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# NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS OF ARBITRARY ORDER AND MULTI-STRIP BOUNDARY CONDITIONS

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Abstract. We study boundary value problems of nonlinear fractional differential equations and inclusions of order $q \in (m-1, m]$, $m \geq 2$ with multi-strip boundary conditions. Multi-strip boundary conditions may be regarded as the generalization of multi-point boundary conditions. Our problem is new in the sense that we consider a nonlocal strip condition of the form:

$$x(1) = \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} x(s) ds,$$

which can be viewed as an extension of a multi-point nonlocal boundary condition:

$$x(1) = \sum_{i=1}^{n-2} \alpha_i x(\eta_i).$$

In fact, the strip condition corresponds to a continuous distribution of the values of the unknown function on arbitrary finite segments $(\zeta_i, \eta_i)$ of the interval $[0, 1]$ and the effect of these strips is accumulated at $x = 1$. Such problems occur in the applied fields such as wave propagation and geophysics. Some new existence and uniqueness results are obtained by using a variety of fixed point theorems. Some illustrative examples are also discussed.

## 1. Introduction

In recent years, boundary value problems for nonlinear fractional differential equations have been addressed by several researchers. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, see [28]. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [17, 25, 29, 30]. For

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some recent development on the topic, see [1, 2, 3, 12, 15, 16] and the references therein.

In the first part of this paper, we consider the following nonlinear fractional BVP of an arbitrary order with multi-strip boundary conditions:

$$\begin{gathered} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1,\ m-1 < q \le m,\ m \ge 2,\ m \in \mathbb{N}, \\ x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \dots, x^{(m-2)}(0) = 0, \\ x(1) = \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} x(s)ds, \quad 0 < \zeta_i < \eta_i < 1,\ i = 1, 2, \dots, (n-2), \end{gathered} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q$, $f$ is a given continuous function and $\alpha_i \in \mathbb{R}$ satisfy the condition:

$$\sum_{i=1}^{n-2} \alpha_i(\eta_i^m - \zeta_i^m) \neq m.$$

The strip boundary condition in problem (1.1) can be regarded as a multi-point nonlocal integral boundary condition. Integral boundary conditions have various applications in applied sciences such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to the papers [6, 18] and references therein. Regarding the application of the strip conditions of fixed size, we know that such conditions appear in the mathematical modeling of real world problems, for example, see [7, 13]. Thus, the present idea of nonlocal strip conditions will be quite fruitful in modeling the strip problems as one can choose an arbitrary set of strips of desired size, which can be fixed according to the requirement by fixing the nonlocal parameters involved in the problem. Furthermore, these conditions can be understood in the sense that the controllers at the end-points of the interval dissipate/absorb energy due to the sensors of finite lengths (continuous distribution of intermediate points of arbitrary length: subsegments of the interval) located at the intermediate positions of the interval.

Recently nonlocal problems with several types of integral boundary conditions have studied in [4, 5, 8, 9, 10, 11, 14, 31].

We prove some new existence and uniqueness results by using a variety of fixed point theorems. In Theorem 3.1 we prove an existence and uniqueness result by using Banach's contraction principle, in Theorem 3.3 we prove the existence of a solution by means of Krasnoselskii's fixed point theorem, while in Theorem 3.5 we prove the existence of a solution via Leray-Schauder nonlinear alternative. The Leray-Schauder degree theory is used in proving the existence result in Theorem 3.6. In Theorem 3.9 we prove an existence and uniqueness result by applying a fixed point theorem of Boyd and Wong [19] for nonlinear contractions.

In the second part of the paper, we consider a nonlinear fractional differential inclusion of an arbitrary order with multi-strip boundary conditions:

$$\begin{gathered} {}^cD^q x(t) \in F(t, x(t)), \quad 0 < t < 1,\ m-1 < q \le m,\ m \ge 2,\ m \in \mathbb{N}, \\ x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \dots, x^{(m-2)}(0) = 0, \\ x(1) = \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} x(s)ds, \quad 0 < \zeta_i < \eta_i < 1,\ i = 1, 2, \dots, (n-2), \end{gathered} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

The aim here is to establish existence results for the problem (1.2), when the right hand side is convex as well as nonconvex valued. In the first result (Theorem 4.8) we consider the case when the right hand side has convex values, and prove an existence result via Nonlinear alternative for Kakutani maps. In the second result (Theorem 4.15), we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result (Theorem 4.19), we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

The methods used are standard, however their exposition in the framework of problems (1.1) and (1.2) is new.

## 2. Preliminaries from fractional calculus

Let us recall some basic definitions of fractional calculus [25, 30].

**Definition 2.1.** For function $g : [0,\infty) \to \mathbb{R}$, at least $n$-times continuously differentiable, the Caputo derivative of fractional order $q$ is defined as

$$ {}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \quad n-1 < q < n,\ n = [q]+1, $$

where $[q]$ denotes the integer part of the real number $q$.

**Definition 2.2.** The Riemann-Liouville fractional integral of order $q$ is defined as

$$ I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, $$

provided the integral exists.

**Lemma 2.3.** *For any $\sigma \in C([0,1],\mathbb{R})$, the unique solution of the boundary value problem*

$$ \begin{gathered} {}^cD^q x(t) = \sigma(t), \quad 0 < t < 1,\ m-1 < q \le m,\ m \ge 2,\ m \in \mathbb{N}, \\ x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \dots, x^{(m-2)}(0) = 0, \\ x(1) = \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} x(s)ds, \quad 0 < \zeta_i < \eta_i < 1,\ i = 1,2,\dots,(n-2), \end{gathered} \tag{2.1} $$

*is given by*

$$ \begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - \frac{mt^{m-1}}{\big(m - \sum_{i=1}^{n-2} \alpha_i(\eta_i^m - \zeta_i^m)\big)} \\ &\quad \times \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \sigma(u)du \Big) ds \Big]. \end{aligned} \tag{2.2} $$

*Proof.* It is well known [25] that the general solution of the fractional differential equation in (2.1) can be written as

$$ x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_0 - c_1 t - c_2 t^2 - \dots - c_{m-1} t^{m-1}, \tag{2.3} $$

where $c_0, c_1, c_2, \dots, c_{m-1}$ are arbitrary constants. Applying the boundary conditions for the problem (2.1), we find that $c_0 = 0, \dots, c_{m-2} = 0$, and

$$\begin{aligned} c_{m-1} = \frac{m}{\left(m - \sum_{i=1}^{n-2} \alpha_i(\eta_i^m - \zeta_i^m)\right)} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \\ - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u)) du \Big) ds \Big]. \end{aligned}$$

Substituting the values of $c_0, c_1, c_2, \dots, c_{m-1}$ in (2.3) yields the solution (2.2). $\square$

## 3. Existence results - the single-valued case

Let $\mathcal{C} = C([0,1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In view of Lemma 2.3, we transform problem (1.1) as

$$x = F(x). \tag{3.1}$$

Here the operator $F : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned} (Fx)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \vartheta t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \\ - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u)) du \Big) ds \Big], \quad t \in [0,1], \end{aligned}$$

where

$$\vartheta = m\Big(m - \sum_{i=1}^{n-2} \alpha_i(\eta_i^m - \zeta_i^m)\Big)^{-1}.$$

For convenience, let us set

$$\Lambda = \frac{1}{\Gamma(q+1)} \Big[ 1 + |\vartheta| \Big\{ 1 + \sum_{i=1}^{n-2} \frac{\alpha_i(\eta_i^{q+1} - \zeta_i^{q+1})}{q+1} \Big\} \Big]. \tag{3.2}$$

### 3.1. **Existence result via Banach's fixed point theorem.**

**Theorem 3.1.** *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function and satisfies the assumption*

(A1) $|f(t,x) - f(t,y)| \le L|x-y|$, *for all* $t \in [0,1]$, $L > 0$, $x, y \in \mathbb{R}$,

*with $L < 1/\Lambda$, where $\Lambda$ is given by (3.2). Then the boundary value problem (1.1) has a unique solution.*

*Proof.* Setting $\sup_{t\in[0,1]} |f(t,0)| = M$ and choosing $r \ge \frac{\Lambda M}{1 - L\Lambda}$, we show that $FB_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$. For $x \in B_r$, we have

$$\begin{aligned} &\|(Fx)(t)\| \\ &\le \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + |\vartheta| t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \\ &\quad + \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u, x(u))| du \Big) ds \Big] \Big\} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{t\in[0,1]}\Big\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds \\
&\quad + |\vartheta| t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds \\
&\quad + \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}(|f(u,x(u))-f(u,0)|+|f(u,0)|)du\Big)ds\Big]\Big\} \\
&\leq (Lr+M)\sup_{t\in[0,T]}\Big\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad + |\vartheta| t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds + \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}du\Big)ds\Big]\Big\} \\
&\leq \frac{(Lr+M)}{\Gamma(q+1)}\Big[1+|\vartheta|\Big\{1+\sum_{i=1}^{n-2}\frac{\alpha_i(\eta_i^{q+1}-\zeta_i^{q+1})}{q+1}\Big\}\Big] \\
&= (Lr+M)\,\Lambda \leq r.
\end{aligned}$$

Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we obtain

$$\begin{aligned}
&\|(Fx)(t)-(Fy)(t)\| \\
&\leq \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds \\
&\quad + |\vartheta| t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds \\
&\quad + \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))-f(u,y(u))|du\Big)ds\Big]\Big\} \\
&\leq L\|x-y\|\sup_{t\in[0,1]}\Big\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad + |\vartheta| t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds + \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}du\Big)ds\Big]\Big\} \\
&\leq \frac{L}{\Gamma(q+1)}\Big[1+|\vartheta|\Big\{1+\sum_{i=1}^{n-2}\frac{\alpha_i(\eta_i^{q+1}-\zeta_i^{q+1})}{q+1}\Big\}\Big]\|x-y\| = L\Lambda\|x-y\|,
\end{aligned}$$

where $\Lambda$ is given by (3.2). Observe that $\Lambda$ depends only on the parameters involved in the problem. As $L<1/\Lambda$, therefore $F$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

### 3.2. **Existence result via Krasnoselskii's fixed point theorem.**

**Lemma 3.2** (Krasnoselskii's fixed point theorem [26]). *Let $M$ be a bounded, closed, convex, and nonempty subset of a Banach space $X$. Let $A,B$ be the operators such that:*

(i) *$Ax+By\in M$ whenever $x,y\in M$;*
(ii) *$A$ is compact and continuous;*
(iii) *$B$ is a contraction mapping.*

*Then there exists $z \in M$ such that $z = Az + Bz$.*

**Theorem 3.3.** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the assumption* (A1)*. Moreover we assume that*

(A2) *$|f(t,x)| \le \mu(t)$, for all $(t,x) \in [0,1] \times \mathbb{R}$, and $\mu \in C([0,1], \mathbb{R}^+)$.*

*If*

$$\frac{L|\vartheta|}{\Gamma(q+1)}\Big(1+\sum_{i=1}^{n-2}\frac{\alpha_i(\eta_i^{q+1}-\zeta_i^{q+1})}{q+1}\Big) < 1, \tag{3.3}$$

*then the boundary value problem* (1.1) *has at least one solution on* $[0,1]$.

*Proof.* By the assumption (A2), we can fix

$$\bar{r} \ge \frac{|\vartheta|\|\mu\|}{\Gamma(q+1)}\Big(1+\sum_{i=1}^{n-2}\frac{\alpha_i(\eta_i^{q+1}-\zeta_i^{q+1})}{q+1}\Big),$$

and consider $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \le \bar{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}}$ as

$$(\mathcal{P}x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds, \quad t \in [0,1],$$

$$\begin{aligned}(\mathcal{Q}x)(t) = -\vartheta t^{m-1}\Big[&\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &-\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\Big)ds\Big], \quad t \in [0,1].\end{aligned}$$

For $x, y \in B_{\bar{r}}$, we find that

$$\|\mathcal{P}x + \mathcal{Q}y\| \le \frac{|\vartheta|\|\mu\|}{\Gamma(q+1)}\Big(1+\sum_{i=1}^{n-2}\frac{\alpha_i(\eta_i^{q+1}-\zeta_i^{q+1})}{q+1}\Big) \le \bar{r}.$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\bar{r}}$. It follows from the assumption (A1) together with (3.3) that $\mathcal{Q}$ is a contraction mapping. Continuity of $f$ implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{P}x\| \le \frac{\|\mu\|}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator $\mathcal{P}$. In view of (A1), we define

$$\sup_{(t,x)\in[0,1]\times B_{\bar{r}}} |f(t,x)| = \overline{f}.$$

Consequently we have

$$\begin{aligned}&\|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\| \\ &= \Big\|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x(s))ds + \int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\Big\| \\ &\le \frac{\overline{f}}{\Gamma(q+1)}|2(t_2-t_1)^q + t_1^q - t_2^q|,\end{aligned}$$

which is independent of $x$ and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Lemma 3.2 are satisfied. So by the conclusion of Lemma 3.2, problem (1.1) has at least one solution on $[0,1]$. $\square$

3.3. **Existence result via Leray-Schauder Alternative.**

**Lemma 3.4** (Nonlinear alternative for single valued maps [23]). . *Let $E$ be a Banach space, $C$ a closed, convex subset of $E$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of $C$) map. Then either*

(i) *$F$ has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of $U$ in $C$) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

**Theorem 3.5.** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function. Assume that:*

(A3) *There exist a function $p \in L^1([0,1], \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x)| \le p(t)\psi(\|x\|)$, for all $(t,x) \in [0,1] \times \mathbb{R}$.*
(A4) *There exists a constant $M > 0$ such that*

$$\frac{M}{\frac{\psi(M)}{\Gamma(q)}\big[\{1+|\vartheta|\}\int_0^1 p(s)ds + |\vartheta|\sum_{i=1}^{n-2}\alpha_i \frac{\eta_i^q - \zeta_i^q}{q}\int_{\zeta_i}^{\eta_i} p(s)ds\big]} > 1.$$

*Then the boundary value problem (1.1) has at least one solution on $[0,1]$.*

*Proof.* Consider the operator $F : \mathcal{C} \to \mathcal{C}$ with $x = F(x)$, where

$$\begin{aligned}(Fx)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &- \sum_{i=1}^{n-2}\alpha_i \int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u,x(u))du\Big)ds\Big], \quad t \in [0,1].\end{aligned}$$

We show that $F$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number $r$, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}|(Fx)(t)| \le &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + |\vartheta| t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\ &- \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}|f(u,x(u))|du\Big)ds\Big] \\ \le &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds + |\vartheta| t^{m-1}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds \\ &+ |\vartheta| t^{m-1}\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)du\Big)ds \\ \le &\frac{\psi(\|x\|)}{\Gamma(q)}\int_0^1 (t-s)^{q-1}p(s)ds + \frac{|\vartheta|\psi(\|x\|)}{\Gamma(q)}\int_0^1 (1-s)^{q-1}p(s)ds \\ &+ \frac{|\vartheta|\psi(\|x\|)}{\Gamma(q)}\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s (s-u)^{q-1}p(u)du\Big)ds \\ \le &\frac{\psi(\|x\|)}{\Gamma(q)}\int_0^1 (t-s)^{q-1}p(s)ds + \frac{|\vartheta|\psi(\|x\|)}{\Gamma(q)}\int_0^1 (1-s)^{q-1}p(s)ds \\ &+ \frac{|\vartheta|\psi(\|x\|)}{\Gamma(q)}\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s (s-u)^{q-1}p(u)du\Big)ds\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\psi(\|x\|)}{\Gamma(q)} \int_0^1 p(s)ds + \frac{|\vartheta|\psi(\|x\|)}{\Gamma(q)} \int_0^1 p(s)ds \\
&\quad + \frac{|\vartheta|\psi(\|x\|)}{\Gamma(q)} \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} p(s)ds \\
&= \frac{\psi(\|x\|)}{\Gamma(q)} \Big[ \{1 + |\vartheta|\} \int_0^1 p(s)ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} p(s)ds \Big].
\end{aligned}
$$

Consequently,

$$
\|Fx\| \le \frac{\psi(r)}{\Gamma(q)} \Big[ \{1 + |\vartheta|\} \int_0^1 p(s)ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} p(s)ds \Big].
$$

Next we show that $F$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where $B_r$ is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$
\begin{aligned}
&|(Fx)(t'') - (Fx)(t')| \\
&= \Big| \frac{1}{\Gamma(q)} \int_0^{t''} (t'' - s)^{q-1} f(s, x(s))ds - \frac{1}{\Gamma(q)} \int_0^{t'} (t' - s)^{q-1} f(s, x(s))ds \\
&\quad - \vartheta\Big((t'')^{m-1} - (t')^{m-1}\Big) \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \\
&\quad - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u, x(u))du \Big) ds \Big] \Big| \\
&\le \Big| \frac{1}{\Gamma(q)} \int_0^{t'} [(t'' - s)^{q-1} - (t' - s)^{q-1}] \psi(r)p(s)ds \Big| \\
&\quad + \Big| \frac{1}{\Gamma(q)} \int_{t'}^{t''} (t'' - s)^{q-1} \psi(r)p(s)ds \Big| \\
&\quad + \Big| \vartheta\Big((t'')^{m-1} - (t')^{m-1}\Big) \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \psi(r)p(s)ds \\
&\quad - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \psi(r)p(u)du \Big) ds \Big] \Big|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' - t' \to 0$. As $F$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3.4) once we have proved the boundendness of the set of all solutions to equations $x = \lambda F x$ for $\lambda \in [0,1]$.

Let $x$ be a solution. Then, for $t \in [0,1]$, and using the computations in proving that $F$ is bounded, we have

$$
\begin{aligned}
|x(t)| &= |\lambda(Fx)(t)| \\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))|ds + |\vartheta| t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u))| du \Big) ds \Big] \\
&\le \frac{\psi(\|x\|)}{\Gamma(q)} \Big[ \{1+|\vartheta|\} \int_0^1 p(s)ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} p(s)ds \Big].
\end{aligned}
$$

Consequently, we have

$$
\frac{\|x\|}{\frac{\psi(\|x\|)}{\Gamma(q)} \big[ \{1+|\vartheta|\} \int_0^1 p(s)ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} p(s)ds \big]} \le 1.
$$

In view of (A4), there exists $M$ such that $\|x\| \neq M$. Let us set

$$
U = \{x \in C([0,1], \mathbb{R}) : \|x\| < M\}.
$$

Note that the operator $F : \overline{U} \to C([0,1], \mathbb{R})$ is continuous and completely continuous. From the choice of $U$, there is no $x \in \partial U$ such that $x = \lambda F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $F$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

3.4. **Existence result via Leray-Schauder degree theory.**

**Theorem 3.6.** *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$. Assume that there exist constants $0 \le \kappa < \frac{1}{\Lambda}$, where $\Lambda$ is given by* (3.2) *and $M > 0$ such that $|f(t,x)| \le \kappa \|x\| + M$ for all $t \in [0,1], x \in \mathbb{R}$. Then the boundary value problem* (1.1) *has at least one solution.*

*Proof.* In view of the fixed point problem (3.1), we just need to prove the existence of at least one solution $x \in \mathbb{R}$ satisfying (3.1). Define a suitable ball $B_R \subset C[0,1]$ with radius $R > 0$ as

$$
B_R = \{x \in \mathcal{C} : \|x\| < R\},
$$

where $R$ will be fixed later. Then, it is sufficient to show that $F : \overline{B}_R \to \mathcal{C}$ satisfies

$$
x \neq \lambda F x, \quad \forall x \in \partial B_R \quad \text{and} \quad \forall \lambda \in [0,1]. \tag{3.4}
$$

Let us set

$$
H(\lambda, x) = \lambda F x, \quad x \in \mathcal{C}, \quad \lambda \in [0,1].
$$

Then, by the Arzelá-Ascoli Theorem, $h_\lambda(x) = x - H(\lambda, x) = x - \lambda F x$ is completely continuous. If (3.4) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$
\begin{aligned}
\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda F, B_R, 0) = \deg(h_1, B_R, 0) \\
&= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,
\end{aligned}
$$

where $I$ denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(t) = x - \lambda F x = 0$ for at least one $x \in B_R$. To prove (3.4), we assume that $x = \lambda F x$ for some $\lambda \in [0,1]$ and for all $t \in [0,1]$ so that

$$
\begin{aligned}
&|x(t)| = |\lambda (Fx)(t)| \\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds + |\vartheta| t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&+ \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u,x(u))| du \Big) ds \Big]
\end{aligned}
$$

$$\begin{aligned}
&\le (\kappa\|x\|+M)\Big\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad + |\vartheta| t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds + \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}du\Big)ds\Big]\Big\} \\
&\le \frac{(\kappa\|x\|+M)}{\Gamma(q+1)}\Big[1+|\vartheta|\big\{1+\sum_{i=1}^{n-2}\frac{\alpha_i(\eta_i^{q+1}-\zeta_i^{q+1})}{q+1}\big\}\Big] \\
&= (\kappa\|x\|+M)\Lambda,
\end{aligned}$$

which, on taking norm ($\sup_{t\in[0,1]}|x(t)| = \|x\|$) and solving for $\|x\|$, yields

$$\|x\| \le \frac{M\Lambda}{1-\kappa\Lambda}.$$

Letting $R = \frac{M\Lambda}{1-\kappa\Lambda}+1$, (3.4) holds. This completes the proof. □

3.5. **Existence result via nonlinear contractions.**

**Definition 3.7.** Let $E$ be a Banach space and let $F: E \to E$ be a mapping. $F$ is said to be a nonlinear contraction if there exists a continuous nondecrasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ with the property:

$$\|Fx - Fy\| \le \Psi(\|x-y\|), \quad \forall x, y \in E.$$

**Lemma 3.8** (Boyd and Wong [19]). *Let $E$ be a Banach space and let $F: E \to E$ be a nonlinear contraction. Then $F$ has a unique fixed point in $E$.*

**Theorem 3.9.** *Assume that:*

(A5) $|f(t,x) - f(t,y)| \le h(t)\frac{|x-y|}{H^*+|x-y|}$, $t \in [0,1]$, $x, y \ge 0$, *where* $h: [0,1] \to \mathbb{R}^+$ *is continuous and*

$$\begin{aligned}
H^* = \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)ds + |\vartheta|\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)ds \\
+ \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}h(u)du\Big)ds\Big].
\end{aligned} \tag{3.5}$$

*Then the boundary value problem* (1.1) *has a unique solution.*

*Proof.* We define the operator $F: \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}
Fx(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds + \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\
+ \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u))du\Big)ds\Big], \quad t \in [0,1].
\end{aligned}$$

Let a continuous nondecreasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ be defined by

$$\Psi(\xi) = \frac{H^*\xi}{H^*+\xi}, \quad \forall \xi \ge 0.$$

Let $x, y \in \mathcal{C}$. Then

$$|f(s,x(s)) - f(s,y(s))| \le \frac{h(s)}{H^*}\Psi(\|x-y\|)$$

so that

$$\begin{aligned}
&|Fx(t) - Fy(t)| \\
&\le \int_0^t \frac{(1-s)^{q-1}}{\Gamma(q)} h(s) \frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|} ds \\
&\quad + |\vartheta| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s) \frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|} ds \\
&\quad + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(m) \frac{|x(m)-y(m)|}{H^* + |x(m)-y(m)|} dm \Big) ds,
\end{aligned}$$

for $t \in [0,1]$. In view of (3.5), it follows that $\|Fx - Fy\| \le \Psi(\|x-y\|)$ and hence $F$ is a nonlinear contraction. Thus, by Lemma 3.8, the operator $F$ has a unique fixed point in $\mathcal{C}$, which in turn is a unique solution of problem (1.1). □

3.6. **Examples.** For the forthcoming examples, we consider the following boundary conditions:

$$x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \quad x(1) = \sum_{i=1}^{3} \alpha_i \int_{\zeta_i}^{\eta_i} x(s) ds, \tag{3.6}$$

where $\zeta_1 = 1/16$, $\zeta_2 = 5/16$, $\zeta_3 = 9/16$, $\eta_1 = 1/4$, $\eta_2 = 1/2$, $\eta_3 = 3/4$, $\alpha_1 = 1/3$, $\alpha_2 = 2/3$, $\alpha_3 = 1$.

**Example 3.10.** Consider the fractional differential equation

$${}^cD^{7/2} x(t) = L\big(\cos t + \tan^{-1} x(t)\big), \quad 0 < t < 1, \tag{3.7}$$

subject to the strip boundary conditions (3.6).

Here, $q = 7/2$, $m = 4$ and $f(t,x) = L\big(\cos t + \tan^{-1} x(t)\big)$. Clearly

$$|f(t,x) - f(t,y)| \le L|\tan^{-1} x - \tan^{-1} y| \le L|x-y|,$$

$$\vartheta = m\Big(m - \sum_{i=1}^{n-2} \alpha_i(\eta_i^m - \zeta_i^m)\Big)^{-1} = 1.0675,$$

$$\Lambda = \frac{1}{\Gamma(q+1)} \Big[1 + |\vartheta| \Big\{1 + \sum_{i=1}^{3} \frac{\alpha_i(\eta_i^{q+1} - \zeta_i^{q+1})}{(q+1)}\Big\}\Big] \approx \frac{34}{105\sqrt{\pi}}.$$

With $L < \frac{1}{\Lambda} \approx 105\sqrt{\pi}/34$, all the assumptions of Theorem 3.1 are satisfied. Hence, there exists a unique solution for problem (3.7)-(3.6) on $[0,1]$.

**Example 3.11.** Consider the equation

$${}^cD^{7/2} x(t) = \frac{1}{4\pi} \sin(2\pi x) + \frac{x^2}{1+x^2}, \quad 0 < t < 1, \tag{3.8}$$

subject to the strip boundary conditions (3.6).

Here, $q = 7/2$, $m = 4$ and $f(t,x) = \frac{1}{4\pi}\sin(2\pi x) + \frac{x^2}{1+x^2}$. Observe that

$$|f(t,x)| = |\frac{1}{4\pi} \sin(2\pi x) + \frac{x^2}{1+x^2}| \le \frac{1}{2}\|x\| + 1.$$

with $\kappa = \frac{1}{2} < \frac{1}{\Lambda} \approx 105\sqrt{\pi}/34$ and $M = 1$. Thus, the conclusion of Theorem 3.6 applies to problem (3.8)-(3.6).

**Example 3.12.** Let us consider the fractional differential equation

$$ {}^cD^{7/2}x(t) = \frac{t|x|}{1+|x|}, \quad 0 < t < 1, \tag{3.9}$$

subject to the strip boundary conditions (3.6).

Here, $q = 7/2$, $m = 4$ and $f(t,x) = \frac{t|x|}{1+|x|}$. We choose $h(t) = (1+t)$ and find that

$$\begin{aligned} H^* = &\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)ds + |\vartheta| \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)ds \\ &+ \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} h(u)du \Big) ds \Big] = 0.222818. \end{aligned}$$

Clearly,

$$|f(t,x) - f(t,y)| = \Big|\frac{t(|x|-|y|)}{1+|x|+|y|+|x||y|}\Big| \le \frac{(1+t)|x-y|}{0.222818+|x-y|}.$$

Thus, the conclusion of Theorem 3.9 applies and problem (3.9)-(3.6) has a unique solution.

## 4. Existence results - the multi-valued case

**Definition 4.1.** A function $x \in C([0,1],\mathbb{R})$ with its Caputo derivative of order $q$ existing on $[0,1]$ is a solution of the problem (1.2) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \vartheta t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &- \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u)du \Big) ds \Big]. \end{aligned}$$

### 4.1. The Carathéodory case.

In this subsection, we are concerned with the existence of solutions for the problem (1.2) when the right hand side has convex values. We first recall some preliminary facts. For a normed space $(X, \|\cdot\|)$, let

$$\begin{gathered} P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}, \\ P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}, \\ P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}, \\ P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}. \end{gathered}$$

**Definition 4.2.** A multi-valued map $G : X \to \mathcal{P}(X)$ :

- (i) is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$;
- (ii) is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in $X$ for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);
- (iii) is called upper semi-continuous (u.s.c.) on $X$ if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of $x_0$ such that $G(\mathcal{N}_0) \subseteq N$;
- (v) is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$;

(v) has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator $G$ will be denoted by $FixG$.

**Remark 4.3.** It is known that, if the multi-valued map $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

**Definition 4.4.** A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \mapsto d(y, G(t)) = \inf\{\|y-z\| : z \in G(t)\}$ is measurable.

**Definition 4.5.** A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$;

Further a Carathéodory function $F$ is called $L^1-$Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,1],\mathbb{R}^+)$ such that
$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t)$$
for all $\|x\| \leq \alpha$ and for a. e. $t \in [0,1]$.

For each $y \in C([0,1],\mathbb{R})$, define the set of selections of $F$ by
$$S_{F,y} := \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,1]\}.$$
The consideration of this subsection is based on the following lemmas.

**Lemma 4.6** (Nonlinear alternative for Kakutani maps [23])**.** *Let $E$ be a Banach space, $C$ a closed convex subset of $E$, $U$ an open subset of $C$ and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is a upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of $C$. Then either*

(i) *$F$ has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

**Lemma 4.7** ([27])**.** *Let $X$ be a Banach space. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an $L^1-$ Carathéodory multivalued map and let $\Theta$ be a linear continuous mapping from $L^1([0,1],\mathbb{R})$ to $C([0,1],\mathbb{R})$. Then the operator*
$$\Theta \circ S_F : C([0,1],\mathbb{R}) \to P_{cp,c}(C([0,1],\mathbb{R})), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$
*is a closed graph operator in $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$.*

**Theorem 4.8.** *Assume that* (A4) *holds. In addition we assume that:*

(H1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact convex values;*
(H2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1],\mathbb{R}^+)$ such that*
$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|) \quad \textit{for each } (t,x) \in [0,1] \times \mathbb{R}.$$

*Then the boundary value problem* (1.2) *has at least one solution on* $[0,1]$.

*Proof.* Define an operator $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ by
$$\Omega(x) = \Big\{h \in C([0,1],\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds$$

$$- \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u)du \Big) ds \Big],\ 0 \le t \le 1 \Big\},$$

for $f \in S_{F,x}$. We will show that $\Omega$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\Omega$ is convex for each $x \in C([0,1],\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,1]$, we have

$$\begin{aligned} h_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s) ds - \vartheta t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_i(s) ds \\ - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f_i(u) du \Big) ds \Big], \quad i = 1, 2. \end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,1]$, we have

$$\begin{aligned} &[\omega h_1 + (1-\omega) h_2](t) \\ &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [\omega f_1(s) + (1-\omega) f_2(s)] ds \\ &\quad - \vartheta t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} [\omega f_1(s) + (1-\omega) f_2(s)] ds \\ &\quad - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} [\omega f_1(s) + (1-\omega) f_2(s)] du \Big) ds \Big]. \end{aligned}$$

Since $S_{F,x}$ is convex ($F$ has convex values), therefore it follows that $\omega h_1 + (1-\omega) h_2 \in \Omega(x)$.

Next, we show that $\Omega$ maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number $r$, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds - \vartheta t^{m-1} \Big[ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s) ds \\ - \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u) du \Big) ds \Big]. \end{aligned}$$

Then, as in Theorem 3.5,

$$\begin{aligned} |h(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)| ds + |\vartheta| t^{m-1} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s)| ds \\ &\quad + |\vartheta| t^{m-1} \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big( \int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |f(u)| du \Big) ds \\ &\le \frac{\psi(\|x\|)}{\Gamma(q)} \Big[ \{1 + |\vartheta|\} \int_0^1 p(s) ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} p(s) ds \Big]. \end{aligned}$$

Thus,

$$\|h\| \le \frac{\psi(r)}{\Gamma(q)} \Big[ \{1 + |\vartheta|\} \int_0^1 p(s) ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_1^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} p(s) ds \Big].$$

Now we show that $\Omega$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t',t''\in[0,1]$ with $t'<t''$ and $x\in B_r$, where $B_r$ is a bounded set of $C([0,1],\mathbb{R})$. For each $h\in\Omega(x)$, we obtain

$$\begin{aligned}
|h(t'')-h(t')| &= \Big|\frac{1}{\Gamma(q)}\int_0^{t''}(t''-s)^{q-1}f(s)ds-\frac{1}{\Gamma(q)}\int_0^{t'}(t'-s)^{q-1}f(s)ds\\
&\quad -\vartheta\Big((t'')^{m-1}-(t')^{m-1}\Big)\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&\quad -\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u)du\Big)ds\Big]\Big|\\
&\le \Big|\frac{1}{\Gamma(q)}\int_0^{t'}[(t''-s)^{q-1}-(t'-s)^{q-1}]\psi(r)p(s)ds\Big|\\
&\quad +\Big|\frac{1}{\Gamma(q)}\int_{t'}^{t''}(t''-s)^{q-1}\psi(r)p(s)ds\Big|\\
&\quad +\Big|\vartheta\Big((t'')^{m-1}-(t')^{m-1}\Big)\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\psi(r)p(s)ds\\
&\quad \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}\psi(r)p(u)du\Big)ds\Big]\Big|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_{r'}$ as $t''-t'\to 0$. As $\Omega$ satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega: C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\Omega$ has a closed graph. Let $x_n\to x_*, h_n\in\Omega(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega(x_*)$. Associated with $h_n\in\Omega(x_n)$, there exists $f_n\in S_{F,x_n}$ such that for each $t\in[0,1]$,

$$\begin{aligned}
h_n(t) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_n(s)ds-\vartheta t^{m-1}\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f_n(s)ds\\
&\quad -\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f_n(u)du\Big)ds\Big].
\end{aligned}$$

Thus we have to show that there exists $f_*\in S_{F,x_*}$ such that for each $t\in[0,1]$,

$$\begin{aligned}
h_*(t) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_*(s)ds-\vartheta t^{m-1}\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f_*(s)ds\\
&\quad -\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f_*(u)du\Big)ds\Big].
\end{aligned}$$

Let us consider the continuous linear operator $\Theta: L^1([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ given by

$$\begin{aligned}
f\mapsto\Theta(f) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds-\vartheta t^{m-1}\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&\quad -\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u)du\Big)ds\Big].
\end{aligned}$$

Observe that

$$\begin{aligned}
&\|h_n(t) - h_*(t)\| \\
&= \Big\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s) - f_*(s))ds - \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}(f_n(s) - f_*(s))ds \\
&\quad - \sum_{i=1}^{n-2}\alpha_i \int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}(f_n(u) - f_*(u))du\Big)ds\Big]\Big\| \to 0,
\end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 4.7 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned}
h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\
- \sum_{i=1}^{n-2}\alpha_i \int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f_*(u)du\Big)ds\Big],
\end{aligned}$$

for some $f_* \in S_{F,x_*}$.

Finally, we discuss a priori bounds on solutions. Let $x$ be a solution of (1.2). Then there exists $f \in L^1([0,1],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$\begin{aligned}
x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds \\
- \sum_{i=1}^{n-2}\alpha_i \int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u)du\Big)ds\Big].
\end{aligned}$$

In view of (H2), and using the computations in second step above, for each $t \in [0,1]$, we obtain

$$\begin{aligned}
|x(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s)|ds + |\vartheta| t^{m-1}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s)|ds \\
&\quad + |\vartheta| t^{m-1}\sum_{i=1}^{n-2}\alpha_i \int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}|f(u)|du\Big)ds \\
&\le \frac{\psi(\|x\|)}{\Gamma(q)}\Big[\{1+|\vartheta|\}\int_0^1 p(s)ds + |\vartheta|\sum_{i=1}^{n-2}\alpha_i \frac{\eta_i^q - \zeta_i^q}{q}\int_{\zeta_i}^{\eta_i} p(s)ds\Big].
\end{aligned}$$

Consequently,

$$\frac{\|x\|}{\frac{\psi(\|x\|)}{\Gamma(q)}\Big[\{1+|\vartheta|\}\int_0^1 p(s)ds + |\vartheta|\sum_{i=1}^{n-2}\alpha_i \frac{\eta_1^q - \zeta_i^q}{q}\int_{\zeta_i}^{\eta_i} p(s)ds\Big]} \le 1.$$

In view of (A4), there exists $M$ such that $\|x\| \ne M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of $U$, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.6), we deduce that $\Omega$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.2). This completes the proof. $\square$

4.2. **The lower semi-continuous case.** Here, we study the case when $F$ is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [20] for lower semi-continuous maps with decomposable values.

**Definition 4.9.** Let $X$ be a nonempty closed subset of a Banach space $E$ and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. $G$ is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set $B$ in $E$.

**Definition 4.10.** Let $A$ be a subset of $[0,1]\times\mathbb{R}$. $A$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if $A$ belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

**Definition 4.11.** A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

**Definition 4.12.** Let $Y$ be a separable metric space and $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say $N$ has a property (BC) if $N$ is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1]\times\mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with $F$ as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with $F$.

**Definition 4.13.** Let $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say $F$ is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

**Lemma 4.14** ([20])**.** *Let $Y$ be a separable metric space and $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then $N$ has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

**Theorem 4.15.** *Assume that* (H2), (H3) *and the following condition holds:*

(H4) $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
  (a) $(t,x) \mapsto F(t,x)$ *is* $\mathcal{L}\otimes\mathcal{B}$ *measurable,*
  (b) $x \mapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$.

*Then the boundary value problem* (1.2) *has at least one solution on* $[0,1]$.

*Proof.* It follows from (H2) and (H4) that $F$ is of l.s.c. type. Then, by Lemma 4.14, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that

$f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$. Consider the problem

$$\begin{gathered} {}^cD^q x(t) = f(x(t)), \quad 0<t<1, \quad m-1<q\le m,\ m\ge 2,\ m\in\mathbb{N}, \\ x(0)=0, \quad x'(0)=0, \quad x''(0)=0, \quad \dots, \quad x^{(m-2)}(0)=0, \\ x(1)=\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i} x(s)ds, \quad 0<\zeta_i<\eta_i<1,\ i=1,2,\dots,(n-2), \end{gathered} \tag{4.1}$$

in the space $C([0,1],\mathbb{R})$. It is clear that if $x$ is a solution of the problem (4.1), then $x$ is a solution to the problem (1.2). In order to transform the problem (4.1) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) = \Big\{ &\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(x(s))ds - \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(x(s))ds \\ &- \sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(x(u))du\Big)ds\Big],\ 0\le t\le 1.\Big\} \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.8. So we omit it. This completes the proof. $\square$

4.3. **The Lipschitz case.** Now we prove the existence of solutions for the problem (1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [22].

Let $(X,d)$ be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B) = \max\big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\big\},$$

where $d(A,b)=\inf_{a\in A} d(a;b)$ and $d(a,B)=\inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized metric space (see [24]).

**Definition 4.16.** A multivalued operator $N : X\to P_{cl}(X)$ is called:

(a) $\gamma$-Lipschitz if and only if there exists $\gamma>0$ such that

$$H_d(N(x),N(y))\le \gamma d(x,y) \text{ for each } x,y\in X;$$

(b) a contraction if and only if it is $\gamma$-Lipschitz with $\gamma<1$.

**Lemma 4.17** (Covitz-Nadler, [22]). *Let $(X,d)$ be a complete metric space. If $N : X\to P_{cl}(X)$ is a contraction, then* $\mathrm{Fix}N\ne\emptyset$.

**Definition 4.18.** *A measurable multi-valued function $F:[0,1]\to\mathcal{P}(X)$ is said to be integrably bounded if there exists a function $h\in L^1([0,1],X)$ such that for all $v\in F(t)$, $\|v\|\le h(t)$ for a.e. $t\in[0,1]$.*

**Theorem 4.19.** *Assume that the following conditions hold:*

(H5) *$F:[0,1]\times\mathbb{R}\to P_{cp}(\mathbb{R})$ is such that $F(\cdot,x):[0,1]\to P_{cp}(\mathbb{R})$ is measurable for each $x\in\mathbb{R}$;*

(H6) *$H_d(F(t,x),F(t,\bar{x}))\le m(t)|x-\bar{x}|$ for almost all $t\in[0,1]$ and $x,\bar{x}\in\mathbb{R}$ with $m\in L^1([0,1],\mathbb{R}^+)$ and $d(0,F(t,0))\le m(t)$ for almost all $t\in[0,1]$.*

*Then the boundary-value problem* (1.2) *has at least one solution on* $[0,1]$ *if*

$$\frac{1}{\Gamma(q)}\Big[\{1+|\vartheta|\}\int_0^1 m(s)ds+|\vartheta|\sum_{i=1}^{n-2}\alpha_i\frac{\eta_1^q-\zeta_i^q}{q}\int_{\zeta_i}^{\eta_i}m(s)ds\Big]<1.$$

*Proof.* We transform the problem (1.2) into a fixed point problem. Consider the set-valued map $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined at the beginning of the proof of Theorem 4.8. It is clear that the fixed point of $\Omega$ are solutions of the problem (1.2).

Note that, by the assumption (H5), since the set-valued map $F(\cdot,x)$ is measurable, it admits a measurable selection $f:[0,1]\to\mathbb{R}$ (see [21, Theorem III.6]). Moreover, from assumption (H6)

$$|f(t)|\le m(t)+m(t)|x(t)|,$$

i.e. $f(\cdot)\in L^1([0,1],\mathbb{R})$. Therefore the set $S_{F,x}$ is nonempty. Also note that since $S_{F,x}\neq\emptyset$, therefore $\Omega(x)\neq\emptyset$ for any $x\in C([0,1],\mathbb{R})$.

Now we show that the operator $\Omega$ satisfies the assumptions of Lemma 4.17. To show that $\Omega(x)\in P_{cl}((C[0,1],\mathbb{R}))$ for each $x\in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\ge 0}\in\Omega(x)$ be such that $u_n\to u$ $(n\to\infty)$ in $C([0,1],\mathbb{R})$. Then $u\in C([0,1],\mathbb{R})$ and there exists $v_n\in S_{F,x}$ such that, for each $t\in[0,1]$, we have

$$\begin{aligned}u_n(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_n(s)ds-\vartheta t^{m-1}\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_n(s)ds\\ -\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v_n(u)du\Big)ds\Big].\end{aligned}$$

As $F$ has compact values, we may pass onto a subsequence (if necessary) to obtain that $v_n$ converges to $v$ in $L^1([0,1],\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in[0,1]$,

$$\begin{aligned}u_n(t)\to u(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v(s)ds-\vartheta t^{m-1}\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v(s)ds\\ -\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)du\Big)ds\Big].\end{aligned}$$

Hence, $u\in\Omega(x)$ and $\Omega(x)$ is closed.

Next we show that $\Omega$ is a contraction on $C([0,1],\mathbb{R})$; i.e., there exists $\gamma<1$ such that

$$H_d(\Omega(x),\Omega(\bar{x}))\le\gamma\|x-\bar{x}\|_\infty\quad\text{for each}\quad x,\bar{x}\in C([0,1],\mathbb{R}).$$

Let $x,\bar{x}\in C([0,1],\mathbb{R})$ and $h_1\in\Omega(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[0,1]$,

$$\begin{aligned}h_1(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_1(s)ds-\vartheta t^{m-1}\Big[\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_1(s)ds\\ -\sum_{i=1}^{n-2}\alpha_i\int_{\zeta_i}^{\eta_i}\Big(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v_1(u)du\Big)ds\Big].\end{aligned}$$

By (H6), we have

$$H_d(F(t,x),F(t,\bar{x}))\le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable ([21, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for $U$. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds - \vartheta t^{m-1}\Big[\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} v_2(s)ds \\ &- \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} v_2(u)du\Big)ds\Big]. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| \le &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\ &+ |\vartheta| t^{m-1} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\ &+ |\vartheta| t^{m-1} \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} \Big(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |v_1(u) - v_2(u)|du\Big)ds \\ \le &\frac{\|x - \overline{x}\|}{\Gamma(q)} \Big[\{1 + |\vartheta|\} \int_0^1 m(s)ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} m(s)ds\Big]. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le \frac{1}{\Gamma(q)} \Big[\{1 + |\vartheta|\} \int_0^1 m(s)ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} m(s)ds\Big] \|x - \overline{x}\|.$$

Analogously, interchanging the roles of $x$ and $\overline{x}$, we obtain

$$\begin{aligned} &H_d(\Omega(x), \Omega(\bar{x})) \\ &\le \gamma \|x - \bar{x}\| \\ &\le \frac{1}{\Gamma(q)} \Big[\{1 + |\vartheta|\} \int_0^1 m(s)ds + |\vartheta| \sum_{i=1}^{n-2} \alpha_i \frac{\eta_i^q - \zeta_i^q}{q} \int_{\zeta_i}^{\eta_i} m(s)ds\Big] \|x - \overline{x}\|. \end{aligned}$$

Since $\Omega$ is a contraction, it follows by Lemma 4.17 that $\Omega$ has a fixed point $x$ which is a solution of (1.2). This completes the proof. □

## 4.4. **Example.**

**Example 4.20.** Consider the strip fractional boundary value problem

$$\begin{gathered} {}^cD^{7/2} x(t) \in F(t, x(t)), \quad 0 < t < 1, \\ x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \quad x(1) = \sum_{i=1}^{3} \alpha_i \int_{\zeta_i}^{\eta_i} x(s)ds, \end{gathered} \tag{4.2}$$

Here, $q = 7/2$, $m = 4$, $\zeta_1 = 1/16$, $\zeta_2 = 5/16$, $\zeta_3 = 9/16$, $\eta_1 = 1/4$, $\eta_2 = 1/2$, $\eta_3 = 3/4$, $\alpha_1 = 1/3$, $\alpha_2 = 2/3$, $\alpha_3 = 1$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \big[\frac{|x|^3}{|x|^3+3} + 3t^3 + 5, \frac{|x|}{|x|+1} + t + 1\big].$$

For $f \in F$, we have

$$|f| \le \max\Big(\frac{|x|^3}{|x|^3+3} + 3t^3 + 5, \frac{|x|}{|x|+1} + t + 1\Big) \le 9, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le 9 = p(t)\psi(\|x\|), \quad x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|) = 9$. Further, using the condition

$$\frac{M}{\frac{\psi(M)}{\Gamma(q)}\big[\{1+|\vartheta|\}\int_0^1 p(s)ds + |\vartheta|\sum_{i=1}^{n-2}\alpha_i\frac{\eta_i^q-\zeta_i^q}{q}\int_{\zeta_i}^{\eta_i} p(s)ds\big]} > 1,$$

we find that $M > 5.6427$. Clearly, all the conditions of Theorem 4.8 are satisfied. So there exists at least one solution of the problem (4.2) on $[0,1]$.

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