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GROWTH OF ENTIRE SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, by means of the normal family theory, we estimate the growth order of entire solutions of some algebraic differential equations and extend the result by Qi et al [14]. We also give some examples to show that our results occur in some special cases.

1. INTRODUCTION AND MAIN RESULTS

Let $f(z)$ be a holomorphic function on the complex plane. We use the standard notation of Nevanlinna theory and denote the order of $f(z)$ by $\rho(f)$ (see Hayman [11], He [12], Laine [13] and Yang [15]). Let D be a domain in the complex plane. A family $\mathcal{F}$ of meromorphic functions in D is normal, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges locally uniformly by spherical distance to a function $g(z)$ meromorphic in D ($g(z)$ is permitted to be identically infinity).

We define spherical derivative of the meromorphic function $f(z)$ as follows

$$f^{\sharp}(z) := \frac{|f'(z)|}{1 + |f(z)|^2}.$$

An algebraic differential equation for $w(z)$ is of the form

$$P(z, w, w', \dots, w^{(k)}) = 0, \tag{1.1}$$

where P is a polynomial in each of its variables.

It is one of the important and interesting subjects to study the growth of meromorphic the solution $w(z)$ of differential equation (1.1) in the complex plane.

In 1956, Goldberg [8] proved that the meromorphic solutions have finite growth order when $k = 1$. Some alternative proofs of this result have been given by Bank and Kaufman [1], by Barsegian [2].

In 1998, Barsegian [4, 3] introduced an essentially new type of weight for differential monomial below and gave the estimates first time for the growth order of meromorphic solutions of large classes of complex differential equations of higher

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degrees by using his initial method [5]. Later Bergweiler [6] reproved Barsegian's result by using Zalcman's Lemma.

To state the result, we first introduce some notation [4]: $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $t_j \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ for $j = 1, 2, \dots, n$, and put $t = (t_1, t_2, \dots, t_n)$. Define $M_t[w]$ by

$$M_t[w](z) := [w'(z)]^{t_1}[w''(z)]^{t_2} \dots [w^{(n)}]^{t_n},$$

with the convention that $M_{\{0\}}[w] = 1$. We call $p(t) := t_1 + 2t_2 + \dots + nt_n$ the weight of $M_t[w]$. A differential polynomial $P[w]$ is an expression of the form

$$P[w](z) := \sum_{t \in I} a_t(z, w(z)) M_t[w](z), \tag{1.2}$$

where the a_t are rational in two variables and I is a finite index set. The total weight $W(P)$ of $P[w]$ is given by $W(P) := \max_{t \in I} p(t)$.

Definition 1.1. $\deg_{z,\infty} a_t$ denotes the degree at infinity in variable z concerning $a_t(z, w)$. $\deg_{z,\infty} a := \max_{t \in I} \max\{\deg_{z,\infty} a_t, 0\}$.

In 2009, the general estimates of growth order of meromorphic solutions $w(z)$ of the same equations, which depend on the degrees at infinity of coefficients of differential polynomial in z, by Yuan et al [16, 17].

Theorem 1.2 ([16]). *Let $w(z)$ be meromorphic in the complex plane and let $P[w]$ be a differential polynomial. If $w(z)$ satisfies the differential equation $[w'(z)]^n = P[w]$ where $n \in \mathbb{N}$ and $n > W(P)$, then the growth order $\rho := \rho(w)$ of $w(z)$ satisfies*

$$\rho \le 2 + \frac{2 \deg_{z,\infty} a}{n - W(P)}.$$

.

Barsegian [3] and Bergweiler [6] proved $\rho < \infty$ under the same conditions as Theorem 1.2. Gu et al [9] considered the case of entire solutions. In 2012, Qi et al [14] gave a small upper bound of the growth order of the entire solutions.

Theorem 1.3. *Let $k, q, n \in \mathbb{N}$, and $P[w]$ be a differential polynomial with the form (1.2). Suppose that $w(z)$ is an entire function whose all zeros have multiplicity at least and $nkq > W(P)$. If $w(z)$ satisfies the differential equation $[Q(w^{(k)}(z))]^n = P[w]$, then the growth order $\rho := \rho(w)$ of $w(z)$ satisfies*

$$\rho \le 1 + \frac{\deg_{z,\infty} a}{nqk - W(P)},$$

where $Q(z)$ is a polynomial of degree q.

In 2011, Gu et al [9] obtained Theorem 1.3 when $k = 1, q = 1$. In this article, we extend Theorem 1.3, and obtain the following result.

Theorem 1.4. *Let $k, m, n, q \in \mathbb{N}$, and let $P[w]$ be a differential polynomial. If an entire function $w(z)$ whose all zeros have multiplicities at least k satisfies the differential equation $[([Q(w^{(k-1)}(z))]^n)']^m = P[w]$ and $(nqk - nq + 1)m > W(P)$, then the growth order $\rho := \rho(w)$ of $w(z)$ satisfies*

$$\rho \le 1 + \frac{\deg_{z,\infty} a}{(nqk - nq + 1)m - W(P)},$$

where $Q(z)$ is a polynomial of degree q.

The following examples show that the result is sharp in special cases.

Example 1.5 ([12])**.** For $n > 0$, let $w(z) = \cos z^{\frac{n}{2}}$, then $\rho(w) = \frac{n}{2}$ and w satisfies the algebraic differential equation:

$$[(w^2)']^2 = n^2 z^{n-2} w^2 (1 - w^2) = 0.$$

when $n = 1$ or 2, $\deg_{z,\infty} a = 0$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of (1.3) satisfies $\rho(w) \le 1$; when $n \ge 3$, $\deg_{z,\infty} a = n - 2$, and the growth order $\rho(w)$ of any entire solution $w(z)$ of above equation satisfies $\rho(w) \le \frac{n}{2}$.

Example 1.6. For $n = 2$, entire function $w(z) = e^{z^2}$ satisfies the algebraic differential equation

$$[(w')^2]' = 8zw^2 + 8z^2 w' w.$$

We know $k = 2$, $m = 1$, $n = 2$, $q = 1$, $\deg_{z,\infty} a = 2$, $W(P) = 1$, and then $\rho = 2 \le 1 + \frac{2}{3-1} = 2$. This example illustrates that Theorem 1.4 is an extension result of Theorems 1.2 and 1.3, and our result is sharp in the special cases.

Set

$$\begin{gathered} (Q(w_2^{(k)}(z)))^{m_1} = a(z) w_1^{(n)} \\ (w_1^{(n)})^{m_2} = P(w_2), \end{gathered} \tag{1.3}$$

where $m_1, m_2 \in \mathbb{N}$, $Q(z)$ and $a(z)$ are two polynomials of degrees q and $D(a)$, respectively. In 2012, Qi et al proved the following theorem.

Theorem 1.7 ([14])**.** *Let $k, m_1, m_2, n, q \in \mathbb{N}$, and let $w = (w_1, w_2)$ be a pair of entire solutions of system (1.3), if $m_1 m_2 q k > W(P)$, and all zeros of w_2 have multiplicity at least k, then the growth orders $\rho(w_i)$ of $w_i(z)$ for $i = 1, 2$ satisfy*

$$\rho(w_1) = \rho(w_2) \le 1 + \frac{\deg_{z,\infty} a + D(a)}{m_1 m_2 q k - W(P)}.$$

In 2009, Gu et al [10] obtained Theorem 1.7 when $k = 1$, $q = 1$ Now we consider the similar result to Theorem 1.4 for the system of the algebraic differential equations

$$\begin{gathered} [([Q(w_2^{(k-1)})]^{m_3})']^{m_1} = a(z) R(w_1^{(n)}), \\ (R(w_1^{(n)}))^{m_2} = P[(w_2], \end{gathered} \tag{1.4}$$

where $R(z)$ is a polynomial, too. We obtain the following result.

Theorem 1.8. *Let $k, n, q, m_1, m_2, m_3 \in \mathbb{N}$, and let $w = (w_1, w_2)$ be a pair of entire solutions of system (1.4). If $(m_3 q k - m_3 q + 1) m_1 m_2 > W(P)$, and all zeros of w_2 have multiplicity at least k, then the growth orders $\rho(w_i)$ of $w_i(z)$ for $i = 1, 2$ satisfy*

$$\rho(w_1) = \rho(w_2) \le 1 + \frac{\deg_{z,\infty} a + m_2 D(a)}{(m_3 q k - m_3 q + 1) m_1 m_2 - W(P)}.$$

Example 1.9. *The entire functions $w_1(z) = e^z + c$, $w_2(z) = e^z$ satisfy the algebraic differential equation system*

$$\begin{gathered} [(w_2^{(k-1)})^2]' = 2(w_1^{(n)})^2 \\ \{(w_1^{(n)})^2\}^3 = (w_2)^3 (w_2')^2, \end{gathered}$$

where c is a constant, $m_1 = 1$, $m_2 = 3$, $m_3 = 2$, $q = 1$, $D(a) = 0$, $W(P) = 2$, $\deg_{z,\infty} a = 0$ and $(m_3 q k - m_3 q + 1) m_1 m_2 = 3(2k - 1) > 2 = W(P)$. So $\rho(w_1) = \rho(w_2) = 1 \le 1$. So the conclusion of Theorem 1.8 may be applied.

2. Main Lemmas

To prove our result, we need the following lemmas. Lemma 2.1 extends the result by Zalcman [18] concerning normal family.

Lemma 2.1 ([19]). *Let $\mathcal{F}$ be a family of meromorphic(or analytic) functions on the unit disc. Then $\mathcal{F}$ is not normal on the unit disc if and only if there exist*

(a) *a number $r \in (0,1)$;*
(b) *points z_n with $|z_n| < r$;*
(c) *functions $f_n \in \mathcal{F}$;*
(d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := f_n(z_n + \rho_n\zeta)$ converges locally uniformly to a nonconstant meromorphic (or entire) function $g(\zeta)$, its order is at most 2. In particular, we may choose w_n and ρ_n, such that

$$\rho_n \leq \frac{2}{f_n^\sharp(w_n)}, f_n^\sharp(w_n) \geq f_n^\sharp(0).$$

Lemma 2.2 ([7]). *Let $f(z)$ be holomorphic in the complex plane, $\sigma > -1$. If $f^\sharp(z) = O(r^\sigma)$, then $T(r,f) = O(r^{\sigma+1})$.*

Lemma 2.3 ([9]). *Let $f(z)$ be holomorphic in whole complex plane with growth order $\rho := \rho(f) > 1$, then for each $0 < \mu < \rho - 1$, there exists a sequence $a_n \to \infty$, such that*

$$\lim_{n\to\infty} \frac{f^\sharp(a_n)}{|a_n|^\mu} = +\infty. \tag{2.1}$$

3. Proofs of theorems

Proof of Theorem 1.4. Suppose that the conclusion of theorem is not true, then there exists an entire solution $w(z)$ satisfies the equation $[([Q(w^{(k-1)}(z))]^n)']^m = P[w]$, such that

$$\rho > 1 + \frac{\deg_{z,\infty} a}{(nqk - nq + 1)m - W(P)}. \tag{3.1}$$

By Lemma 2.2 we know that for each $0 < \mu < \rho - 1$, there exists a sequence of points $a_j \to \infty (j \to \infty)$, such that (2.1) is valid. This implies that the family $\{w_j(z) := w(a_j + z)\}_{j\in\mathbb{N}}$ is not normal at $z = 0$. By Lemma 2.1, there exist sequences $\{b_j\}$ and $\{\rho_j\}$ such that

$$|a_j - b_j| < 1, \quad \rho_j \to 0, \tag{3.2}$$

and $g_j(\zeta) := w_j(b_j - a_j + \rho_j\zeta) = w(b_j + \rho_j\zeta)$ converges locally uniformly to a nonconstant entire function $g(\zeta)$, which order is at most 2, all zeros of $g(\zeta)$ have multiplicity at least k . In particular, we may choose b_j and ρ_j, such that

$$\rho_j \leq \frac{2}{w^\sharp(b_j)}, \quad w^\sharp(b_j) \geq w^\sharp(a_j). \tag{3.3}$$

According to (2.1) and (3.1)–(3.3), we can get the following conclusion: For any fixed constant $0 \leq \mu < \rho - 1$, we have

$$\lim_{j\to\infty} b_j^\mu \rho_j = 0. \tag{3.4}$$

In the differential equation $[([Q(w^{(k-1)}(z))]^n)']^m = P[w]$, we now replace z by $b_j + \rho_j\zeta$. Assuming that $P[w]$ has the form (1.2). Then we obtain

$$[([Q(w^{(k-1)}(b_j+\rho_j\zeta))]^n)']^m = \sum_{r\in I} a_r(b_j+\rho_j\zeta, g_j(\zeta))\rho_j^{-p(r)} M_r[g_j](\zeta).$$

From

$$\begin{aligned}&([Q(w^{(k-1)}(b_j+\rho_j\zeta))]^n)' \\ &= n[Q(w^{(k-1)}(b_j+\rho_j\zeta))]^{n-1}Q'(w^{(k-1)}(b_j+\rho_j\zeta))(w^{(k)}(b_j+\rho_j\zeta)),\end{aligned}$$

we have

$$\begin{aligned}&([Q(w^{(k-1)}(b_j+\rho_j\zeta))]^n)' \\ &= \rho_j^{-(nqk-nq+1)} g_j^{(k)}(\zeta)[nq(g_j^{(k-1)})^{(nq-1)(k-1)}(\zeta) + H(\rho_j, g_j^{(k-1)}(\zeta))],\end{aligned}$$

where $H(s,t)$ is a polynomial in two variables, whose degree $\deg_s H$ in s satisfies $\deg_s H \geq 1$. Hence we deduce that

$$\begin{aligned}&\rho_j^{-(nqk-nq+1)m}\{g_j^{(k)}(\zeta)[nq(g_j^{(k-1)})^{(nq-1)(k-1)}(\zeta) + H(\rho_j, g_j^{(k-1)}(\zeta))]\}^m \\ &= \sum_{r\in I} a_r(b_j+\rho_j\zeta, g_j(\zeta))\rho_j^{-p(r)} M_r[g_j](\zeta).\end{aligned}$$

Therefore,

$$\begin{aligned}&\{g_j^{(k)}(\zeta)[nq(g_j^{(k-1)})^{(nq-1)(k-1)}(\zeta) + H(\rho_j, g_j^{(k-1)}(\zeta))]\}^m \\ &= \sum_{r\in I} \frac{a_r(b_j+\rho_j\zeta, g_j(\zeta))}{b_j^{\deg a_r}}[b_j^{\frac{\deg a_r}{(nqk-nq+1)m-p(r)}}\rho_j]^{(nqk-nq+1)m-p(r)} M_r[g_j](\zeta).\end{aligned} \tag{3.5}$$

Because $0 \leq \mu = \frac{\deg_{z,\infty} a_r}{(nqk-nq+1)m-p(r)} \leq \frac{\deg_{z,\infty} a}{(nqk-nq+1)m-W(P)} < \rho - 1$, $p(r) < (nqk - nq + 1)m$, for every fixed $\zeta \in$ C, if ζ is not the zero of $g(\zeta)$, by (3.4) then we can get $g^{(k)}(\zeta) = 0$ from (3.5). By the all zeros of $g(\zeta)$ have multiplicity at least k, this is a contradiction. The proof is complete. □

Proof of Theorem 1.8. By the first equation of the systems of algebraic differential equations (1.4), we know

$$R(w_1^{(n)}) = \frac{[([Q(w_2^{(k-1)})]^{m_3})']^{m_1}}{a(z)}.$$

Therefore, $\rho(w_1) = \rho(w_2)$.

If w_2 is a rational function, then w_1 must be a rational function, so that the conclusion of Theorem 1.8 is right. If w_2 is a transcendental entire function, by the systems of algebraic differential equations (1.4), then we have

$$[([Q(w_2^{(k-1)})]^{m_3})']^{m_1 m_2} = (a(z))^{m_2} P[w_2]. \tag{3.6}$$

Suppose that the conclusion of Theorem 1.8 is not true, then there exists an entire vector $w(z) = (w_1(z), w_2(z))$ which satisfies the system of equations (1.4) such that

$$\rho := \rho(w_2) > 1 + \frac{\deg_{z,\infty} a + m_2 D(a)}{(m_3qk - m_3q + 1)m_1m_2 - W(P)}, \quad . \tag{3.7}$$

By Lemma 2.2 we know that for each $0 < \mu < \rho - 1$, there exists a sequence of points $a_j \to \infty (j \to \infty)$, such that (2.1) is right. This implies that the family

$\{w_j(z) := w(a_j + z)\}_{j\in\mathbb{N}}$ is not normal at $z = 0$. By Lemma 2.1, there exist sequences $\{b_j\}$ and $\{\rho_j\}$ such that

$$|a_j - b_j| < 1, \quad \rho_j \to 0, \tag{3.8}$$

and $g_j(\zeta) := w_{2,j}(b_j - a_j + \rho_j\zeta) = w_2(b_j + \rho_j\zeta)$ converges locally uniformly to a nonconstant entire function $g(\zeta)$, which order is at most 2, all zeros of $g(\zeta)$ have multiplicity at least k . In particular, we may choose b_j and ρ_j, such that

$$\rho_j \le \frac{2}{w_2^\sharp(b_j)}, \quad w_2^\sharp(b_j) \ge w_2^\sharp(a_j). \tag{3.9}$$

According to (3.6) and (3.7)–(3.9), we can get the following conclusion: For any fixed constant $0 \le \mu < \rho - 1$, we have

$$\lim_{m\to\infty} b_j^\mu \rho_j = 0. \tag{3.10}$$

In the differential equation (3.6) we now replace z by $b_j + \rho_j\zeta$, then we obtain

$$\begin{aligned}
&[([Q(w^{(k-1)}(b_j + \rho_j\zeta))]^{m_3})']^{m_1m_2}\\
&= (a(b_j + \rho_j\zeta))^{m_2} \sum_{r\in I} a_r(b_j + \rho_j\zeta, g_j(\zeta))\rho_j^{-p(r)} M_r[g_j](\zeta).
\end{aligned}$$

By

$$\begin{aligned}
&([Q(w^{(k-1)}(b_j + \rho_j\zeta))]^{m_3})'\\
&= m_3[Q(w^{(k-1)}(b_j + \rho_j\zeta))]^{m_3-1} Q'(w^{(k-1)}(b_j + \rho_j\zeta))(w^{(k)}(b_j + \rho_j\zeta)),
\end{aligned}$$

we obtain

$$\begin{aligned}
&([Q(w^{(k-1)}(b_j + \rho_j\zeta))]^{m_3})'\\
&= \rho_j^{-(m_3qk-m_3q+1)} g_j^{(k)}(\zeta)[m_3q(g_j^{(k-1)})^{(m_3q-1)(k-1)}(\zeta) + H(\rho_j, g_j^{(k-1)}(\zeta))],
\end{aligned}$$

where $H(s,t)$ is a polynomial in two variables, whose degree $\deg_s H$ in s satisfies $\deg_s H \ge 1$. Therefore,

$$\begin{aligned}
&\rho_j^{-(m_3qk-m_3q+1)m_1m_2}\{g_j^{(k)}(\zeta)[m_3q(g_j^{(k-1)})^{(m_3q-1)(k-1)}(\zeta) + H(\rho_j, g_j^{(k-1)}(\zeta))]\}^{m_1m_2}\\
&= \sum_{r\in I}(a(b_j + \rho_j\zeta))^{m_2} a_r(b_j + \rho_j\zeta, g_j(\zeta))\rho_j^{-p(r)} M_r[g_j](\zeta).
\end{aligned}$$

Thus

$$\begin{aligned}
&\{g_j^{(k)}(\zeta)[m_3q(g_j^{(k-1)})^{(m_3q-1)(k-1)}(\zeta) + H(\rho_j, g_j^{(k-1)}(\zeta))]\}^{m_1m_2}\\
&= \sum_{r\in I} \frac{(a(b_j + \rho_j\zeta))^{m_2} a_r(b_j + \rho_j\zeta, g_j(\zeta))}{b_j^{\deg a_r + m_2D(a)}}\\
&\quad \times [b_j^{\frac{\deg a_r + m_2D(a)}{(m_3qk-m_3q+1)m_1m_2-p(r)}}\rho_j]^{(m_3qk-m_3q+1)m_1m_2-p(r)} M_r[g_j](\zeta).
\end{aligned} \tag{3.11}$$

For every fixed $\zeta \in \mathrm{C}$, if ζ is not a zero of $g(\zeta)$, for $j \to \infty$ and

$$\begin{aligned}
0 \le \mu &= \frac{\deg_{z,\infty} a_r + m_2D(a)}{(m_3qk - m_3q + 1)m_1m_2 - p(r)}\\
&\le \frac{\deg_{z,\infty} a + m_2D(a)}{(m_3qk - m_3q + 1)m_1m_2 - \deg P(w_2)} < \rho - 1
\end{aligned}$$

then we have $(g^{(k)})^{m_1 m_2} = 0$, which contradicts with all zeros of $g(\zeta)$ have multiplicity at least k. So

$$\rho(w_2) \le 1 + \frac{\deg_{z,\infty} a + m_2 D(a)}{(m_3 qk - m_3 q + 1)m_1 m_2 - \deg P(w_2)}.$$

□

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