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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE GENERALIZED DAMPED BOUSSINESQ EQUATION

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Abstract. We consider the Cauchy problem for the n-dimensional generalized damped Boussinesq equation. Based on decay estimates of solutions to the corresponding linear equation, we define a solution space with time weighted norms. Under small condition on the initial value, the existence and asymptotic behavior of global solutions in the corresponding Sobolev spaces are established by the contraction mapping principle.

1. Introduction

We study the Cauchy problem of the generalized damped Boussinesq equation in n space dimensions

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t - \alpha\Delta^3 u + \beta\Delta^2 u - \Delta u = \Delta f(u) \tag{1.1}$$

with the initial value

$$t = 0: \quad u = u_0(x), \quad u_t = u_1(x). \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$, a, b, α, β are positive constants. The nonlinear term $f(u) = O(u^{1+\theta})$ and θ is a positive integer.

The first initial boundary value problem for

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t - \alpha\Delta^3 u + \beta\Delta^2 u - \Delta u = \gamma\Delta(u^2) \tag{1.3}$$

in a unit circle was investigated in [16], where a, b, α, β are positive constants and γ is a constant. The existence and the uniqueness of strong solution was established and the solutions were constructed in the form of series in the small parameter present in the initial conditions. The long-time asymptotics was also obtained in the explicit form. In [1], the authors considered the initial-boundary value problem for (1.3) in the unit ball $B \subset \mathbb{R}^3$, similar results were established. It is well-known that the equation (1.3) is closely contacted with many wave equations. For example, the equation (which we call the Bq equation)

$$u_{tt} - u_{xx} + u_{xxxx} = (u^2)_{xx},$$

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which was derived by Boussinesq in 1872 to describe shallow water waves. The improved Bq equation(which we call IBq equation) is

$$u_{tt} - u_{xx} - u_{xxtt} = (u^2)_{xx}.$$

A modification of the IBq equation analogous of the MKdV equation yields

$$u_{tt} - u_{xx} - u_{xxtt} = (u^3)_{xx},$$

which we call the IMBq equation (see [5]). (1.1) is a higher order wave equation. In [8], we considered the Cauchy problem for the Cahn-Hilliard equation with inertial term. Combining high frequency, low frequency technique and energy methods, we obtained global existence and asymptotic behavior of solutions. Wang, Liu and Zhang [13] investigated a fourth wave equation that is of the regularity-loss type. Based on the decay property of the solution operators, global existence and asymptotic behavior of solutions are obtained. For global existence and asymptotic behavior of solutions to higher order wave equations, we refer to [2]-[3] and [6]-[15] and references therein.

The main purpose of this paper is to establish global existence and asymptotic behavior of solutions to (1.1), (1.2) by using the contraction mapping principle. Firstly, we consider the decay property of the following linear equation

$$u_{tt} - a\Delta u_{tt} - 2b\Delta u_t - \alpha\Delta^3 u + \beta\Delta^2 u - \Delta u = 0. \tag{1.4}$$

We obtain the following decay estimate of solutions to (1.4) associated with initial condition (1.2),

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{1}{2}}(\|u_0\|_{\dot{H}_1^{-1}} + \|u_1\|_{\dot{H}_1^{-2}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \tag{1.5}$$

$(k \le s+2)$,

$$\|\partial_x^h u_t(t)\|_{L^2} \le C(1+t)^{-\frac{n}{4}-\frac{h}{2}-1}(\|u_0\|_{\dot{H}_1^{-1}} + \|u_1\|_{\dot{H}_1^{-2}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \tag{1.6}$$

$(h \le s)$ Based on the estimates (1.5) and (1.6), we define a solution space with time weighted norms. Then global existence and asymptotic behavior of classical solutions to (1.1), (1.2) are obtained by using the contraction mapping principle.

We give notation which is used in this paper. Let $\mathscr{F}[u]$ denote the Fourier transform of u defined by

$$\hat{u}(\xi) = \mathscr{F}[u] = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)dx,$$

and we denote its inverse transform by $\mathscr{F}^{-1}$.

For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. The usual Sobolev space of s is defined by $H_p^s = (I-\Delta)^{-s/2}L^p$ with the norm $\|f\|_{H_p^s} = \|(I-\Delta)^{s/2}f\|_{L^p}$; the homogeneous Sobolev space of s is defined by $\dot{H}_p^s = (-\Delta)^{-s/2}L^p$ with the norm $\|f\|_{\dot{H}_p^s} = \|(-\Delta)^{s/2}f\|_{L^p}$; especially $H^s = H_2^s, \dot{H}^s = \dot{H}_2^s$. Moreover, we know that $H_p^s = L^p \cap \dot{H}_p^s$ for $s \ge 0$.

Finally, in this paper, we denote every positive constant by the same symbol C or c without confusion. $[\cdot]$ is the Gauss symbol.

The article is organized as follows. In Section 2 we derive the solution formula of our semi-linear problem. We study the decay property of the solution operators appearing in the solution formula in section 3. Then, in Section 4, we discuss the linear problem and show the decay estimates. Finally, we prove global existence and asymptotic behavior of solutions for the Cauchy problem (1.1), (1.2) in Section 5.

2. Solution formula

The aim of this section is to derive the solution formula for problem (1.1), (1.2). We first investigate the equation (1.4). Taking the Fourier transform, we have

$$(1+a|\xi|^2)\hat{u}_{tt}+2b|\xi|^2\hat{u}_t+(\alpha|\xi|^6+\beta|\xi|^4+|\xi|^2)\hat{u}=0. \tag{2.1}$$

The corresponding initial value are

$$t=0: \quad \hat{u}=\hat{u}_0(\xi), \quad \hat{u}_t=\hat{u}_1(\xi). \tag{2.2}$$

The characteristic equation of (2.1) is

$$(1+a|\xi|^2)\lambda^2+2b|\xi|^2\lambda+\alpha|\xi|^6+\beta|\xi|^4+|\xi|^2=0. \tag{2.3}$$

Let $\lambda=\lambda_\pm(\xi)$ be the corresponding eigenvalues of (2.3), we obtain

$$\lambda_\pm(\xi)=\frac{-b|\xi|^2\pm|\xi|\sqrt{-1-(a+\beta-b^2)|\xi|^2-(\alpha+a\beta)|\xi|^4-a\alpha|\xi|^6}}{1+a|\xi|^2}. \tag{2.4}$$

The solution to the problem (2.1)-(2.2) is given in the form

$$\hat{u}(\xi,t)=\hat{G}(\xi,t)\hat{u}_1(\xi)+\hat{H}(\xi,t)\hat{u}_0(\xi), \tag{2.5}$$

where

$$\hat{G}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(e^{\lambda_+(\xi)t}-e^{\lambda_-(\xi)t}) \tag{2.6}$$

and

$$\hat{H}(\xi,t)=\frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(\lambda_+(\xi)e^{\lambda_-(\xi)t}-\lambda_-(\xi)e^{\lambda_+(\xi)t}). \tag{2.7}$$

We define $G(x,t)$ and $H(x,t)$ by

$$G(x,t)=\mathscr{F}^{-1}[\hat{G}(\xi,t)](x), \quad H(x,t)=\mathscr{F}^{-1}[\hat{H}(\xi,t)](x),$$

respectively, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Then, applying $\mathscr{F}^{-1}$ to (2.5), we obtain

$$u(t)=G(t)*u_1+H(t)*u_0. \tag{2.8}$$

By the Duhamel principle, we obtain the solution formula to (1.1), (1.2),

$$u(t)=G(t)*u_1+H(t)*u_0+\int_0^t G(t-\tau)*(I-a\Delta)^{-1}\Delta f(u)(\tau)d\tau. \tag{2.9}$$

3. Decay Property

The aim of this section is to establish decay estimates of the solution operators $G(t)$ and $H(t)$ appearing in the solution formula (2.8).

Lemma 3.1. *The solution of problem* (2.1), (2.2) *satisfies*

$$|\xi|^2(1+|\xi|^2)|\hat{u}(\xi,t)|^2+|\hat{u}_t(\xi,t)|^2\le Ce^{-c\omega(\xi)t}(|\xi|^2(1+|\xi|^2)|\hat{u}_0(\xi)|^2+|\hat{u}_1(\xi)|^2), \tag{3.1}$$

for $\xi\in\mathbb{R}^n$ *and* $t\ge0$, *where* $\omega(\xi)=\frac{|\xi|^2}{1+|\xi|^2}$.

Proof. Multiplying (2.1) by $\bar{\hat{u}}_t$ and taking the real part yields

$$\frac{1}{2}\frac{d}{dt}\{(1+a|\xi|^2)|\hat{u}_t|^2+(\alpha|\xi|^6+\beta|\xi|^4+|\xi|^2)|\hat{u}|^2\}+2b|\xi|^2|\hat{u}_t|^2=0. \tag{3.2}$$

Multiplying (2.1) by $\bar{\hat{u}}$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\{b|\xi|^2|\hat{u}|^2+2(1+a|\xi|^2)Re(\hat{u}_t\bar{\hat{u}})\}+(\alpha|\xi|^6+\beta|\xi|^4+|\xi|^2)|\hat{u}|^2-(1+a|\xi|^2)|\hat{u}_t|^2=0. \tag{3.3}$$

Multiplying both sides of (3.2) and (3.3) by $(1+a|\xi|^2)$ and $b|\xi|^2$ respectively, summing up the products yields

$$\frac{d}{dt}E+F=0, \tag{3.4}$$

where

$$E=\frac{1}{2}(1+a|\xi|^2)^2|\hat{u}_t|^2+(1+a|\xi|^2)(\alpha|\xi|^6+\beta|\xi|^4+|\xi|^2)|\hat{u}|^2+b^2|\xi|^4|\hat{u}|^2 \\ +b|\xi|^2(1+a|\xi|^2)\operatorname{Re}(\hat{u}_t\bar{\hat{u}})$$

and

$$F=b|\xi|^2(\alpha|\xi|^6+\beta|\xi|^4+|\xi|^2)|\hat{u}|^2+b|\xi|^2(1+a|\xi|^2)|\hat{u}_t|^2.$$

A simple computation implies that

$$C(1+|\xi|^2)^2E_0\le E\le C(1+|\xi|^2)^2E_0, \tag{3.5}$$

where

$$E_0=|\xi|^2(1+|\xi|^2)|\hat{u}|^2+|\hat{u}_t|^2.$$

Note that $F\ge c|\xi|^2E_0$. It follows from (3.5) that

$$F\ge c\omega(\xi)E, \tag{3.6}$$

where

$$\omega(\xi)=\frac{|\xi|^2}{1+|\xi|^2}.$$

Using (3.4) and (3.6), we obtain

$$\frac{d}{dt}E+c\omega(\xi)E\le 0.$$

Thus $E(\xi,t)\le e^{-cw(\xi)t}E(\xi,0)$, which together with (3.5) proves the desired estimates (3.1). Then the proof is complete. □

Lemma 3.2. *Let $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ be the fundamental solution of* (1.4) *in the Fourier space, which are given in* (2.6) *and* (2.7)*, respectively. Then we have the estimates*

$$|\xi|^2(1+|\xi|^2)|\hat{G}(\xi,t)|^2+|\hat{G}_t(\xi,t)|^2\le Ce^{-c\omega(\xi)t} \tag{3.7}$$

and

$$|\xi|^2(1+|\xi|^2)|\hat{H}(\xi,t)|^2+|\hat{H}_t(\xi,t)|^2\le C|\xi|^2(1+|\xi|^2)e^{-c\omega(\xi)t} \tag{3.8}$$

for $\xi\in\mathbb{R}^n$ and $t\ge 0$, where $\omega(\xi)=\frac{|\xi|^2}{1+|\xi|^2}$.

Proof. If $\hat{u}_0(\xi)=0$, from (2.5), we obtain

$$\hat{u}(\xi,t)=\hat{G}(\xi,t)\hat{u}_1(\xi),\quad \hat{u}_t(\xi,t)=\hat{G}_t(\xi,t)\hat{u}_1(\xi).$$

Substituting the equalities into (3.1) with $\hat{u}_0(\xi)=0$, we obtain (3.7). In what follows, we consider $\hat{u}_1(\xi)=0$, it follows from (2.5) that

$$\hat{u}(\xi,t)=\hat{H}(\xi,t)\hat{u}_0(\xi),\hat{u}_t(\xi,t)=\hat{H}_t(\xi,t)\hat{u}_0(\xi).$$

Substituting the equalities into (3.1) with $\hat{u}_1(\xi)=0$, we obtain the desired estimate (3.8). The Lemma is proved. □

Lemma 3.3. *Let $k\ge 0$ and $1\le p\le 2$. Then we have*

$$\|\partial_x^k G(t)*\phi\|_{L^2}\le C(1+t)^{-(\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{k}{2}+\frac{l}{2}-\frac{1}{2})}\|\phi\|_{\dot{H}_p^{-l}}+Ce^{-ct}\|\partial_x^{(k-2)_+}\phi\|_{L^2},\quad(3.9)$$

$$\|\partial_x^k H(t)*\phi\|_{L^2}\le C(1+t)^{-(\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{k}{2}+\frac{l}{2})}\|\phi\|_{\dot{H}_p^{-l}}+Ce^{-ct}\|\partial_x^k\phi\|_{L^2}\quad(3.10)$$

$$\|\partial_x^k G_t(t)*\phi\|_{L^2}\le C(1+t)^{-(\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{k}{2}+\frac{l}{2})}\|\phi\|_{\dot{H}_p^{-l}}+Ce^{-ct}\|\partial_x^k\phi\|_{L^2},\quad(3.11)$$

$$\|\partial_x^k H_t(t)*\phi\|_{L^2}\le C(1+t)^{-(\frac{n}{2}(\frac{1}{p}-\frac{1}{2})+\frac{k}{2}+\frac{l}{2}+\frac{1}{2})}\|\phi\|_{\dot{H}_p^{-l}}+Ce^{-ct}\|\partial_x^{k+2}\phi\|_{L^2}\quad(3.12)$$

$$\|\partial_x^k G(t)*(I-a\Delta)^{-1}\Delta g\|_{L^2}\le C(1+t)^{-(\frac{n}{4}+\frac{k}{2}+\frac{1}{2})}\|g\|_{L^1}+Ce^{-ct}\|\partial_x^k g\|_{L^2},\quad(3.13)$$

$$\|\partial_x^k G_t(t)*(I-a\Delta)^{-1}\Delta g\|_{L^2}\le C(1+t)^{-(\frac{n}{4}+\frac{k}{2}+1)}\|g\|_{L^1}+Ce^{-ct}\|\partial_x^k g\|_{L^2},\quad(3.14)$$

where $(k-2)_+=\max\{0,k-2\}$.

Proof. Firstly, we prove (3.9). By the Plancherel theorem and (3.7), we obtain

$$\begin{aligned}
&\|\partial_x^k G(t)*\phi\|_{L^2}^2\\
&=\int_{|\xi|\le R_0}|\xi|^{2k}|\hat{G}(\xi,t)|^2|\hat{\phi}(\xi)|^2d\xi+\int_{|\xi|\ge R_0}|\xi|^{2k}|\hat{G}(\xi,t)|^2|\hat{\phi}(\xi)|^2d\xi\\
&\le C\int_{|\xi|\le R_0}|\xi|^{2k-2}e^{-c|\xi|^2t}|\hat{\phi}(\xi)|^2d\xi\\
&\quad+Ce^{-ct}\int_{|\xi|\ge R_0}|\xi|^{2k}(|\xi|^2(1+|\xi|^2))^{-1}|\hat{\phi}(\xi)|^2d\xi\\
&\le C\||\xi|^{-l}\hat{\phi}(\xi)\|_{L^{p'}}^2\Big(\int_{|\xi|\le R_0}|\xi|^{(2k-2+2l)q}e^{-cq|\xi|^2t}d\xi\Big)^{1/q}\\
&\quad+Ce^{-ct}\|\partial_x^{(k-2)_+}\phi\|_{L^2}^2,
\end{aligned}\quad(3.15)$$

where R_0 is a small positive constant and $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{2}{p'}+\frac{1}{q}=1$. It follows from Hausdorff-Young inequality that

$$\|\,|\xi|^{-l}\hat{\phi}(\xi)\|_{L^{p'}}\le C\|(-\Delta)^{-\frac{l}{2}}\phi\|_{L^p}.\quad(3.16)$$

By a straight computation, we obtain

$$\begin{aligned}
\Big(\int_{|\xi|\le R_0}|\xi|^{(2k-2+2l)q}e^{-cq|\xi|^2t}d\xi\Big)^{1/q}&\le C(1+t)^{-(\frac{n}{2q}+k-1+l)}\\
&\le C(1+t)^{-(n(\frac{1}{p}-\frac{1}{2})+k-1+l)}.
\end{aligned}\quad(3.17)$$

Combining (3.15), (3.16) and (3.17) yields (3.9).

Similarly, using (3.7) and (3.8), respectively, we can prove (3.10)-(3.12).

In what follows, we prove (3.13). By the Plancherel theorem, (3.7), and Hausdorff-Young inequality, we have

$$
\begin{aligned}
&\|\partial_x^k G(t) * (I-a\Delta)^{-1}\Delta g\|_{L^2}^2\\
&= \int_{|\xi|\le R_0} |\xi|^{2k}|\hat G(\xi,t)|^2|\xi|^4(1+a|\xi|^2)^{-2}|\hat g(\xi)|^2 d\xi\\
&\quad + \int_{|\xi|\ge R_0} |\xi|^{2k}|\hat G(\xi,t)|^2|\xi|^4(1+|\xi|^2)^{-2}|\hat g(\xi)|^2 d\xi\\
&\le C\int_{|\xi|\le R_0} |\xi|^{2k+2}e^{-c|\xi|^2 t}|\hat g(\xi)|^2 d\xi + Ce^{-ct}\int_{|\xi|\ge R_0}|\xi|^{2k}|\hat g(\xi)|^2 d\xi\\
&\le C\|\hat g(\xi)\|_{L^\infty}^2\int_{|\xi|\le R_0}|\xi|^{2k+2}e^{-c|\xi|^2 t}d\xi + Ce^{-ct}\|\partial_x^k g\|_{L^2}^2\\
&\le C(1+t)^{-(\frac n2+k+1)}\|g\|_{L^1}^2 + Ce^{-ct}\|\partial_x^k g\|_{L^2}^2.
\end{aligned}
$$

where R_0 is a small positive constant. Thus (3.13) follows. Similarly, we can prove (3.14). Thus we have completed the proof of lemma. □

4. Decay estimate for solutions to the linear equation

Theorem 4.1. *Assume that* $u_0 \in H^{s+2}(\mathbb{R}^n)\cap \dot H_1^{-1}(\mathbb{R}^n)$, $u_1 \in H^s(\mathbb{R}^n)\cap \dot H_1^{-2}(\mathbb{R}^n)$ $(s\ge [\frac n2]+5)$. *Then the classical solution* $u(x,t)$ *to* (1.4) *associated with initial condition* (1.2)*, which is given by the formula* (2.8)*, satisfies the decay estimates*

$$
\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-\frac n4-\frac k2-\frac12}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}}+\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}) \tag{4.1}
$$

for $k\le s+2$,

$$
\|\partial_x^h u_t(t)\|_{L^2} \le C(1+t)^{-\frac n4-\frac h2-1}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}}+\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}) \tag{4.2}
$$

for $h\le s$,

$$
\|\partial_x^m u(t)\|_{L^\infty} \le C(1+t)^{-\frac n2-\frac m2-\frac12}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}}+\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}) \tag{4.3}
$$

for $m\le s+1-[\frac n2]$.

Proof. Firstly, we prove (4.1). Using (3.9) and (3.10), we obtain

$$
\begin{aligned}
&\|\partial_x^k u(t)\|_{L^2}\\
&\le \|\partial_x^k G(t)*u_1\|_{L^2} + C\|\partial_x^h H(t)*u_0\|_{L^2}\\
&\le C(1+t)^{-\frac n4-\frac k2-\frac12}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}}) + Ce^{-ct}(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s})\\
&\le C(1+t)^{-\frac n4-\frac k2-\frac12}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}}+\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}).
\end{aligned}
$$

Similar to the proof of (4.1), using (3.11) and (3.12), we can prove (4.2). In what follows, we prove (4.3). Using (4.1) and Gagliardo-Nirenberg inequality, it is not difficult to get (4.3). The Lemma is proved. □

5. Existence of global solution and asymptotic behavior

The purpose of this section is to prove the existence and asymptotic behavior of global solutions to the Cauchy problem (1.1), (1.2). We need the following Lemma, which come from [4] (see also [17]).

Lemma 5.1. *Let s and θ be positive integers, $\delta > 0$, $p, q, r \in [1, \infty]$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{r}$, and let $k \in \{0, 1, 2, \cdots, s\}$. Assume that $F(v)$ is a class of C^s and satisfies*

$$|\partial_v^l F(v)| \le C_{l,\delta}|v|^{\theta+1-l}, \quad |v| \le \delta, \quad 0 \le l \le s, l < \theta + 1$$

and

$$|\partial_v^l F(v)| \le C_{l,\delta}, \ \ |v| \le \delta, l \le s, \theta + 1 \le l.$$

If $v \in L^p \cap W^{k,q} \cap L^\infty$ and $\|v\|_{L^\infty} \le \delta$, then

$$\|F(v)\|_{W^{k,r}} \le C_{k,\delta}\|v\|_{W^{k,q}}\|v\|_{L^p}\|v\|_{L^\infty}^{\theta-1},$$
$$\|\partial_x^\alpha F(v)\|_{L^r} \le C_{k,\delta}\|\partial_x^\alpha v\|_{L^q}\|v\|_{L^p}\|v\|_{L^\infty}^{\theta-1}, \quad |\alpha| \le k.$$

Lemma 5.2. *Let s and θ be positive integers, $\delta > 0$, $p, q, r \in [1, \infty]$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{r}$, and let $k \in \{0, 1, 2, \cdots, s\}$. Let $F(v)$ be a function that satisfies the assumptions of Lemma 5.1. Moreover, assume that*

$$|\partial_v^s F(v_1) - \partial_v^s F(v_2)| \le C_\delta(|v_1| + |v_2|)^{\max\{\theta-s,\theta\}}|v_1 - v_2|, \quad |v_1| \le \delta, \quad |v_2| \le \delta.$$

If $v_1, v_2 \in L^p \cap W^{k,q} \cap L^\infty$ and $\|v_1\|_{L^\infty} \le \delta, \|v_2\|_{L^\infty} \le \delta$, then for $|\alpha| \le k$, we have

$$\begin{aligned}
&\|\partial_x^\alpha (F(v_1) - F(v_2))\|_{L^r} \\
&\le C_{k,\delta}\{(\|\partial_x^\alpha v_1\|_{L^q} + \|\partial_x^\alpha v_2\|_{L^q})\|v_1 - v_2\|_{L^p} \\
&\quad + (\|v_1\|_{L^p} + \|v_2\|_{L^p})\|\partial_x^\alpha (v_1 - v_2)\|_{L^q}\}(\|v_1\|_{L^\infty} + \|v_2\|_{L^\infty})^{\theta-1}.
\end{aligned}$$

Based on the estimates (4.1)-(4.3) of solutions to (1.4) associated with initial condition (1.2), we define the following solution space

$$X = \{u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0, \infty); H^s(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sup_{t\ge 0}\{\sum_{k\le s+2}(1+t)^{\frac{n}{4}+\frac{k}{2}+\frac{1}{2}}\|\partial_x^k u(t)\|_{L^2} + \sum_{h\le s}(1+t)^{\frac{n}{4}+\frac{h}{2}+1}\|\partial_x^h u_t(t)\|_{L^2}\},$$

For $R > 0$, we define $X_R = \{u \in X : \|u\|_X \le R\}$. For $m \le s + 1 - [\frac{n}{2}]$, using Gagliardo-Nirenberg inequality, we obtain

$$\|\partial_x^m u(t)\|_{L^\infty} \le C(1+t)^{-(\frac{n}{2}+\frac{m}{2}+\frac{1}{2})}\|u\|_X. \tag{5.1}$$

Theorem 5.3. *Assume that $u_0 \in H^{s+2}(\mathbb{R}^n) \cap \dot{H}_1^{-1}(\mathbb{R}^n)$, $u_1 \in H^s(\mathbb{R}^n) \cap \dot{H}_1^{-2}(\mathbb{R}^n)$ ($s \ge [\frac{n}{2}] + 5$) and integer $\theta \ge 2$. Let $f(u)$ be a function of class C^{s+2} and satisfy Lemmas 5.1 and 5.2. Put*

$$E_0 = \|u_0\|_{\dot{H}_1^{-1}} + \|u_1\|_{\dot{H}_1^{-2}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}.$$

If E_0 is suitably small, the Cauchy problem (1.1)-(1.2) has a unique global classical solution $u(x,t)$ satisfying $u \in C([0, \infty); H^{s+2}(\mathbb{R}^n))$, $u_t \in C([0, \infty); H^s(\mathbb{R}^n))$, $u_{tt} \in L^\infty([0, \infty); H^{s-2}(\mathbb{R}^n))$. Moreover, the solution satisfies the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \le CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{1}{2}}, \tag{5.2}$$
$$\|\partial_x^h u_t(t)\|_{L^2} \le CE_0(1+t)^{-\frac{n}{4}-\frac{h}{2}-1} \tag{5.3}$$

for $k \le s + 2$ and $h \le s$.

Proof. Define the mapping

$$\Psi(u)=G(t)*u_1+H(t)*u_0+\int_0^t G(t-\tau)*(I-a\Delta)^{-1}\Delta f(u(\tau))d\tau. \tag{5.4}$$

Using (3.9)-(3.10), (3.13), Lemma 5.1 and (5.1), for $k\le s+2$ we obtain

$$\begin{aligned}
&\|\partial_x^k\Psi(u)\|_{L^2}\\
&\le C\|\partial_x^kG(t)*u_1\|_{L^2}+C\|\partial_x^kH(t)*u_0\|_{L^2}\\
&\quad+C\int_0^t\|\partial_x^kG(t-\tau)*(I-a\Delta)^{-1}\Delta f(u(\tau))\|_{L^2}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac12}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}})+Ce^{-ct}(\|u_0\|_{H^{s+2}+\|u_1\|_{H^s}})\\
&\quad+C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac12}\|f(u)\|_{L^1}d\tau\\
&\quad+C\int_{t/2}^t(1+t-\tau)^{-\frac{n}{4}-\frac12}\|\partial_x^kf(u)\|_{L^1}d\tau\\
&\quad+C\int_0^te^{-c(t-\tau)}\|\partial_x^kf(u)\|_{L^2}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac12}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}})+Ce^{-ct}(\|u_0\|_{H^{s+2}+\|u_1\|_{H^s}})\\
&\quad+C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac12}\|u\|_{L^2}^2\|u\|_{L^\infty}^{\theta-1}d\tau\\
&\quad+C\int_{t/2}^t(1+t-\tau)^{-\frac{n}{4}-\frac12}\|\partial_x^ku\|_{L^2}^2\|u\|_{L^\infty}^{\theta-1}d\tau+C\int_0^te^{-c(t-\tau)}\|\partial_x^ku\|_{L^2}\|u\|_{L^\infty}^{\theta}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac12}(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}})+Ce^{-ct}(\|u_0\|_{H^{s+2}+\|u_1\|_{H^s}})\\
&\quad+CR^{\theta+1}\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac12}(1+\tau)^{-(\frac{n}{2}+1)}(1+\tau)^{-(\frac{n}{2}+\frac12)(\theta-1)}d\tau\\
&\quad+CR^{\theta+1}\int_{t/2}^t(1+t-\tau)^{-\frac{n}{4}-\frac12}(1+\tau)^{-\frac{n}{2}-k-1}(1+\tau)^{-(\frac{n}{2}+\frac12)(\theta-1)}d\tau\\
&\quad+CR^{\theta+1}\int_0^te^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac12}(1+\tau)^{-(\frac{n}{2}+\frac12)\theta}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac12}\{(\|u_0\|_{\dot H_1^{-1}}+\|u_1\|_{\dot H_1^{-2}}+\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s})+R^{\theta+1}\}.
\end{aligned}$$

Thus

$$(1+t)^{\frac{n}{4}+\frac{k}{2}+\frac12}\|\partial_x^k\Psi(u)\|_{L^2}\le CE_0+CR^{\theta+1}. \tag{5.5}$$

It follows from (5.4) that

$$\Psi(u)_t=G_t(t)*u_1+H_t(t)*u_0+\int_0^tG_t(t-\tau)*(I-a\Delta)^{-1}\Delta f(u(\tau))d\tau. \tag{5.6}$$

Using (3.11)-(3.12), (3.14) Lemma 5.1 and (5.1), for $h\le s$ we have

$$\begin{aligned}
&\|\partial_x^h\Psi(u)_t\|_{L^2}\\
&\le C\|\partial_x^hG_t(t)*u_1\|_{L^2}+C\|\partial_x^hH_t(t)*u_0\|_{L^2}\\
&\quad+C\int_0^t\|\partial_x^hG_t(t-\tau)*(I-a\Delta)^{-1}\Delta f(u(\tau))\|_{L^2}d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{n}{4}-\frac{h}{2}-1}(\|u_0\|_{\dot{H}_1^{-1}}+\|u_1\|_{\dot{H}_1^{-2}})+Ce^{-ct}(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s})\\
&\quad +C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{h}{2}-1}\|f(u)\|_{L^1}d\tau\\
&\quad +C\int_{t/2}^t(1+t-\tau)^{-\frac{n}{4}-1}\|\partial_x^h f(u)\|_{L^1}d\tau+C\int_0^t e^{-c(t-\tau)}\|\partial_x^h f(u)\|_{L^2}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-\frac{h}{2}-1}(\|u_0\|_{\dot{H}_1^{-1}}+\|u_1\|_{\dot{H}_1^{-2}})+Ce^{-ct}(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s})\\
&\quad +C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{h}{2}-1}\|u\|_{L^2}^2\|u\|_{L^\infty}^{\theta-1}d\tau\\
&\quad +C\int_{t/2}^t(1+t-\tau)^{-\frac{n}{4}-1}\|\partial_x^h u\|_{L^2}^2\|u\|_{L^\infty}^{\theta-1}d\tau+C\int_0^t e^{-c(t-\tau)}\|\partial_x^h u\|_{L^2}\|u\|_{L^\infty}^{\theta}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-\frac{h}{2}-1}(\|u_0\|_{\dot{H}_1^{-1}}+\|u_1\|_{\dot{H}_1^{-2}})+Ce^{-ct}(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s})\\
&\quad +CR^{\theta+1}\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{h}{2}-1}(1+\tau)^{-(\frac{n}{2}+1)}(1+\tau)^{-(\frac{n}{2}+\frac{1}{2})(\theta-1)}d\tau\\
&\quad +CR^{\theta+1}\int_{t/2}^t(1+t-\tau)^{-\frac{n}{4}-1}(1+\tau)^{-\frac{n}{2}-h-1}(1+\tau)^{-(\frac{n}{2}+\frac{1}{2})(\theta-1)}d\tau\\
&\quad +CR^{\theta+1}\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{4}-\frac{h}{2}-1}(1+\tau)^{-(\frac{n}{2}+\frac{1}{2})\theta}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-\frac{h}{2}-1}\{(\|u_0\|_{\dot{H}_1^{-1}}+\|u_1\|_{\dot{H}_1^{-2}}+\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s})+R^{\theta+1}\}.
\end{aligned}
$$

Thus

$$(1+t)^{\frac{n}{4}+\frac{h}{2}+1}\|\partial_x^h\Psi(u)_t\|_{L^2}\le CE_0+CR^{\theta+1}. \tag{5.7}$$

Combining (5.5), (5.7) and taking E_0 and R suitably small yields

$$\|\Psi(u)\|_X\le R. \tag{5.8}$$

For $\tilde{u},\bar{u}\in X_R$, by using (5.4), we have

$$\Psi(\tilde{u})-\Psi(\bar{u})=\int_0^t G(t-\tau)*(I-a\Delta)^{-1}\Delta[f(\tilde{u})-f(\bar{u})]d\tau. \tag{5.9}$$

Using (5.9), (3.13) and Lemma 5.2, (5.1), for $k\le s+2$ we obtain

$$
\begin{aligned}
&\|\partial_x^k\Psi(\tilde{u})-\Psi(\bar{u}))\|_{L^2}\\
&\le \int_0^t\|\partial_x^k G(t-\tau)*(I-a\Delta)^{-1}\Delta[f(\tilde{u})-f(\bar{u})]\|_{L^2}d\tau\\
&\le C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac{1}{2}}\|(f(\tilde{u})-f(\bar{u}))\|_{L^1}d\tau\\
&\quad +C\int_{t/2}^t(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|\partial_x^k(f(\tilde{u})-f(\bar{u}))\|_{L^1}d\tau\\
&\quad +C\int_0^t e^{-c(t-\tau)}\|\partial_x^k(f(\tilde{u})-f(\bar{u}))\|_{L^2}d\tau\\
&\le C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac{1}{2}}(\|\tilde{u}\|_{L^2}+\|\bar{u}\|_{L^2})\|\tilde{u}-\bar{u}\|_{L^2}\\
&\quad \times(\|\tilde{u}\|_{L^\infty}+\|\bar{u}\|_{L^\infty})^{\theta-1}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{t/2}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\{(\|\partial_x^k\tilde{u}\|_{L^2}+\|\partial_x^k\tilde{u}\|_{L^2})\|\tilde{u}-\bar{u}\|_{L^2}\\
&+(\|\tilde{u}\|_{L^2}+\|\bar{u}\|_{L^2})\|\partial_x^k(\tilde{u}-\bar{u})\|_{L^2}\}(\|\tilde{u}\|_{L^\infty}+\|\bar{u}\|_{L^\infty})^{\theta-1}d\tau\\
&+C\int_0^t e^{-c(t-\tau)}\{(\|\partial_x^k\tilde{u}\|_{L^2}+\|\partial_x^k\tilde{u}\|_{L^2})\|\tilde{u}-\bar{u}\|_{L^\infty}\\
&+(\|\tilde{u}\|_{L^\infty}+\|\bar{u}\|_{L^\infty})\|\partial_x^k(\tilde{u}-\bar{u})\|_{L^2}\}(\|\tilde{u}\|_{L^\infty}+\|\bar{u}\|_{L^\infty})^{\theta-1}d\tau\\
\le{}& CR^\theta\|\tilde{u}-\bar{u}\|_X\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{k}{2}-\frac{1}{2}}(1+\tau)^{-(\frac{n}{2}+\frac{1}{2})\theta}d\tau\\
&+CR^\theta\|\tilde{u}-\bar{u}\|_X\int_{t/2}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}(1+\tau)^{-(\frac{\theta}{2}(n+1)+\frac{k+1}{2})}d\tau\\
&+CCR^\theta\|\tilde{u}-\bar{u}\|_X\int_0^t e^{-c(t-\tau)}(1+\tau)^{-(\frac{n}{4}+\frac{n}{2}\theta+\frac{k}{2}+\frac{1}{2})}d\tau\\
\le{}& CR^\theta(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{1}{2}}\|\tilde{u}-\bar{u}\|_X,
\end{aligned}
$$

which implies

$$
(1+t)^{\frac{n}{4}+\frac{k}{2}+\frac{1}{2}}\|\partial_x^k(\Psi(\tilde{u})-\Psi(\bar{u}))\|_{L^2}\le CR^\theta\|\tilde{u}-\bar{u}\|_X. \tag{5.10}
$$

Similarly for $h\le s$, from (5.6), (3.14) and (5.1), we have

$$
\begin{aligned}
\|\partial_x^h(\Psi(\tilde{u})-\Psi(\bar{u}))_t\|_{L^2}&\le\int_0^t\|\partial_x^hG_t(t-\tau)*(I-a\Delta)^{-1}\Delta[f(\tilde{u})-f(\bar{u})]\|_{L^2}d\tau\\
&\le C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{h}{2}-1}\|(f(\tilde{u})-f(\bar{u}))\|_{L^1}d\tau\\
&\quad+C\int_{t/2}^{t}(1+t-\tau)^{-\frac{n}{4}-1}\|\partial_x^h(f(\tilde{u})-f(\bar{u}))\|_{L^1}d\tau\\
&\quad+C\int_0^t e^{-c(t-\tau)}\|\partial_x^h(f(\tilde{u})-f(\bar{u}))\|_{L^2}d\tau\\
&\le CR^\theta(1+t)^{-\frac{n}{4}-\frac{h}{2}-1}\|\tilde{u}-\bar{u}\|_X,
\end{aligned}
$$

which implies

$$
(1+t)^{\frac{n}{4}+\frac{h}{2}+1}\|\partial_x^h(\Psi(\tilde{u})-\Psi(\bar{u}))_t\|_{L^2}\le CR^\theta\|\tilde{u}-\bar{u}\|_X. \tag{5.11}
$$

Using (5.10), (5.11) and taking R suitably small yields

$$
\|\Psi(\tilde{u})-\Psi(\bar{u})\|_X\le\frac{1}{2}\|\tilde{u}-\bar{u}\|_X. \tag{5.12}
$$

From (5.8) and (5.12), we know that Ψ is strictly contracting mapping. Consequently, we conclude that there exists a fixed point $u\in X_R$ of the mapping Ψ, which is a classical solution to (1.1), (1.2). This completes the proof. □

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