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NONEXISTENCE OF SELF-SIMILAR SINGULARITIES IN IDEAL VISCOELASTIC FLOWS

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Abstract. We prove the nonexistence of finite time self-similar singularities in an ideal viscoelastic flow in $\mathbb{R}^3$. We exclude the occurrence of Leray-type self-similar singularities under suitable integrability conditions on velocity and deformation tensor. We also prove the nonexistence of asymptotically self-similar singularities in our system. The present work extends the results obtained by Chae in the case of magnetohydrodynamics (MHD).

1. Introduction

We prove the nonexistence of finite time self-similar singularity in the Cauchy problem of the ideal viscoelastic flow in $\mathbb{R}^3$:

$$\begin{gathered} u_t + (u\cdot\nabla)u = -\nabla p + \nabla\cdot FF^T,\\ F^k_t + (u\cdot\nabla)F^k = (F^k\cdot\nabla)u,\\ \operatorname{div}(u) = 0,\\ u(x,0)=u_0,\quad F(x,0)=F_0(x). \end{gathered} \tag{1.1}$$

Here $x\in\mathbb{R}^3$ is the spatial variable and $t\ge 0$ is the time, $u=u(x,t)=(u^1,u^2,u^3)$ is the velocity of the flow, $P=P(x,t)$ is a prescribed scalar pressure, $F=F(x,t)\in\mathbb{R}^{3\times 3}$ is the deformation tensor with F^k being the k-th column of F. The initial data (u_0,F_0) is assumed to satisfy the following condition:

$$\operatorname{div}(u_0)=\operatorname{div}(F^k_0)=0. \tag{1.2}$$

By taking divergence on the second equation in (1.1) and using $\operatorname{div}(u)=0$,

$$\operatorname{div}(F^k)_t + (u\cdot\nabla)(\operatorname{div}(F^k)) = 0,$$

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so F is "divergence free" for later time. In view of the above, we can rewrite (1.1)-(1.2) into the following equivalent form:

$$\begin{gathered}u_t + (u \cdot \nabla)u = -\nabla p + \sum (F^k \cdot \nabla)F^k, \\ F^k_t + (u \cdot \nabla)F^k = (F^k \cdot \nabla)u, \\ \operatorname{div}(u) = \operatorname{div}(F^k) = 0, \\ u(x,0) = u_0, \quad F(x,0) = F_0(x).\end{gathered} \tag{1.3}$$

We refer to [2] or [6] for a more detailed derivation and physical discussion of the above system.

The study of incompressible fluids can be dated back to 18th century when Euler [7] considered the well-known Euler's equations:

$$\begin{gathered}u_t + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div}(u) = 0.\end{gathered} \tag{1.4}$$

The local-in-time existence of (1.4) can be found in Kato [10], yet it is still interesting to know whether or not finite time singularities do occur in the above system. There seems to be no satisfactory result until Beale et al [1] discovered that, if a solution u for (1.4) possesses a singularity at a finite time $T > 0$, then it is necessary to have

$$\int_0^T \|\nabla \times u(\cdot,t)\|_{L^\infty} dt = +\infty.$$

The above blow-up criterion is later generalized to many other related systems, for example ideal magnetohydrodynamics (MHD) (Caflisch-Klapper-Steel [3]) and ideal viscoelastic flow (Hu and Hynd [8]). With a clever argument suggested by Chae in [4]-[5], he applied the results in Caflisch et al [3] to exclude the possibility of a finite time apparition of self-similar singularities in both viscous and ideal MHD model. The goal of our present work is to extend Chae's results to ideal viscoelastic flow based on the blow-up criterion shown by Hu and Hynd [8].

We now give a precise formulation of our result. To begin with, we observe that if (u, F, p) is a solution of (1.3), then for any $\lambda > 0$, $(u^{(\lambda)}, F^{(\lambda)}, p^{(\lambda)})$ is also a solution with the initial data $(u_0^{(\lambda)}, F_0^{(\lambda)})$, where

$$\begin{gathered}u^{(\lambda)}(x,t) = \lambda u(\lambda x, \lambda^2 t), \\ F^{(\lambda)}(x,t) = \lambda F(\lambda x, \lambda^2 t), \\ p^{(\lambda)}(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \\ (u_0^{(\lambda)}(x), F_0^{(\lambda)}(x)) = (\lambda u_0(\lambda x), \lambda F_0(\lambda x)).\end{gathered}$$

Therefore, if there exists a self-similar blowing up solution $(u(x,t), F(x,t))$ of (1.3), then it has to be of the form

$$u(x,t) = \frac{1}{\sqrt{T_* - t}} U\Big(\frac{x}{\sqrt{T_* - t}}\Big), \tag{1.5}$$

$$F^k(x,t) = \frac{1}{\sqrt{T_* - t}} E^k\Big(\frac{x}{\sqrt{T_* - t}}\Big), \tag{1.6}$$

$$p(x,t) = \frac{1}{T_* - t} P\Big(\frac{x}{\sqrt{T_* - t}}\Big), \tag{1.7}$$

when t is being close to the possible blow-up time T_*. By substituting (1.5)-(1.7) into (1.3), we find that $(U, E, P) = (U(y), E(y), P(y))$ satisfies of the stationary system

$$\begin{gathered}\frac{1}{2}U + \frac{1}{2}(y \cdot \nabla)U + (U \cdot \nabla)U = -\nabla P + \sum_k (E^k \cdot \nabla)E^k, \\ \frac{1}{2}E^k + \frac{1}{2}(y \cdot \nabla)E^k + (U \cdot \nabla)E^k = (E^k \cdot \nabla)U, \\ \operatorname{div}(U) = \operatorname{div}(E^k) = 0.\end{gathered} \tag{1.8}$$

Conversely, if (U, E, P) is a smooth solution of (1.8), then (u, F, p) as defined by (1.5)-(1.7) is a classical solution of (1.3) which blows up at $t = T_*$. Leray [9] was the first one to study self-similar singularities of the form similar to (1.5)-(1.7) in Navier-Stokes equations, while Chae [4]-[5] considered those in the case of (MHD). We apply those concepts to viscoelastic flow which will be given in later sections.

The following theorems are the main results of this paper:

Theorem 1.1. *Suppose there exists $T_* > 0$ such that we have a representation of a solution (u, F) to (1.3) by (1.5)-(1.7) for all $t \in (0, T_*)$ with (U, E) satisfying the following conditions:*

$$(U, E) \in (C_0^1(\mathbb{R}^3))^2, \tag{1.9}$$

$$\begin{gathered}\text{there exists } q > 0 \text{ such that } (\Omega, E) \in (L^r(\mathbb{R}^3))^2 \text{ for all } r \in (0, q), \\ \text{where } \Omega = \nabla \times U.\end{gathered} \tag{1.10}$$

Then we have $U = E = 0$.

The above theorem can be proved in the same way as in Chae [5, pp. 1014–1017], so we omit the proof here. By changing the decay conditions on (U, E), we can derive similar results about the nonexistence of self-similar singularities. The following theorem is reminiscent of [4, Theorem 1.2].

Theorem 1.2. *Suppose there exists $T_* > 0$ such that we have a representation of a solution (u, F) to (1.3) by (1.5)-(1.7) for all $t \in (0, T_*)$ with (U, E) satisfying the following conditions:*

$$(U, E) \in (H^m(\mathbb{R}^3))^2 \quad \text{for } m > \frac{3}{2} + 1, \tag{1.11}$$

$$\|\nabla U\|_{L^\infty} + \|\nabla E\|_{L^\infty} < \varepsilon, \tag{1.12}$$

where $\varepsilon > 0$ is a constant as chosen in Theorem 2.1. Then $U = E = 0$.

Using Theorems 1.1-1.2, we can obtain the following result about asymptotically self-similar singularity, which is reminiscent of [4, Theorem 1.3]. We refer to Chae [5] (especially section 2) for more detailed descriptions of asymptotically self-similar singularities.

Theorem 1.3. *Given $T > 0$ and $m > \frac{3}{2} + 1$, let $(u, F) \in ((C([0, T); H^m(\mathbb{R}^3)))^2$ be a classical solutions to (1.3). Suppose there exists functions U, E satisfying (1.7)–(1.8) as in Theorem 1.1 such that*

$$\begin{aligned}&\sup_{0<t<T} \frac{1}{T-t} \Big\| u(\cdot, t) - \frac{1}{\sqrt{T-t}} U\Big(\frac{\cdot}{\sqrt{T-t}}\Big) \Big\|_{L^1} \\ &+ \sup_{0<t<T} \frac{1}{T-t} \Big\| F(\cdot, t) - \frac{1}{\sqrt{T-t}} E\Big(\frac{\cdot}{\sqrt{T-t}}\Big) \Big\|_{L^1} < \infty,\end{aligned} \tag{1.13}$$

and

$$\begin{aligned} &\lim_{t\nearrow T}(T-t)\big\|\nabla u(\cdot,t)-\frac{1}{\sqrt{T-t}}\nabla U\Big(\frac{\cdot}{\sqrt{T-t}}\Big)\big\|_{L^\infty} \\ &+\lim_{t\nearrow T}(T-t)\big\|\nabla F(\cdot,t)-\frac{1}{\sqrt{T-t}}\nabla E\Big(\frac{\cdot}{\sqrt{T-t}}\Big)\big\|_{L^\infty}=0, \end{aligned} \tag{1.14}$$

then $U=E=0$.

Moreover, there exists $\delta>0$ *such that* (u,F) *can be extended to a solution of* (1.3) *in* $[0,T+\delta]\times\mathbb{R}^3$ *with* $(u,F)\in C([0,T+\delta];H^m(\mathbb{R}^3))$.

This article is organized as follows. In section 2 we first prove Theorem 2.1 about a continuation principle for local solution of (1.3) with the help of an auxiliary lemma. We then begin the proofs of Theorems 1.2-1.3 in section 3 which basically follow the arguments as given in Chae [4]-[5].

2. Continuation Principle for Local Solution

In this section we show the continuation principle for local solution of (1.3) with the help of an energy estimate as given in Lemma 2.2. The result is reminiscent of [4, Lemma 2.1] except that we apply the blow-up criterion for singularities in ideal viscoelastic flow proved by Hu and Hynd [8]. We begin with the following theorem.

Theorem 2.1. *Given* $T>0$ *and* $m>\frac{3}{2}+1$, *let* $(u,F)\in((C([0,T);H^m(\mathbb{R}^3)))^2$ *be a classical solutions to* (1.3). *There exists* $\varepsilon>0$ *and* $\delta>0$ *such that if*

$$\sup_{0\le t<T}(T-t)[\|\nabla u(\cdot,t)\|_{L^\infty}+\|\nabla F(\cdot,t)\|_{L^\infty}]<\varepsilon, \tag{2.1}$$

then (u,F) *can be extended to a solution of* (1.3) *in* $[0,T+\delta]\times\mathbb{R}^3$ *satisfying* $(u,F)\in C([0,T+\delta];H^m(\mathbb{R}^3))$.

Theorem 2.1 can be proved by an argument developed by Chae [4]. It requires an energy estimate which is given by the following lemma:

Lemma 2.2. *Assume that the hypotheses and notation are as in Theorem 2.1. Then for* $0\le t<T$, *we have the following estimates*

$$\begin{aligned} &\frac{d}{dt}\Big(\|u\|_{H^m}^2+\sum_k\|F^k\|_{H^m}^2\Big) \\ &\le M\Big(\|\nabla u\|_{L^\infty}+\sum_k\|\nabla F^k\|_{L^\infty}^2\Big)\Big(\|u\|_{H^m}^2+\sum_k\|F^k\|_{H^m}^2\Big), \end{aligned} \tag{2.2}$$

where $M>0$ *is a generic constant which depends only on* m.

The proof of the above lemma is exactly as in Hu and Hynd [8, pp. 4–6].

Proof of Theorem 2.1. We follow the proof of [4, Lemma 2.1]. We claim

$$\int_0^T(\|\nabla u\|_{L^\infty}^2+\sum_k\|\nabla F^k\|_{L^\infty}^2)ds<\infty, \tag{2.3}$$

and if (2.3) holds, then we have

$$\int_0^T(\|\nabla\times u\|_{L^\infty}+\sum_k\|\nabla\times F^k\|_{L^\infty})ds<\infty,$$

and so by the blow-up criterion derived in [8], there exists $\delta > 0$ such that (u, F) can be continuously extended to a solution of (1.3) in $[0, T+\delta] \times \mathbb{R}^3$ with $(u, F) \in (C([0, T+\delta]; H^m(\mathbb{R}^3)))^2$.

To prove (2.3), we apply (2.2) as in Lemma 2.2 to obtain

$$\begin{aligned} &\frac{d}{dt}\Big[(T-t)(\|u\|_{H^m}^2 + \sum_k \|F^k\|_{H^m}^2)\Big] + \|u\|_{H^m}^2 + \sum_k \|F^k\|_{H^m}^2 \\ &\le M(T-t)\Big(\|\nabla u\|_{L^\infty} + \sum_k \|F^k\|_{L^\infty}\Big)\Big(\|u\|_{H^m}^2 + \sum_k \|F^k\|_{H^m}^2\Big). \end{aligned} \tag{2.4}$$

If we choose $\varepsilon < 1/(2M)$, then (2.1) and (2.4) imply

$$\frac{d}{dt}\Big[(T-t)(\|u\|_{H^m}^2 + \sum_k \|F^k\|_{H^m}^2)\Big] + \frac{1}{2}\Big(\|u\|_{H^m}^2 + \sum_k \|F^k\|_{H^m}^2\Big) \le 0.$$

Upon integrating the above,

$$\int_0^T (\|u\|_{H^m}^2 + \sum_k \|F^k\|_{H^m}^2) ds \le 2T(\|u_0\|_{H^m}^2 + \sum_k \|F_0^k\|_{H^m}^2) < \infty \tag{2.5}$$

By the Sobolev embedding theorem, there exists $l \ge 1$ and $\beta \in (0, 1)$ such that $H^m(\mathbb{R}^3) \hookrightarrow C^{l,\beta}(\mathbb{R}^3)$, so we obtain from (2.5) that

$$\begin{aligned} \int_0^T (\|\nabla u\|_{L^\infty} + \sum_k \|\nabla F^k\|_{L^\infty}) ds &\le \int_0^T (\|u\|_{H^m} + \sum_k \|F^k\|_{H^m}) ds \\ &\le \sqrt{T}\Big[\int_0^T (\|u\|_{H^m}^2 + \sum_k \|F^k\|_{H^m}^2) ds\Big]^{1/2} < \infty, \end{aligned}$$

and hence (2.3) follows. □

3. Nonexistence of Asymptotically Self-similar Singularities

In this section we prove Theorems 1.2-1.3 as stated in section 1. These arguments are reminiscent of Chae [4] and hence we omit some of those technical details.

Proof of Theorem 1.2. By the definitions (1.5)–(1.6) for $t \in (0, T_*)$,

$$\begin{gathered} \|U(\cdot, t)\|_{L^\infty} = (T_* - t)\|\nabla u(\cdot, t)\|_{L^\infty}, \\ \|\nabla E(\cdot, t)\|_{L^\infty} = (T_* - t)\|\nabla F(\cdot, t)\|_{L^\infty}. \end{gathered}$$

So assumption (1.12) implies

$$\sup_{0 \le t < T_*} (T_* - t)\,[\|\nabla u(\cdot, t)\|_{L^\infty} + \|\nabla F(\cdot, t)\|_{L^\infty}] < \varepsilon.$$

By Theorem 2.1, (u, F) can be continuously extended to a solution of (1.3) in $[0, T_* + \delta] \times \mathbb{R}^3$, which is impossible unless $u = F = 0$. Therefore $U = E = 0$ as required. □

Proof of Theorem 1.3. Define $(\tilde{u}, \tilde{F}, \tilde{p})$ by

$$\tilde{u}(y, s) = (\sqrt{T-t})u(x, t), \quad \tilde{F}(y, s) = (\sqrt{T-t})F(x, t), \quad \tilde{p}(y, s) = (\sqrt{T-t})p(x, t),$$

where

$$y = \frac{x}{\sqrt{T-t}}, \quad s = \frac{1}{2}\log\Big(\frac{T}{T-t}\Big).$$

Assumptions (1.12)-(1.14) can then be rewritten as

$$\sup_{0<s<\infty} \|\tilde{u}(\cdot,s) - U(\cdot)\|_{L^1} + \sup_{0<s<\infty} \|\tilde{F}(\cdot,s) - E(\cdot)\|_{L^1} < \infty \tag{3.1}$$

and

$$\lim_{s\to\infty} \|\nabla\tilde{u}(\cdot,s) - \nabla U(\cdot)\|_{L^\infty} + \|\nabla\tilde{F}(\cdot,s) - \nabla E(\cdot)\|_{L^\infty} = 0, \tag{3.2}$$

and we obtain $\lim_{s\to\infty} \|\tilde{u}(\cdot,t) - U(\cdot)\|_{H^1(B_R)} = \lim_{s\to\infty} \|\tilde{F}(\cdot,t) - E(\cdot)\|_{H^1(B_R)} = 0$ for all $R > 0$ where $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$. Hence $U, E \in C_0^1(\mathbb{R}^3)$. Apply the same argument as in Chae [4] pp.451–453, we have that (U, E) is a classical solution of (1.3) satisfying (1.10), and so Theorem 1.1 implies $U = E = 0$.

Next we put $U = E = 0$ into (3.2), then

$$\lim_{s\to\infty} \|\nabla\tilde{u}(\cdot,s)\|_{L^\infty} + \|\nabla\tilde{F}(\cdot,s)\|_{L^\infty} = 0,$$

and so there exists $s' > 0$ such that

$$\sup_{s'\le s<\infty} \|\nabla\tilde{u}(\cdot,s)\|_{L^\infty} + \|\nabla\tilde{F}(\cdot,s)\|_{L^\infty} < \varepsilon,$$

where $\varepsilon > 0$ is as chosen in Theorem 2.1. Let $t' = T(1-e^{-2s'})$, then (u, F) satisfies

$$\sup_{t'\le t<T} (T-t)(\|\nabla u(\cdot,t)\|_{L^\infty} + \|\nabla F(\cdot,t)\|_{L^\infty}) < \varepsilon,$$

and by Theorem 2.1, there exists $\delta > 0$ such that (u, F) can be extended to a solution of (1.3) in $[0, T + \delta] \times \mathbb{R}^3$ with $(u, F) \in C([0, T + \delta]; H^m(\mathbb{R}^3))$. This completes the proof of Theorem 1.3. □

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