiv (0, 0, \dots, 0)^T := \mathbf{0}$, which implies that $\phi(t) \in V_1$. This tells us that

$$\operatorname{Im} L \subset V_1.$$

On the other hand, if $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))^T \in V_1 \backslash \{\mathbf{0}\}$ then we have $\int_0^t \varphi(s)\,\mathrm{d}s \in AP(\mathbb{R}, \mathbb{R}^n)$. Indeed, if $\lambda \neq 0$ then we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big[\int_0^t \varphi(s)\,\mathrm{d}s\Big] e^{-i\lambda t}\,\mathrm{d}t = \frac{1}{i\lambda} \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(s) e^{-i\lambda t}\,\mathrm{d}s.$$

It follows that

$$\Lambda\Big[\int_0^t \varphi(s)\,\mathrm{d}s - m\Big(\int_0^t \varphi(s)\,\mathrm{d}s\Big)\Big] = \Lambda(\varphi).$$

Thus

$$\int_0^t \varphi(s)\,\mathrm{d}s - m\Big(\int_0^t \varphi(s)\,\mathrm{d}s\Big) \in V_1 \subset \mathbb{X}.$$

Note that $\int_0^t \varphi(s)\,\mathrm{d}s - m(\int_0^t \varphi(s)\,\mathrm{d}s)$ is the primitive of $\varphi(t)$ in $\mathbb{X}$, so we have $\varphi(t) \in \operatorname{Im} L$. Hence, we deduce that $V_1 \subset \operatorname{Im} L$, which completes the proof of our claim. Therefore, $\operatorname{Im} L = V_1$.

Furthermore, one can easily show that $\operatorname{Im} L$ is closed in $\mathbb{Y}$ and

$$\dim \ker L = n = \operatorname{codim} \operatorname{Im} L.$$

Therefore, L is a Fredholm mapping of index zero. The proof is complete. $\square$

Lemma 3.3. *Let $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$, $Q : \mathbb{Y} \to \mathbb{Y}$ such that $Ny = (G_1 y, G_2 y, \dots, G_n y)^T$, $y = (x_1, x_2, \dots, x_n)^T \in \mathbb{X}$, where, for $i = 1, 2, \dots, n$, $t \in \mathbb{R}$,*

$$\begin{aligned} G_i y(t) &= r_i(t) - a_{ii}(t) \exp\{x_i(t)\} \\ &\quad - \sum_{j=1, j\neq i}^{n} a_{ij}(t) \exp\big\{x_j\big(t - \tau_j(t, \exp\{x_1(t)\}, \dots, \exp\{x_n(t)\})\big)\big\}, \end{aligned}$$

$Py = m(y)$, $y \in \mathbb{X}$, $Qz = m(z)$, and $z \in \mathbb{Y}$. Then N is L-compact on $\bar{\Omega}$, where Ω is any open bounded subset of $\mathbb{X}$.

Proof. The projections P and Q are continuous such that $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q$. It is clear that

$$(I - Q)V_2 = \{\mathbf{0}\} \quad \text{and} \quad (I - Q)V_1 = V_1.$$

Therefore, $\operatorname{Im}(I - Q) = V_1 = \operatorname{Im} L$. In view of

$$\operatorname{Im} P = \ker L \quad \text{and} \quad \operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q),$$

we can conclude that the generalized inverse (of L) $K_P : \operatorname{Im} L \to \ker P \cap \operatorname{Dom} L$ exists and is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - m\Big[\int_0^t z(s)\,\mathrm{d}s\Big].$$

Thus

$$QNy = (H_1 y, H_2 y, \dots, H_n y)^T,$$
$$K_P(I-Q)Ny = f[y(t)] - Qf[y(t)],$$

where

$$f[y(t)] = \int_0^t [Ny(s) - QNy(s)]\,\mathrm{d}s$$

and

$$\begin{aligned} H_i y = m[G_i y] = m\Big[& r_i(t) - a_{ii}(t)\exp\{x_i(t)\} \\ & - \sum_{j=1, j\neq i}^{n} a_{ij}(t)\exp\big\{x_j\big(t - \tau_j(t, \exp\{x_1(t)\}, \dots, \exp\{x_n(t)\})\big)\big\}\Big] \end{aligned}$$

for $i = 1, 2, \dots, n$. QN and $(I-Q)N$ are obviously continuous. Now we claim that K_P is also continuous. By our hypothesis, for any $\varepsilon < 1$ and any compact set $S \subset C([-\sigma, 0], \mathbb{R}^n)$, let $l(\varepsilon, S)$ be the inclusion interval of $E\{F, \varepsilon, S\}$. Suppose that $\{z_n(t)\} \subset \operatorname{Im} L = V_1$ and $z_n(t)$ uniformly converges to $z_0(t)$. Since $\int_0^t z_n(s)\,\mathrm{d}s \in \mathbb{Y}\ (n = 0, 1, 2, \dots)$, there exists ρ, $(0 < \rho < \varepsilon)$ such that $E\{F, \rho, S\} \subset E\{\int_0^t z_n(s)\,\mathrm{d}s, \varepsilon\}$. Let $l(\rho, S)$ be the inclusion interval of $E\{F, \rho, S\}$ and

$$l = \max\{l(\rho, S),\, l(\varepsilon, S)\}.$$

It is easy to see that l is the inclusion interval of both $E\{F, \varepsilon, S\}$ and $E\{F, \rho, S\}$. Hence, for all $t \notin [0, l]$, there exists $\tau_t \in E\{F, \rho, S\} \subset E\{\int_0^t z_n(s)\,\mathrm{d}s, \varepsilon\}$ such that $t + \tau_t \in [0, l]$. Therefore, by the definition of almost periodic functions we observe that

$$\begin{aligned} &\Big\| \int_0^t z_n(s)\,\mathrm{d}s \Big\| \\ &= \sup_{t\in\mathbb{R}} \Big| \int_0^t z_n(s)\,\mathrm{d}s \Big| \\ &\le \sup_{t\in[0,l]} \Big| \int_0^t z_n(s)\,\mathrm{d}s \Big| + \sup_{t\notin[0,l]} \Big| \Big(\int_0^t z_n(s)\,\mathrm{d}s - \int_0^{t+\tau_t} z_n(s)\,\mathrm{d}s \Big) + \int_0^{t+\tau_t} z_n(s)\,\mathrm{d}s \Big| \\ &\le 2 \sup_{t\in[0,l]} \Big| \int_0^t z_n(s)\,\mathrm{d}s \Big| + \sup_{t\notin[0,l]} \Big| \int_0^t z_n(s)\,\mathrm{d}s - \int_0^{t+\tau_t} z_n(s)\,\mathrm{d}s \Big| \\ &\le 2 \int_0^t |z_n(s)|\,\mathrm{d}s + \varepsilon. \end{aligned} \tag{3.2}$$

By applying (3.2), we conclude that $\int_0^t z(s)\,\mathrm{d}s\ (z \in \operatorname{Im} L)$ is continuous and consequently K_P and $K_P(I-Q)Ny$ are also continuous.

From (3.2), we also have that $\int_0^t z(s)\,\mathrm{d}s$ and $K_P(I-Q)Ny$ are uniformly bounded in $\bar{\Omega}$. In addition, we can easily conclude that $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)Ny$ is equicontinuous in $\bar{\Omega}$. Hence by the Arzelà-Ascoli theorem, we can immediately conclude that $K_P(I-Q)N(\bar{\Omega})$ is compact. Thus N is L-compact on $\bar{\Omega}$. The proof is complete. $\square$

Theorem 3.4. *If* (H1) *holds and the following conditions are satisfied:*

(H2) $m[r_i] > 0$, $i = 1, 2, \dots, n$.

(H3) $\sum_{j=1}^{n} m[a_{ij}] > 0$, $i = 1, 2, \dots, n$.
(H4) *The system of linear algebraic equations*

$$m[r_i] = \sum_{j=1}^{n} m[a_{ij}]v_j, \quad i = 1, 2, \dots, n \tag{3.3}$$

has a unique solution $(v_1^*, v_2^*, \dots, v_n^*)^T \in \mathbb{R}^n$ *with* $v_i^* > 0$, $i = 1, 2, \dots, n$.

Then (1.1) *has at least one positive almost periodic solution.*

Proof. To apply the continuation theorem of coincidence degree theory, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ the same as those in Lemma 3.1 and the mappings L, N, P, Q the same as those defined in Lemmas 3.2 and 3.3, respectively. Thus, we can obtain that L is a Fredholm mapping of index zero and N is a continuous operator which is L-compact on $\bar{\Omega}$. It remains to search for an appropriate open and bounded subset Ω.

Corresponding to the operator equation $Ly = \lambda Ny$, $\lambda \in (0, 1)$, where $y = (x_1, x_2, \dots, x_n)^T$, we obtain, for $i = 1, 2, \dots, n$,

$$\begin{aligned} \dot{x}_i(t) = \lambda\Big[& r_i(t) - a_{ii}(t)\exp\{x_i(t)\} - \sum_{j=1, j\neq i}^{n} a_{ij}(t) \\ & \times \exp\big\{x_j\big(t - \tau_j(t, \exp\{x_1(t)\}, \dots, \exp\{x_n(t)\})\big)\big\}\Big]. \end{aligned} \tag{3.4}$$

Suppose that $y \in \mathbb{X}$ is a solution of (3.4) for a certain $\lambda \in (0, 1)$. For any $t_0 \in \mathbb{R}$, we can choose a point $\tilde{\tau} - t_0 \in [l, 2l] \cap E\{F, \rho, S\}$, where $\rho\,(0 < \rho < \varepsilon)$ satisfies $E\{F, \rho\} \subset E\{y, \varepsilon\}$. Integrating (3.4) from t_0 to $\tilde{\tau}$, we obtain

$$\begin{aligned} & \lambda \int_{t_0}^{\tilde{\tau}} \Big[a_{ii}(s)\exp\{x_i(s)\} \\ & + \sum_{j=1, j\neq i}^{n} a_{ij}(s)\exp\big\{x_j\big(s - \tau_j(s, \exp\{x_1(s)\}, \dots, \exp\{x_n(s)\})\big)\big\}\Big]\mathrm{d}s \\ & \le \lambda \int_{t_0}^{\tilde{\tau}} r_i(s)\,\mathrm{d}s + \Big|\int_{t_0}^{\tilde{\tau}} \dot{x}_i(s)\,\mathrm{d}s\Big| \le \lambda \int_{t_0}^{\tilde{\tau}} r_i(s)\,\mathrm{d}s + \varepsilon, \quad i = 1, 2, \dots, n. \end{aligned} \tag{3.5}$$

Hence, from (3.4) and (3.5), we obtain

$$\begin{aligned} \int_{t_0}^{\tilde{\tau}} |\dot{x}_i(s)|\,\mathrm{d}s & \le \lambda \int_{t_0}^{\tilde{\tau}} r_i(s)\,\mathrm{d}s + \lambda \int_{t_0}^{\tilde{\tau}} \Big[a_{ii}(s)\exp\{x_i(s)\} \\ & \quad + \sum_{j=1, j\neq i}^{n} a_{ij}(s)\exp\big\{x_j\big(s - \tau_j(s, \exp\{x_1(s)\}, \dots, \exp\{x_n(s)\})\big)\big\}\Big]\,\mathrm{d}s \\ & \le 2\lambda \int_{t_0}^{\tilde{\tau}} r_i(s)\,\mathrm{d}s + \varepsilon \le 2\lambda \int_{t_0}^{\tilde{\tau}} r_i(s)\,\mathrm{d}s + 1 := A_i, \quad i = 1, 2, \dots, n. \end{aligned}$$

Therefore, for $\tilde{\tau} \ge t_0 + l$, we can easily have

$$\int_{t_0}^{t_0+l} |\dot{x}_i(t)|\,\mathrm{d}t \le A_i, \quad i = 1, 2, \dots, n.$$

Denote

$$\bar{\theta} = \max_{1\le i\le n} \sup_{t\in\mathbb{R}} x_i(t), \quad \underline{\theta} = \min_{1\le i\le n} \inf_{t\in\mathbb{R}} x_i(t), \quad i = 1, 2, \dots, n.$$

In view of (3.4), for $i=1,2,\dots,n$, we obtain

$$\begin{aligned} m[r_i] = m\Big[&a_{ii}(t)\exp\{x_i(t)\} \\ &+ \sum_{j=1,j\neq i}^{n} a_{ij}(t)\exp\big\{x_j\big(t-\tau_j(t,\exp\{x_1(t)\},\dots,\exp\{x_n(t)\})\big)\big\}\Big]. \end{aligned} \tag{3.6}$$

From (3.6), one has

$$m[r_i] \geq \sum_{j=1}^{n} m[a_{ij}]\exp\{\underline{\theta}\}, \quad i=1,2,\dots,n,$$

or

$$\underline{\theta} \leq \ln\frac{m[r_i]}{\sum_{j=1}^{n} m[a_{ij}]}, \quad i=1,2,\dots,n.$$

Consequently, by Lemma 2.2, for any $\varepsilon>0$, there exists a ξ_ε^i such that

$$\begin{aligned} x_i(t) &\leq x_i(\xi_\varepsilon^i) + \int_{t_0}^{t_0+l} |\dot{x}_i(t)|\,\mathrm{d}t < (\underline{\theta}+\varepsilon) + A_i \\ &< \ln\frac{m[r_i]}{\sum_{j=1}^{n} m[a_{ij}]} + 1 + A_i, \quad i=1,2,\dots,n. \end{aligned} \tag{3.7}$$

Similarly, we obtain

$$m[r_i] \leq \Big\{\sum_{j=1}^{n} m[a_{ij}]\Big\}\exp\{\bar{\theta}\}, \quad i=1,2,\dots,n,$$

so

$$\bar{\theta} \geq \ln\frac{m[r_i]}{\sum_{j=1}^{n} m[a_{ij}]}, \quad i=1,2,\dots,n.$$

By Lemma 2.2, for any $\varepsilon>0$, there exists a η_ε^i such that

$$\begin{aligned} x_i(t) &\geq x_i(\eta_\varepsilon^i) - \int_{t_0}^{t_0+l} |\dot{x}_1(t)|\,\mathrm{d}t > (\bar{\theta}-\varepsilon) - A_i \\ &\geq \ln\frac{m[r_i]}{\sum_{j=1}^{n} m[a_{ij}]} - A_i - 1, \quad i=1,2,\dots,n. \end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that for $i=1,2,\dots,n$,

$$\begin{aligned} &\sup_{t\in\mathbb{R}} |x_i(t)| \\ &\leq \max\Big\{\Big|\ln\frac{m[r_i]}{\sum_{j=1}^{n} m[a_{ij}]} + (A_i+1)\Big|, \Big|\ln\frac{m[r_i]}{\sum_{j=1}^{n} m[a_{ij}]} - (A_i+1)\Big|\Big\} := M_i. \end{aligned} \tag{3.9}$$

Clearly, $M_i (i=1,2,\dots,n))$ are independent of the choice of λ. Take $M = \max_{1\leq i\leq n}\{M_i\}+K$, where $K>0$ is taken sufficiently large such that the unique solution $(v_1^*,v_2^*,\dots,v_n^*)^T$ of system (3.3) satisfies $\|(v_1^*,v_2^*,\dots,v_n^*)^T\| < M$. Next, take

$$\Omega = \big\{y(t) = (x_1(t),x_2(t),\dots,x_n(t))^T \in \mathbb{X} : \|x\| < M\big\},$$

then it is clear that Ω satisfies condition (1) of Lemma 2.2. When $y\in\partial\Omega\cap\ker L$, then y is a constant vector with $\|y\|=M$. Hence

$$QNy = (H_1y, H_2y, \dots, H_ny)^T \neq \mathbf{0},$$

where

$$H_i y = m[G_i y] = m\big[r_i\big] - \sum_{j=1}^{n} m[a_{ij}]\exp\{x_j\},\ i = 1, 2, \dots, n,$$

which implies that condition (2) of Lemma 2.1 is satisfied. Furthermore, take $J : \operatorname{Im} Q \to \ker L$ such that $J(z) = z$ for $z \in \mathbb{Y}$. In view of (H4), by a straightforward computation, we find

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \operatorname{sgn}\{(-1)^n[\det(m(a_{ij}))]e^{\Sigma_{i=1}^{n} v_i^*}\} \neq 0.$$

Therefore, condition (3) of Lemma 2.1 holds. Hence, $Ly = Ny$ has at least one solution in $\operatorname{Dom} L \cap \bar{\Omega}$. In other words, (3.1) has at least one almost periodic solution $x(t)$, that is, (1.1) has at least one positive almost periodic solution $(u_1(t), \dots, u_n(t))^T$. The proof is complete. □

4. An example and simulation

Consider the Lotka-Volterra system

$$\begin{aligned}\dot{u}(t) &= u(t)\big[3 - \cos\sqrt{2}t - (3 - \cos t)u(t) - (2 + \sin t)v\big(t - \tau_1(t, u(t), v(t))\big)\big],\\ \dot{v}(t) &= v(t)\big[2 - \sin\sqrt{3}t - (1 - \sin t)v(t) - (3 - \cos\sqrt{2}t)u(t - \tau_2(t, u(t), v(t)))\big],\end{aligned}$$

where $\tau_i \in C(\mathbb{R}^3, \mathbb{R})$ $(i = 1, 2)$ are almost periodic in t uniformly for $(u, v)^T \in \mathbb{R}^2$.

One can calculate that $m[r_1] = 3$, $m[r_2] = 2$, $m[a_{11}] = 3$, $m[a_{22}] = 1$, $m[a_{12}] = 2$ $m[a_{21}] = 3$. It is easy to check that (H1)–(H4) are satisfied. By Theorem 3.4, Equation (1.1) has at least one positive almost periodic solution $(u(t), v(t))^T$. We take $\tau_1(t, u(t), v(t)) = \exp\{\sin\sqrt{2}v(t) + \cos\sqrt{3}u(t)\}\cos t$ and $\tau_2(t, u(t), v(t)) = \exp\{\sin\sqrt{3}v(t) + \cos u(t)\}\sin\sqrt{2}t$. Figure 1 shows the numerical simulation which illustrates the effectiveness of our results.

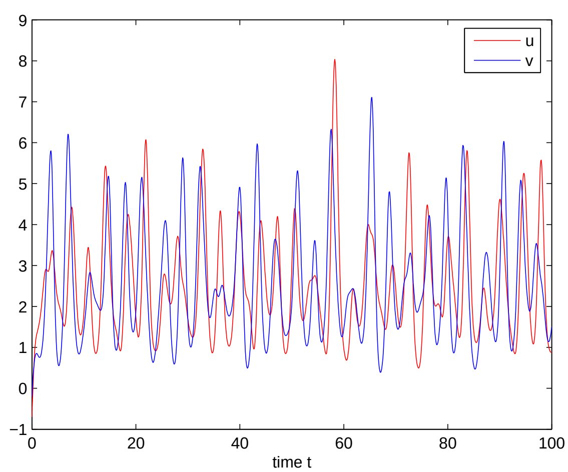

Figure 1. Population density for the two species u, v

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Yongkun Li

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China

E-mail address: yklie@ynu.edu.cn

Chao Wang

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China

E-mail address: super2003050239@163.com