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BEHAVIOR OF THE ENERGY FOR LAMÉ SYSTEMS IN BOUNDED DOMAINS WITH NONLINEAR DAMPING AND EXTERNAL FORCE

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Abstract. We study behavior of the energy for solutions to a Lamé system on a bounded domain, with localized nonlinear damping and external force. The equation is set up in three dimensions and under a microlocal geometric condition. More precisely, we prove that the behavior of the energy is determined by a solution to a forced differential equation, an it depends on the L^2 norm of the force.

1. Introduction and statement of the problem

Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Let us consider the Lamé system with localized nonlinear damping and external force,

$$\begin{gathered}
\partial_t^2 u - \Delta_e u + a(x) g(\partial_t u) = f(t,x), \quad \text{in } \mathbb{R}_+ \times \Omega, \\
u = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \\
u(0,x) = \varphi_1(x), \quad \partial_t u(0,x) = \varphi_2(x) \quad \text{in } \Omega.
\end{gathered} \tag{1.1}$$

Here Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda+\mu)\nabla \operatorname{div} u, \quad u = (u_1,u_2,u_3),$$

and we assume that the Lamé constants λ and μ satisfy the conditions

$$\mu > 0, \quad \lambda + 2\mu > 0. \tag{1.2}$$

Moreover, $a(x) \in L^\infty(\Omega)$ is a nonnegative real function, f is in $(L^2_{\rm loc}(\mathbb{R}_+, L^2(\Omega)))^3$ and

$$g(\partial_t u) = (g_1(\partial_t u_1), g_2(\partial_t u_2), g_3(\partial_t u_3)),$$

where $g_i : \mathbb{R} \to \mathbb{R}$ is a continuous monotone increasing function satisfying $g_i(0) = 0$ and the following growth assumption:

$$c_1 s^2 \le g_i(s)s \le c_2 s^2, \quad |s| \ge 1, \quad \text{for } i = 1,2,3, \tag{1.3}$$

with $c_1, c_2 > 0$. We can find applications for this system in geophysics and seismic waves propagation. In the case $\lambda + \mu = 0$ we obtain a vector wave equation and we aim in this article to generalize some well known results for the wave equation.

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In this framework, due to the nonlinear semi-group theory, it is well known that, for every $\varphi = (\varphi_1, \varphi_2) \in \mathcal{H} = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3$, the system (1.1) admits a unique global solution $u(t,x)$ such that

$$u \in C^0(\mathbb{R}_+, (H_0^1(\Omega))^3) \cap C^1(\mathbb{R}_+, (L^2(\Omega))^3). \tag{1.4}$$

The energy of u at time t is defined by

$$E_u(t) = \frac{1}{2} \int_\Omega (\mu |\nabla u|^2 + (\lambda + \mu)|\operatorname{div} u|^2 + |\partial_t u|^2)(t,x) dx, \tag{1.5}$$

and the following energy functional law holds

$$\begin{aligned} &E_u(t) + \int_s^t \int_\Omega a(x) g(\partial_t u(\sigma,x)) \cdot \partial_t u(\sigma,x)\, dx\, d\sigma \\ &= E_u(s) + \int_s^t \int_\Omega f(t,x) \cdot \partial_t u(\sigma,x)\, dx\, d\sigma, \end{aligned} \tag{1.6}$$

for every $t \geq s \geq 0$.

For the literature we quote essentially the result of Bisognin et al [12] which established that the solutions of a system in elasticity theory with a nonlinear localized dissipation decay in an algebraic rate to zero using some energy identities associated with localized multipliers. For more results on the energy decay for the Lamé system with linear or nonlinear damping we refer the reader to Alabau and Komornik [1, 2], Alabau [3], Guesmia [14], Horn [16, 17] and references therein. We note that the method used in these papers is based on technical multipliers. In the same spirit, we can also quote the work of Guesmia [15] for the observability, exact controllability and internal or boundary stabilization of general elasticity systems with variable coefficients depending on both time and space variables. See also the work of Bellassoued [4] which investigate the decay property of the solutions to the initial-boundary value problem for the elastic wave equation with a local time-dependent nonlinear damping. We note moreover that Burq and Lebeau [5] introduced the microlocal defect measures attached to sequences of solutions of the Lamé system and proved a propagation result when the energy of the longitudinal component goes to zero. Finally, Daoulatli et al [8] adapted the Lax-Philips theory, and under the assumption (GC), gave the rate of decay of the local energy for solutions of the Lamé system on exterior domain with nonlinear localized damping. Let us indicated that all the result above are without external force and no result seems to be known when $f \neq 0$. We specially mention the result of Daoulatli [7], which study the behavior of the energy of solutions of the wave equation with localized damping and an external force on compact Riemannian manifold with boundary.

The main purpose of this work is to give the behavior of the energy of solutions of (1.1). First we recall the following definition.

Definition 1.1. We will call *generalized bicharacteristic path* any curve which consists of generalized bicharacteristics of the principal symbol p (where $p(t,x;\tau,\xi) = (\mu|\xi|^2 - \tau^2)^2((\lambda+2\mu)|\xi|^2 - \tau^2))$, with possibility of moving from a characteristic manifold to another, at each point of $T^*(\partial\Omega)$, in the way indicated in [8].

Remark 1.2. A *generalized* geodesic path is constituted of segments living in Ω, that intersect the boundary transversally (at hyperbolic points for $p_L(t,x;\tau,\xi) = c_L^2|\xi|^2 - \tau^2$ or $p_T(t,x;\tau,\xi) = c_T^2|\xi|^2 - \tau^2$ (where $c_L = \sqrt{\lambda + 2\mu}$ and $c_T = \sqrt{\mu}$), or tangentially (at diffractive points). These segments may be connected to arcs of curves living on $\partial\Omega$ which are projections of glancing rays associated to p_L or p_T. The projection of such a generalized bicharacteristic path on $\bar{\Omega}$ will be called a *generalized geodesic path.*

Definition 1.3. Let ω be an open subset of Ω, $T > 0$ and consider the following assumption:

(GC) every generalized geodesic path of Ω, issued at $t = 0$, meets $\mathbb{R}_+ \times \omega$ between the limits 0 and T.

We shall relate the open subset ω with the damper a by $\omega = \{x \in \Omega : a(x) > \mu > 0\}$.

Before stating the main result of this paper, we will define some functions. According to [18] there exists a concave continuous, strictly increasing functions h_i $(i = 1, 2, 3)$, linear at infinity with $h_i(0) = 0$ such that

$$h_i(g_i(s)s) \geq \epsilon_0(|s|^2 + |g_i(s)|^2), \quad |s| \leq \eta, \tag{1.7}$$

for some ϵ_0, $\eta > 0$. For example when g_i is superlinear, odd and the function $s \longmapsto \sqrt{s}g_i(\sqrt{s})$ is convex, then $h_i^{-1}(s) = \sqrt{s}g_i(\sqrt{s})$ when $|s| \leq \eta$. For further information on the construction of a such function we refer the interested reader to [6, 9, 18]. With this function, we define

$$h(s) = s + h_0(s), \quad \text{where } h_0(s) = \sum_{i=1}^{3} m_a(\Omega_T) h_i\Big(\frac{s}{m_a(\Omega_T)}\Big), \tag{1.8}$$

for $s \geq 0$, $dm_a = a(x)\,dx\,dt$ and $\Omega_T = (0, T) \times \Omega$.

In this article, we show that under the assumption (GC) we obtain the following observability inequality:

Non-autonomous observability inequality: There exists a constant $T > 0$ such that the solution $u(t, x)$ to the nonlinear problem (1.1) with initial data $\varphi = (\varphi_0, \varphi_1)$ satisfies

$$E_u(t) \leq C_T h\Big(\int_t^{t+T} \int_\Omega a(x)g(\partial_t u) \cdot \partial_t u + |f(\sigma, x)|^2 \,dx\,d\sigma\Big),$$

for every $t \geq 0$.

From the observability inequality above, we infer that the behavior of the energy depends on $\|f(t, x)\|_{L^2(\Omega)}$. More precisely, we will prove that this behavior is governed by a forced differential equation and depends on

$$\Gamma(t) = 2\Big(\|f(t, .)\|^2_{L^2(\Omega)} + \psi^*(\|f(t, .)\|_{L^2(\Omega)})\Big),$$

where ψ^* is the convex conjugate of the function ψ, defined by

$$\psi(s) = \begin{cases} \frac{1}{2T} h^{-1}\big(\frac{s^2}{8C_T e^T}\big) & s \in \mathbb{R}_+, \\ +\infty & s \in \mathbb{R}_-^*, \end{cases}$$

with $C_T \geq 1$ and $T > 0$. More precisely we have the following theorem.

Theorem 1.4. *Let the function h be defined by* (1.8)*. We assume that the assumption* (GC) *holds and*

$$\Gamma(t) = 2\Big(\|f(t, .)\|^2_{L^2(\Omega)} + \psi^*(\|f(t, .)\|_{L^2(\Omega)})\Big) \in L^1_{\rm loc}(\mathbb{R}_+).$$

Let $u(t)$ be the solution to (1.1) *with initial condition $(\varphi_0, \varphi_1) \in \mathcal{H}$. Then*

$$E_u(t) \le 2e^T (S(t-T) + \int_{t-T}^t \Gamma(s) ds), \quad t \ge T, \tag{1.9}$$

where $S(t)$ is the positive solution of the ordinary differential equation

$$\frac{dS}{dt} + \frac{1}{4T} h^{-1}(\frac{1}{K} S) = \Gamma(t), \quad S(0) = E_u(0), \tag{1.10}$$

with $K \ge 2C_T$. Moreover,

- *If there exists $C > 0$, such that $\int_{t-T}^t \Gamma(\tau) d\tau \le C$, for every $t \ge T$. Then $E_u(t)$ is bounded.*
- *If $\int_{t-T}^t \Gamma(\tau) d\tau \to 0$ as $t \to +\infty$, and if $E_u(t)$ admits a limit at infinity, then the limit is zero.*
- *If $\Gamma \in L^1(\mathbb{R}_+)$, then $E_u(t) \to 0$ as $t \to +\infty$.*
- *If $\int_{t-T}^t \Gamma(\tau) d\tau \to +\infty$ as $t \to +\infty$, then $S(t) \to +\infty$ as $t \to +\infty$.*

We discuss now the methods used for establishing the main result. We note that the present work is compared to the work of [7] and [8]. Here, we follow the same program and we study the behavior of the energy for the Lamé system with Dirichlet boundary condition in a bounded domain and by adding the external force. We consider the notion of bicharacteristic path and we adapt for our context a propagation result for the microlocal defect measures attached to sequences of solutions of (1.1). We deduce then a nonlinear observability estimate which is needed to prove Theorem 1.4.

2. Proof of the main result

Before presenting the proof of our main theorem, we introduce some notation and recall some results from the literature.

Proposition 2.1. *Let u be a solution of* (1.1) *with initial data in the energy space. Then*

$$E_u(t) \le (1 + \frac{1}{\epsilon}) e^{\epsilon(t-s)} \Big(E_u(s) + \frac{1}{\epsilon} \int_s^t \int_\Omega |f(\sigma, x)|^2 \, dx \, d\sigma \Big), \tag{2.1}$$

for every $\epsilon > 0$ and for every $t \ge s \ge 0$.

Proof. Let $t \ge s \ge 0$. From the energy identity (1.6), we infer that

$$E_u(t) \le E_u(s) + \int_s^t \int_\Omega f(t, x) \cdot \partial_t u(\sigma, x) \, dx \, d\sigma.$$

Using Young's inequality, we obtain

$$E_u(t) \le E_u(s) + \frac{1}{\epsilon} \int_s^t \int_\Omega |f(\sigma, x)|^2 \, dx \, d\sigma + \epsilon \int_s^t E_u(\sigma) d\sigma,$$

for every $\epsilon > 0$. Now Gronwall's inequality gives

$$E_u(t) \le e^{\epsilon(t-s)} \Big(E_u(s) + \frac{1}{\epsilon} \int_s^t \int_\Omega |f(\sigma, x)|^2 \, dx \, d\sigma \Big).$$

$\square$

By analogy with [8, Proposition 5.1], we obtain the following result.

Proposition 2.2. *Let* (u_n) *be a bounded sequence of solutions of the linear Lamé system*

$$\begin{gathered}\partial_t^2 u_n - \Delta_e u_n = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u_n = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \\ (u_n(0,x), \partial_t u_n(0,x)) = \varphi_n(x) \quad \text{in } \Omega.\end{gathered} \tag{2.2}$$

with initial data in $\mathcal{H}$*, weakly converging to* 0 *in* $\mathcal{H}$*. We assume that* (GC) *holds and that* $\partial_t u_n \to 0$ *in* $(L^2_{\rm loc}(]0,T[\times\omega))^3$*. Then there exists a subsequence (still denoted* (u_n)*) such that* $u_n \to 0$ *in* $(H^1_{\rm loc}(]0,T[, H^1(\Omega)))^3$*.*

Before giving the proof of Proposition 2.2, we recall some facts on microlocal defect measures associated to bounded sequences of solutions to the linear Lamé system with Dirichlet boundary conditions. We give them within their original statement [8], and we note that (with obvious modifications of their proofs) all these results remain valid in our situation.

We consider the linear Lamé system on $\mathbb{R} \times \Omega$.

$$\begin{gathered}\partial_t^2 u - \Delta_e u = 0, \quad \text{in } \mathbb{R} \times \Omega, \\ u = 0 \quad \text{on } \mathbb{R} \times \partial\Omega, \\ (u(0,x), \partial_t u(0,x) = (\varphi_1(x), \varphi_2(x)) \in (H^1_0(\Omega))^3 \times (L^2(\Omega))^3.\end{gathered} \tag{2.3}$$

We decompose first the solution of system (2.3) into

$$u = u_L + u_T, \tag{2.4}$$

where the longitudinal wave u_L and the transversal wave u_T, respectively, satisfies the wave system

$$\begin{gathered}(\partial_t^2 - c_L^2\Delta) u_L = 0, \quad \operatorname{rot} u_L = 0, \\ (\partial_t^2 - c_T^2\Delta) u_T = 0, \quad \operatorname{div} u_T = 0, \\ u = u_L + u_T = 0 \quad \text{on } \mathbb{R} \times \partial\Omega,\end{gathered} \tag{2.5}$$

with $c_L = \sqrt{\lambda + 2\mu}$ and $c_T = \sqrt{\mu}$. Moreover, if $(u_n)_n$ is a bounded sequence of solutions of (2.3) weakly converging to 0 in $(H^1_{\rm loc}(\mathbb{R}_t, H^1(\Omega)))^3$, the sequences (u_{n_L}) and (u_{n_T}) are also of bounded energy and weakly converging to 0 in $(H^1_{\rm loc}(\mathbb{R}_t, H^1(\Omega)))^3$. In this way, according to [5], we can attach to (u_{n_L}) (resp. (u_{n_T})) a microlocal defect measure ν_L (resp. ν_T). These measures are orthogonal in the measure theory sense (see [5, Proposition 4.4] or [11, Lemme 3.30]). In addition, ν_L is supported in the characteristic set

$$\begin{aligned}\operatorname{Char}\mathcal{L} &= (\operatorname{Char}\mathcal{L})_\Omega \cup (\operatorname{Char}\mathcal{L})_{\partial\Omega} \\ &= \{(t,x,\tau,\xi) : x \in \Omega,\ t > 0,\ c_L^2|\xi|^2 - \tau^2 = 0\} \\ &\quad \cup \{(t,y,\tau,\eta) : y \in \partial\Omega,\ t > 0,\ r_L := \tau^2 - c_L^2|\eta|^2 \geq 0\},\end{aligned}$$

and ν_T is supported in

$$\operatorname{Char}\mathcal{T} = \{(t,x,\tau,\xi);\ x \in \Omega,\ t > 0,\ c_T^2|\xi|^2 - \tau^2 = 0\}.$$

This fact is known as the elliptic regularity theorem for the m.d.m's.

Let us now analyze the propagation properties of the measures ν_L and ν_T. In the interior, i.e. in $T^*(\mathbb{R} \times \Omega)$, we are in presence of two waves which propagate independently, so we have at our disposal the classical measures propagation theorem of [13]. Near the boundary $\partial\Omega$, we have to take into account, the nature of the bicharacteristics hitting $\partial\Omega$.

Take ρ in $\operatorname{Char} P_{\partial\Omega} = \{(t,y,\tau,\eta);\, y \in \partial\Omega,\, t > 0,\, r_T := \tau^2 - c_T^2|\eta|^2 \geq 0\}$; for $r_{L,T} = r_{L,T}(\rho) \geq 0$, we denote $\gamma^-_{L,T}$ (resp. $\gamma^+_{L,T}$) the (longitudinal/transversal) incoming (resp. outgoing) bicharacteristic to (resp. from) ρ (this half bicharacteristic does not contain ρ). Following then word by word the argument developed in [5, proof of Theorem 4], we have

Proposition 2.3. *With the notation above, we have*

1. $r_L < 0$, *ρ is an elliptic point for the longitudinal wave. Hence, $\nu_L = 0$ near ρ and*
   (a) $\nu_T = 0$ *near ρ if $r_T < 0$,*
   (b) ν_T *propagates from γ^-_T to γ^+_T if $0 \leq r_T$.*
2. $0 < r_L \leq r_T$, *ρ is a hyperbolic point for the longitudinal and the transversal wave. In this case, we obtain: If $\gamma^-_{L,T} \cap \text{support}(\nu_{L,T}) = \emptyset$, then $\nu_{T,L}$ propagates from $\gamma^-_{T,L}$ to $\gamma^+_{T,L}$.*
3. $0 = r_L < r_T$, *ρ is a glancing point for the longitudinal wave. Here we have: If $\gamma^-_L \cap \text{support}\ (\nu_L) = \emptyset$, then ν_T propagates from γ^-_T to γ^+_T.*

As a consequence, using the conservation of the total mass (see [5]), we obtain the following result.

Corollary 2.4. *For $0 \leq r_L$, we have the following equivalence:*

$$(\gamma^-_L \cap \operatorname{support}(\nu_L)) \cup (\gamma^-_T \cap \operatorname{support}(\nu_T)) = \emptyset$$

if and only if

$$(\gamma^+_L \cap \operatorname{support}(\nu_L)) \cup (\gamma^+_T \cap \operatorname{support}(\nu_T)) = \emptyset\,.$$

Proof of Proposition 2.2. Under the decomposition (2.4), it suffices to prove that $u_{n,L,T} \to 0$ in $(H^1_{\rm loc}(]0,T[, H^1(\Omega)))^3$, and thanks to the orthogonality property of the measures ν_L and ν_T and the elliptic regularity theorem, we have $\partial_t u_{n,L,T} \to 0$ in $(L^2_{\rm loc}(]0,T[\times\omega))^3$ and then $\nu_L = \nu_T$ on $]0,T[\times\omega$. Therefore, to prove Proposition (2.2), we have to establish the following implication:

$$\nu_L = \nu_T = 0 \quad \text{on }]0,T[\times\omega \Rightarrow \nu_L = \nu_T = 0 \quad \text{on }]0,T[\times\Omega.$$

We argue by contradiction. Let (u_n) be a bounded sequence of solutions of (2.2) with initial data in $\mathcal{H}$, and $\nu_{L,T}$ the microlocal defect measure associated to $(u_{n,L,T})$. Let $q \in T^*(]0,T[\times\Omega)$ such that $q \in \operatorname{support}(\nu_L) \cup \operatorname{support}(\nu_T)$ and γ a generalized bicaracteristic path starting at q. The geometric assumption saying that any straight line in Ω has only finite order contacts with $\partial\Omega$, we may assume that q is an interior point.

In this way one can find a bicharacteristic γ (γ_L or γ_T) issued from q and traced backward in time, contained in the support of the associated measure (i.e $\gamma_L \subset \operatorname{support}(\nu_L)$ or $\gamma_T \subset \operatorname{support}(\nu_T)$). As γ hits the boundary $\partial\Omega$, we have two possibilities:

(a) γ hits $\partial\Omega$, for the first time, in some point ρ such that $r_L(\rho) < 0$.
(b) γ hits $\partial\Omega$, for the first time, in some point ρ such that $0 \leq r_L(\rho)$.

In the first case, we are near an elliptic point for the longitudinal wave, so the measure is carried by the transversal component and propagates along the reflected bicharacteristic. In the second case, thanks to Proposition 2.3 and Corollary 2.4, one of the two incoming bicharacteristics γ^-_L or γ^-_T at ρ is, locally, in $\operatorname{support}(\nu_L)$ or in $\operatorname{support}(\nu_T)$. Thus, we can construct a bicharacteristic path Γ issued from q

(the union of all these successive rays γ_L or γ_T charged by the measure ν_L or ν_T) contained in support(ν_L)$\cup$ support(ν_T). According to assumption (GC) Γ meets $]0,T[\times\omega$ at $t_0 < T$, and this contradicts the fact that $\Gamma \subset$support(ν_L)$\cup$support(ν_T), since $\nu_L = \nu_T = 0 \quad$ on $]0,T[\times\omega$. The proof of Proposition 2.2 is complete. □

Now, we prove the observability estimate which constitute with the lemma 2.7 below the main ingredient of the proof of Theorem 1.4.

Proposition 2.5. *Let the function h be defined by* (1.8). *We assume that the assumption* (GC) *holds. Then there exists $C_T > 0$, such that the following inequality holds:*

$$E_u(t) \le C_T h\Big(\int_t^{t+T}\int_\Omega a(x)g(\partial_t u)\cdot\partial_t u + |f(\sigma,x)|^2\,dx\,d\sigma\Big), \tag{2.6}$$

for every $t \ge 0$,for every solution u of (1.1) *with initial data in the energy space $\mathcal{H}$, and for every $f$in $(L^2_{\rm loc}(\mathbb{R}_+, L^2(\Omega)))^3$.*

Proof. To prove this result we argue by contradiction. We assume that there exist a sequence $(u_n)_n$ solution of (1.1) with initial data in the energy space, a non-negative sequence $(t_n)_n$and f_n in $(L^2_{\rm loc}(\mathbb{R}_+, L^2(\Omega)))^3$, such that

$$E_{u_n}(t_n) \ge nh\Big(\int_{t_n}^{t_n+T}\int_\Omega a(x)g(\partial_t u_n)\cdot\partial_t u_n + |f_n(\sigma,x)|^2\,dx\,d\sigma\Big).$$

Moreover, u_n has the following regularity:

$$u_n \in C\big(\mathbb{R}_+, (H_0^1(\Omega))^3\big)\cap C^1\big(\mathbb{R}_+, (L^2(\Omega))^3\big).$$

Setting $\alpha_n = (E_{u_n}(t_n))^{1/2} > 0$ and $v_n(t,x) = \frac{u_n(t_n+t,x)}{\alpha_n}$. Then v_n satisfies

$$\begin{gathered}\partial_t^2 v_n - \Delta_e v_n + \frac{1}{\alpha_n}a(x)g(\alpha_n\partial_t v_n) = \frac{1}{\alpha_n}f_n(t_n+t,x), \quad \text{in } \mathbb{R}_+\times\Omega,\\ v_n = 0 \quad \text{on } \mathbb{R}_+\times\partial\Omega,\\ (v_n(0,x), \quad \partial_t v_n(0,x)) = \frac{1}{\alpha_n}(u_n(t_n,x), \partial_t u_n(t_n,x)), \quad \text{in } \Omega.\end{gathered} \tag{2.7}$$

Moreover $E_{v_n}(0) = 1$ and

$$1 \ge \frac{n}{\alpha_n^2}h\Big(\int_0^T\int_\Omega a(x)g(\alpha_n\partial_t v_n)\cdot\alpha_n\partial_t v_n + |f_n(t_n+t,x)|^2\,dx\,dt\Big).$$

Since $h = I + h_0$ and h_0 is non-negative and increasing function and from the inequality above, we infer that

$$\int_0^T\int_\Omega |\frac{1}{\alpha_n}f_n(t_n+t,x)|^2\,dx\,dt \le \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0 \tag{2.8}$$

and

$$\Big[I + \sum_{i=1}^3 m_a(\Omega_T)h_i\circ\frac{I}{m_a(\Omega_T)}\Big]\Big(\int_0^T\int_\Omega a(x)g(\alpha_n\partial_t v_n)\cdot\alpha_n\partial_t v_n\,dx\,dt\Big) \le \frac{\alpha_n^2}{n}. \tag{2.9}$$

Re-using the fact that the function h_0 is non-negative gives

$$\alpha_n^{-1}\int_0^T\int_\Omega a(x)g(\alpha_n\partial_t v_n)\cdot\partial_t v_n\,dx\,dt \underset{n\to+\infty}{\longrightarrow} 0 \tag{2.10}$$

and

$$h_i\Big(\frac{1}{m_a(\Omega_T)}\int_0^T\int_\Omega a(x)g_i(\alpha_n\partial_t v_n)\alpha_n(\partial_t v_n)_i\,dx\,dt\Big)\le\frac{\alpha_n^2}{nm_a(\Omega_T)},\quad i=1,2,3. \tag{2.11}$$

Denote $\Omega_{1,i}=\{(t,x)\in[0,T]\times\Omega:|\alpha_n(\partial_t v_n)_i(t,x)|<\mu\}$ and $\Omega_{2,i}=\Omega_T\backslash\Omega_{1,i}$.

Since g_i has a linear behavior on $\{|s|\ge\eta\}$, using (2.10), we infer that

$$\|a(x)(\partial_t v_n)_i\|^2_{L^2(\Omega_{2,i})}\le c_1\alpha_n^{-1}\int_0^T\int_\Omega a(x)g(\alpha_n\partial_t v_n)\cdot\partial_t v_n\,dx\,d\tau\underset{n\to+\infty}{\longrightarrow}0. \tag{2.12}$$

Moreover, h_i is concave, then using (the reverse) Jensen's inequality

$$\begin{aligned}&h_i\Big(\frac{1}{\mathfrak{m}_a(\Omega_T)}\int_0^T\int_\Omega a(x)g_i(\alpha_n\partial_t v_n)\alpha_n(\partial_t v_n)_i\,dx\,d\tau\Big)\\&\ge\frac{1}{\mathfrak{m}_a(\Omega_T)}\int_{\Omega_T}h_i(g_i(\alpha_n\partial_t v_n)\alpha_n(\partial_t v_n)_i)d\mathfrak{m}_a,\end{aligned}$$

which gives

$$\alpha_n^{-2}\int_{\Omega_{1,i}}h_i(g_i(\alpha_n\partial_t v_n)\alpha_n(\partial_t v_n)_i)d\mathfrak{m}_a\le\frac{1}{n}.$$

Therefore, from (1.7) we obtain

$$\int_{\Omega_{1,i}}a(x)[\alpha_n^{-2}|g_i(\alpha_n(\partial_t v_n)_i)|^2+|(\partial_t v_n)_i|^2]\,dx\,dt\underset{n\to+\infty}{\longrightarrow}0.$$

Combining the estimate above with (2.12) we obtain

$$\|a(x)\partial_t v_n\|_{(L^2(\Omega_T))^3}\underset{n\to+\infty}{\longrightarrow}0 \tag{2.13}$$

and we conclude that

$$\|\frac{1}{\alpha_n}a(x)g(\alpha_n\partial_t v_n)\|_{(L^2(\Omega_T))^3}\underset{n\to+\infty}{\longrightarrow}0. \tag{2.14}$$

Hence, passing to the limit in (2.7), we see that the weak limit $v\in(H^1([0,T]\times\Omega))^3$ satisfies the system

$$\begin{gathered}\partial_t^2 v-\Delta_e v=0\quad\text{in }]0,T[\times\Omega,\\ v=0\quad\text{on }]0,T[\times\Omega,\\ (v(0,x),\partial_t v(0,x))=\psi(x),\quad\text{in }\Omega\,.\end{gathered} \tag{2.15}$$

Moreover, we obtain

$$a(x)\partial_t v=0,\quad\text{on }\Omega_T. \tag{2.16}$$

Now, let w_n be the solution of the system

$$\begin{gathered}\partial_t^2 w_n-\Delta_e w_n=0,\quad\text{in }\mathbb{R}_+\times\Omega,\\ w_n=0,\quad\text{on }\mathbb{R}_+\times\Omega,\\ (w_n(0,x),\quad\partial_t w_n(0,x))=\frac{1}{\alpha_n}(u_n(t_n,x),\partial_t u_n(t_n,x)),\quad\text{in }\mathbb{R}_+\times\Omega.\end{gathered} \tag{2.17}$$

It is clear that the sequence $(w_n)_n$ is bounded in $(H^1_{\rm loc}([0,T]\times\Omega))^3$; moreover, by the hyperbolic energy inequality, (2.8) and (2.14) we infer that

$$\sup_{0\le t\le T}E_{v_n-w_n}(t)\le C(T)\|\frac{1}{\alpha_n}a(x)g(\partial_t v_n)-\frac{1}{\alpha_n}f_n(t_n+t,x)\|^2_{L^2(\Omega_T)}\underset{n\to+\infty}{\longrightarrow}0. \tag{2.18}$$

Consequently, thanks to (2.13), we deduce that

$$\|a(x)\partial_t w_n\|_{(L^2(\Omega_T))^3} \underset{n\to+\infty}{\to} 0, \tag{2.19}$$

to obtain a contradiction we use the following result for which we postpone its proof.

Proposition 2.6. *We assume that the assumption* (GC) *holds. Then there exists* $\alpha_T > 0$*, such that the inequality*

$$E_w(0) \le \alpha_T \Big(\int_0^T \int_\omega |\partial_t w|^2 \,dx\,ds \Big) \tag{2.20}$$

holds for every solution w of

$$\begin{gathered} \partial_t^2 w - \Delta_e w = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ w = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \\ (w(0,x), \partial_t w(0,x)) = (w_0(x), w_1(x)), \quad \text{in } \Omega \end{gathered} \tag{2.21}$$

with initial data in the energy space $\mathcal{H}$.

Now, using (2.19) and Proposition 2.6, we obtain

$$1 = E_{v_n}(0) = E_{w_n}(0) \le \alpha_T \int_0^T \int_\omega |\partial_t w_n|^2 \,dx\,dt \underset{n\to+\infty}{\longrightarrow} 0\,,$$

and this concludes the Proof of Proposition 2.5. □

Proof of Proposition 2.6. We argue by contradiction: we suppose the existence of a sequence (w_n), solutions of (2.21) such that

$$\int_0^T \int_\omega |\partial_t w_n|^2 \,dx\,dt \le \frac{E_{w_n}(0)}{n}.$$

Denote $\alpha_n = E_{w_n}(0)^{1/2}$ and $z_n = \frac{w_n}{\alpha_n}$. Moreover z_n satisfies

$$\begin{gathered} \partial_t^2 z_n - \Delta_e z_n = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ z_n = 0, \quad \text{in } \mathbb{R}_+ \times \partial\Omega, \\ E_{z_n}(0) = 1, \quad \int_0^T \int_\omega |\partial_t z_n|^2 \,dx\,dt \le \frac{1}{n}. \end{gathered} \tag{2.22}$$

The sequence z_n is bounded in $C^0([0,T],(H^1(\Omega))^3) \cap C^1([0,T],(L^2(\Omega))^3)$, then, it admits a subsequence, still denoted by z_n, that is weakly-* convergent in the space $L^\infty([0,T],(H^1(\Omega))^3) \cap W^{1,\infty}((0,T),(L^2(\Omega))^3)$. In this way, $z_n \rightharpoonup z$ in $(H^1([0,T]\times\Omega))^3$. Passing to the limit in the equation satisfied by z_n we obtain

$$\begin{gathered} \partial_t^2 z - \Delta_e z = 0, \quad \text{in }]0,T[\times\Omega, \\ z = 0 \quad \text{in }]0,T[\times\partial\Omega, \\ \partial_t z = 0 \quad \text{on }]0,T[\times\omega. \end{gathered} \tag{2.23}$$

We need to check that the trivial solution, $v = 0$, is the only solution of (2.23) in $C^0([0,T],(H^1(\Omega))^3) \cap C^1([0,T],(L^2(\Omega))^3)$. For this, we identify the function z solution of (2.23) with its initial data $\phi \in \mathcal{H}$, and we consider the space $G = \{\phi \in \mathcal{H}, z \text{ is a solution of } (2.23)\}$.

Every z in G is smooth on $]0,T[\times\omega$; therefore, according to the geometric control condition and the result of [21] on propagation of singularities, G is constituted of smooth functions. Moreover, G is obviously closed in $\mathcal{H}$, and we deduce that it is of finite dimension. On the other hand, $\partial/\partial t$ operates on G, so it admits an eigenvalue λ, and there exists a nonzero function $z_0(x)$ on Ω such that $\Delta_e z_0 = \lambda z_0$, $z_0 \equiv 0$ on ω, $z_0 = 0$ on $\partial\Omega$; and this is impossible by unique continuation property of Δ_e (see, for instance, [10]).

Now, we multiply $E_{z_n}(s)$ by $\varphi(s)$, with $\varphi \in C_0^\infty(]0,T[)$, $\varphi = 1$ on $]\varepsilon, T-\varepsilon[$, $\varphi \geq 0$, and we integrate. This gives

$$
\begin{aligned}
&\int_0^T \varphi(s) E_{z_n}(s) ds \\
&= \frac{1}{2}\int_0^T \int_\Omega (\mu\varphi(s)|\nabla z_n|^2 + (\lambda+\mu)\varphi(s)|\operatorname{div} z_n|^2 + \varphi(s)|\partial_t z_n|^2)(s,x)\, dx\, ds.
\end{aligned}
$$

Proposition 2.2 and (2.22) imply that the second member approaches 0 as $n \to +\infty$. Using the fact that $E_{z_n}(s) = 1$, we obtain $T - 2\varepsilon \to 0$ as $n \to +\infty$ and this gives a contradiction. ☐

We recall now the following lemma due to [7] which is useful to determine the behavior of the energy.

Lemma 2.7. *Let $T > 0$ and*

- *$\Gamma \in L^1_{\rm loc}(\mathbb{R}_+)$ and non-negative. Setting $\delta(t) = \int_t^{t+T} \Gamma(s)ds$, for $t \geq 0$.*
- *$W(t)$ be a non-negative function for $t \in \mathbb{R}_+$. Moreover we assume that there exists a positive, monotone, increasing function α with $\alpha(0) \geq 1$, such that*

$$
W(t) \leq \alpha(t-s)\Big[W(s) + \int_s^t \Gamma(\sigma)d\sigma\Big], \quad \text{for every } t \geq s \geq 0.
$$

- *Suppose that ℓ and $I - \ell : \mathbb{R}_+ \to \mathbb{R}$ are increasing functions with $\ell(0) = 0$ and*

$$
W((m+1)T) + \ell\{W(mT) + \delta(mT)\} \leq W(mT) + \delta(mT), \tag{2.24}
$$

for $m = 0, 1, 2, \dots$, where $\ell(s)$ does not depend on m.

Then

$$
W(t) \leq \alpha(T)\Big(S(t-T) + \int_{t-T}^t \Gamma(s)ds\Big), \quad \forall t \geq T,
$$

where $S(t)$ is the non negative solution of the differential equation

$$
\frac{dS}{dt} + \frac{1}{T}\ell(S) = \Gamma(t); \quad S(0) = W(0). \tag{2.25}
$$

Moreover, we assume that ℓ is continuous, strictly increasing and $\lim_{s\to+\infty} \ell(s) = +\infty$

- *If there exists $C > 0$, such that $\int_{t-T}^t \Gamma(\tau)d\tau \leq C$, for every $t \geq T$. Then $S(t)$ is bounded.*
- *If $\int_{t-T}^t \Gamma(\tau)d\tau \to 0$ as $t \to +\infty$, and if $S(t)$ admits a limit at infinity, then this limit is zero.*
- *If $\Gamma \in L^1(\mathbb{R}_+)$, then $S(t) \to 0$ as $t \to +\infty$.*
- *We assume that $\lim_{s\to+\infty}(I-\ell)(s) = +\infty$, then if $\int_{t-T}^t \Gamma(\tau)d\tau \to +\infty$ as $t \to +\infty$, we have $S(t) \to +\infty$ as $t \to +\infty$.*

We can now proceed the proof of the main result of this article.

Proof of Theorem 1.4. We assume that the assumption (GC) holds. Let u be a solution of (1.1) with initial data in the energy space. Then according to Proposition 2.5, we have

$$E_u(t) \le C_T h\Big(\int_t^{t+T}\int_\Omega a(x)g(\partial_t u)\cdot\partial_t u\,dx\,d\sigma + \int_t^{t+T}\int_\Omega |f(s,x)|^2\,dx\,ds\Big), \tag{2.26}$$

for some $C_T \ge 1$. The energy identity (1.6) gives

$$\int_t^{t+T}\int_\Omega a(x)g(\partial_t u)\cdot\partial_t u\,dx\,d\sigma \le E_u(t) - E_u(t+T) + \int_t^{t+T}\int_\Omega |f(\sigma,x)\cdot\partial_t u|\,dx\,d\sigma. \tag{2.27}$$

Let ψ be defined by

$$\psi(s) = \begin{cases} \frac{1}{2T}h^{-1}(\frac{s^2}{8C_T e^T}) & s \in \mathbb{R}_+,\\ +\infty & s \in \mathbb{R}_-^*. \end{cases}$$

It is clear that ψ convex is and proper function. Hence, we can apply Young's inequality [20]

$$\begin{aligned}\int_t^{t+T}\int_\Omega |f(\sigma,x)\cdot\partial_t u|\,dx\,d\sigma &\le \int_t^{t+T}\|f(\sigma,.)\|_{L^2}\|\partial_t u(\sigma,.)\|_{L^2}d\sigma\\ &\le \int_t^{t+T}\psi^*(\|f(\sigma,.)\|_{L^2}) + \psi(\|\partial_t u(\sigma,.)\|_{L^2})d\sigma,\end{aligned}$$

where ψ^* is the convex conjugate of the function ψ, defined by $\psi^*(s) = \sup_{y\in\mathbb{R}}[sy - \psi(y)]$

Using the energy inequality (2.1) and the observability estimate (2.26), we infer that

$$\int_t^{t+T}\psi(\|\partial_t u(\sigma,.)\|_{L^2})d\sigma \le \frac{1}{2}\Big(\int_t^{t+T}\int_\Omega g(\partial_s u)\cdot\partial_s u d\mathfrak{m}_a + \int_t^{t+T}\int_\Omega |f(s,x)|^2\,dx\,ds\Big)$$

then (2.27) gives

$$\begin{aligned}&\int_t^{t+T}\int_\Omega a(x)g(\partial_t u)\cdot\partial_t u\,dx\,d\sigma\\ &\le 2\Big(E_u(t) - E_u(t+T) + \int_t^{t+T}\int_\Omega |f(s,x)|^2\,dx\,ds + \int_t^{t+T}\psi^*(\|f(\sigma,.)\|_{L^2})d\sigma\Big).\end{aligned}$$

The inequality above combined with the observability estimate (2.26) and the fact $h = I + \mathfrak{m}_a(\Omega_T)h_0 \circ \frac{I}{\mathfrak{m}_a(\Omega_T)}$ is increasing, gives

$$E_u(t) \le C_T h\Big(4\Big(E_u(t) - E_u(t+T) + 2\int_t^{t+T}\|f(\sigma,.)\|_{L^2}^2 + \psi^*(\|f(\sigma,.)\|_{L^2})d\sigma\Big)\Big).$$

Setting

$$\Gamma(s) = 2(\|f(\sigma,.)\|_{L^2}^2 + \psi^*(\|f(s,.)\|_{L^2})).$$

Therefore,

$$E_u(t) + \int_t^{t+T}\Gamma(s)ds \le Kh\Big(4\Big(E_u(t) - E_u(t+T) + \int_t^{t+T}\Gamma(s)\,dx\,ds\Big)\Big),$$

with $K \ge 2C_T$. Setting $\theta(t) = \int_t^{t+T}\Gamma(s)ds$. Thus

$$E_u(t+T) + \frac{1}{4}h^{-1}\Big(\frac{1}{K}(E_u(t) + \theta(t))\Big) \le E_u(t) + \theta(t), \tag{2.28}$$

for every $t \geq 0$. Take $t = mt$, $m \in \mathbb{N}$,

$$E_u((m+1)T) + \frac{1}{4}h^{-1}\Big(\frac{1}{K}(E_u(mT) + \theta(mT))\Big) \leq E_u(mT) + \theta(mT).$$

Setting $W(t) = E_u(t)$, $\ell(s) = \frac{1}{4}h^{-1} \circ \frac{I}{K}$ and

$$\Gamma(s) = 2(\|f(s,.)\|_{L^2}^2 + \psi^*(\|f(s,.)\|_{L^2})).$$

It is clear that the functions ℓ and $I - \ell$ are increasing on the positive axis and $\ell(0) = 0$. The function $\Gamma \in L^1_{\rm loc}(\mathbb{R}_+)$ and non-negative on $\mathbb{R}_+$. According to lemma 2.7, we obtain

$$E_u(t) \leq 2e^T\Big(S(t-T) + \int_{t-T}^t \Gamma(s)ds\Big), \quad \forall t \geq T,$$

where $S(t)$ is the solution of the following differential equation

$$\frac{dS}{dt} + \frac{1}{T}\ell(S) = \Gamma(t), \quad S(0) = W(0).$$

The function ℓ is continuous, strictly increasing and $\lim_{s\to+\infty}\ell(s) = +\infty$, therefore using Lemma 2.7, we infer that

- If there exists $C > 0$, such that $\int_{t-T}^t \Gamma(\tau)d\tau \leq C$ for every $t \geq T$. Then $S(t)$ is bounded, which gives $E_u(t)$ is bounded.
- We assume that $E_u(t) \to \alpha \geq 0$ as $t \to +\infty$ and $\int_{t-T}^t \Gamma(\tau)d\tau \to 0$ as $t \to +\infty$. Consequently (2.28) gives
$$E_u(t) + \ell\Big(E_u(t-T) + \int_{t-T}^t \Gamma(\tau)d\tau\Big) \leq E_u(t-T) + \int_{t-T}^t \Gamma(\tau)d\tau, \tag{2.29}$$
for every $t \geq T$. Passing to the limit in the inequality above, we infer that $\ell(\alpha) = 0$. Which means $\alpha = 0$. Therefore, if $E_u(t)$ admits a limit at infinity, then the limit is zero.
- If $\Gamma \in L^1(\mathbb{R}_+)$, then $S(t) \to 0$ as $t \to +\infty$, which gives $E_u(t) \to 0$ as $t \to +\infty$.
- Since h^{-1} is linear at infinity, therefore $(I - \ell)$ is positive and linear at infinity, which gives $\lim_{s\to+\infty}(I-\ell)(s) = +\infty$. Thus, if $\int_{t-T}^t \Gamma(\tau)d\tau \to +\infty$ as $t \to +\infty$, we obtain $S(t) \to +\infty$ as $t \to +\infty$.

□

3. Applications

Preliminary results. In the following proposition we give a result on the behavior of the solutions of (1.10) due to [7].

Proposition 3.1. *Let p a differentiable, strictly increasing function on $\mathbb{R}_+$ with $p(0) = 0$. We assume that there exists $m_1 > 0$ such that, $p(x) \leq m_1 x$ for every $x \in [0, \eta]$ for some $0 < \eta << 1$ and that the property*

$$p(Kx) \geq mp(K)p(x), \tag{3.1}$$

holds, for some $m > 0$ and for every $(K, x) \in [1, +\infty[\times\mathbb{R}_+$.

We suppose that $\Gamma \in C^1(\mathbb{R}_+)$ and non-negative.

(1) Let $\tilde{p}$ be a increasing function vanishing at the origin. Let S satisfy the differential equation

$$\frac{dS}{dt} + \tilde{p}(S) = \Gamma(t), \quad S(0) \geq 0. \tag{3.2}$$

Then $S(t) \geq 0$ for every $t \geq 0$.

(2) Let S be a non-negative function, satisfying the differential inequality

$$\frac{dS}{dt} + p(S) \leq \Gamma(t), \quad S(0) \geq 0.$$

(a) *If $\Gamma(t) = 0$, for every $t \geq 0$, then $S(t) \leq \psi^{-1}(t)$, for every $t \geq 0$ where $\psi(x) = \int_x^{S(0)} \frac{ds}{p(s)}$, $x \in]0, S(0)]$.*

(b) *If $\Gamma(t) > 0$, for every $t \geq 0$, and*

(i) *There exist $c > 0$ and $\kappa \geq 1$ such that*

$$\frac{d}{dt} p^{-1}(\Gamma(t)) + c\Gamma(t) \leq 0, \text{ for every } t \geq 0, \tag{3.3}$$

$$mp(\kappa) - \kappa c - 1 \geq 0, \quad \kappa p^{-1} \circ \Gamma(0) \geq S(0), \tag{3.4}$$

then $S(t) \leq \kappa \psi^{-1}(ct)$ for every $t \geq 0$, where

$$\psi(x) = \int_x^{p^{-1} \circ \Gamma(0)} \frac{ds}{p(s)}, \quad x \in]0, p^{-1} \circ \Gamma(0)].$$

Noting that in this case we have $p^{-1} \circ \Gamma(t) \leq \psi^{-1}(ct)$, for every $t \geq 0$.

(ii) *There exist $c > 0$ and $\kappa \geq 1$ such that $\frac{d}{dt} p^{-1}(\Gamma(t)) + c\Gamma(t) \geq 0$, for every $t \geq 0$ and*

$$mp(\kappa) - c\kappa - 1 \geq 0, \quad \kappa p^{-1} \circ \Gamma(0) \geq S(0),$$

then $S(t) \leq \kappa p^{-1} \circ \Gamma(t)$, for every $t \geq 0$. Noting that in this case we have $p^{-1} \circ \Gamma(t) \geq \psi^{-1}(ct)$ for every $t \geq 0$, where

$$\psi(x) = \int_x^{p^{-1} \circ \Gamma(0)} \frac{ds}{p(s)}, \quad x \in]0, p^{-1} \circ \Gamma(0)].$$

Examples. Setting

$$\Gamma(t) = 2\Big(\|f(t,.)\|^2_{L^2(\Omega)} + \psi^*(\|f(t,.)\|_{L^2(\Omega)}) \Big),$$

where ψ^* is the convex conjugate of the function ψ, defined by

$$\psi(s) = \begin{cases} \frac{1}{2T} h^{-1}(\frac{s^2}{8C_T e^T}) & s \in \mathbb{R}_+ \\ +\infty & s \in \mathbb{R}^*_-, \end{cases}$$

and $\psi^*(s) = \sup_{y \in \mathbb{R}} [sy - \psi(y)]$. To obtain the rate of decay, we use proposition 3.1.

g_i is linearly bounded. We have $h(s) = 2s$, then

$$\psi^*\Big(\|f(t,.)\|_{L^2(M)} \Big) \leq C_1 \|f(t,.)\|^2_{L^2(M)},$$

for some $C_1 > 0$. The ODE (1.10) governing the energy bound reduces to

$$\frac{dS}{dt} + CS = \Gamma(t), \tag{3.5}$$

where the constant $C > 0$ and does not depend on $E_u(0)$.

(1) If there are constants $C_0 > 0$ and $\theta \in \mathbb{R}$, such that $\Gamma(t) \le C_0 e^{-\theta t}$. We have

$$\int_{t-T}^{t} e^{-\theta s} ds \le \begin{cases} |\frac{1}{\theta}|[e^{|\theta| T} - 1]e^{-\theta t} & \theta \ne 0 \\ T & \theta = 0 \end{cases}$$

for $t \ge T$.

Multiply both sides of (3.5) by $\exp(Ct)$ and integrate from 0 to t, to obtain

(a) $C > \theta$, $E_u(t) \le c(1 + E_u(0))e^{-\theta t}$ for $t \ge 0$,
(b) $C = \theta$, $E_u(t) \le c(1 + E_u(0))(1 + t)e^{-\theta t}$ for $t \ge 0$,
(c) $C < \theta$, $E_u(t) \le c(1 + E_u(0))e^{-Ct}$ for $t \ge 0$.

(2) If there are constants $C_0 > 0$ and $\theta \in \mathbb{R}$, such that $\Gamma(t) \le C_0(1 + t)^{-\theta}$, then we have

$$\int_{t-T}^{t} (1 + s)^{-\theta} ds \le \begin{cases} T(1 + t - T)^{-\theta} & \theta > 0 \\ T(1 + t)^{-\theta} & \theta \le 0 \end{cases}$$

for $t \ge T$. Therefore,

$$E_u(t) \le \begin{cases} c(1 + E_u(0))(1 + t - T)^{-\theta} & \theta > 0 \\ c(1 + E_u(0))T(1 + t)^{-\theta} & \theta \le 0 \end{cases} \tag{3.6}$$

for $t \ge T$, where $c > 0$.

The nonlinear case. The rate of decay of the energy depends only on the behavior of h^{-1} near zero. To determine it, we have only to find $0 < N_0 \le 1$, such that

$$C_1 h_i^{-1}\big(\frac{s}{2C_2}\big) \le h^{-1}(s) \quad \text{for every } 0 \le s \le N_0,$$

where $C_1 = \min(\mathfrak{m}_a(\Omega_T), 1)$ and $C_2 = \max(\mathfrak{m}_a(\Omega_T), 1)$.

(1) If $\Gamma \in L^1(\mathbb{R}_+)$, we choose $K \ge \max(C_T, \frac{E_u(0)+\|\Gamma\|_{L^1(\mathbb{R}_+)}}{N_0})$. Equation (1.10) governing the energy bound reduces to

$$\frac{dS}{dt} + C_1 h_i^{-1}\big(\frac{S}{2KC_2}\big) \le \Gamma(t) \quad \text{on } [0, +\infty[,$$

with $S(0) = E_u(0)$.

(2) If $\Gamma \in L^1_{\rm loc}(\mathbb{R}_+)$ and

$$\int_{t-T}^{t} \Gamma(\tau) d\tau \le C \quad \text{for every } t \ge T,$$

then $S(t)$ is bounded and therefore there exists $A > 0$ such that $S(t) \le A$, for every $t \ge 0$. We choose $K \ge \max(C_T, \frac{A}{N_0})$. The ODE (1.10) governing the energy bound reduces to

$$\frac{dS}{dt} + C_1 h_i^{-1}\big(\frac{S}{2KC_2}\big) \le \Gamma(t),$$

with $S(0) = E_u(0)$.

(3) If $\Gamma \in L^1_{\rm loc}(\mathbb{R}_+)$ and

$$\int_{t-T}^{t} \Gamma(\tau) d\tau \underset{t \to +\infty}{\longrightarrow} +\infty,$$

then $S(t) \to +\infty$ as $t \to +\infty$. Therefore, there exists $t_0 > 0$ such that $\frac{S(t)}{K} >> 1$ for $t \geq t_0$. Since the function h is strictly increasing and linear at infinity, then the ODE (1.10) governing the energy bound reduces to

$$\frac{dS}{dt} + \frac{C}{K} S \leq \Gamma(t) \quad \text{on } [t_0, +\infty[,$$

with $S(t_0) \leq E_u(0) + \int_0^{t_0} \Gamma(s) ds$.

Example 1: Sublinear near the origin. Assume $g_i(s) = s|s|^{r_0 - 1}$, $|s| < 1$, $r_0 \in (0,1)$. We choose $h_i^{-1}(s) = \sqrt{s} g_i^{-1}(\sqrt{s}) = s^{\frac{1+r_0}{2r_0}}$, for $0 \leq s \leq 1$. We have

$$\psi^*\Big(\|f(t,.)\|_{L^2(\Omega)}\Big) \leq \tilde{C}\Big(\|f(t,.)\|_{L^2(\Omega)}^{r_0+1} + \|f(t,.)\|_{L^2(\Omega)}^2\Big),$$

for some $\tilde{C} > 0$. The ODE (1.10) governing the energy bound reduces to

$$\frac{dS}{dt} + C S^{(1+r_0)/2r_0} \leq \Gamma(t),$$

where C is positive and depends on K.

(1) If there are constants $C_0 > 0$ and $\theta > 0$ such that $\Gamma(t) \leq C_0 (1+t)^{-\theta}$, then

(1) $\theta \in]0, \frac{1+r_0}{1-r_0}]$ implies

$$E_u(t) \leq c(1+t-T)^{-\frac{2r_0\theta}{1+r_0}}, \ t \geq T,$$

where $c > 0$.

(2) $\theta \geq \frac{1+r_0}{1-r_0}$ implies

$$E_u(t) \leq c(1+t-T)^{-\frac{2r_0}{1-r_0}}, \ t > T,$$

with $c > 0$ and depend on $E_u(0)$.

(2) If there are constants $C_0 > 0$ and $\theta > 0$, such that $\Gamma(t) \leq C_0 e^{-\theta t}$, then

$$E_u(t) \leq c(t-T+1)^{-\frac{2r_0}{1-r_0}}, \quad t > T,$$

where c is positive and depends on $E_u(0)$.

Example 2: Different behavior. Assume

$$g_1(s) = \begin{cases} s^2 e^{-1/s^2} & 0 \leq s < 1 \\ -s^2 e^{-1/s^2} & -1 < s < 0 \end{cases}$$

$$g_2(s) = s|s|^{r-1}, \quad |s| < 1, \ r > 1$$

$$g_3(s) = s|s|^{r_0-1}, \quad |s| < 1, \ r_0 \in (0,1).$$

We choose

$$h_1^{-1}(s) = \sqrt{s} g_1(\sqrt{s}) = s^{3/2} e^{-1/s}, \quad 0 < s < \eta << 1,$$

$$h_2^{-1}(s) = \sqrt{s} g_2(\sqrt{s}) = s^{\frac{1+r}{2}}, \quad 0 \leq s \leq \eta,$$

$$h_3^{-1}(s) = \sqrt{s} g_3^{-1}(\sqrt{s}) = s^{\frac{1+r_0}{2r_0}}, \quad 0 \leq s \leq \eta.$$

We have

$$\psi^*(s) \leq \tilde{C}(s|\ln(s)|^{-1/2} + s^{\frac{r+1}{r}} + s^{\frac{r_0-1}{r_0+1}} + s^2),$$

for some $\tilde{C} > 0$ and $s > 0$. The ODE (1.10) governing the energy bound reduces to

$$\frac{dS}{dt} + C S^{3/2} e^{-\frac{1}{CS}} \leq \Gamma(t),$$

where C is positive and depends on K. If there are constants $C_0 > 0$ and $\theta > 0$, such that $\Gamma(t) \leq C_0(1+t)^{-\theta}$, then

$$E_u(t) \leq \frac{c_0}{\ln(ct + c_1)}, \quad t \geq T,$$

with $c, c_0, c_1 > 0$. These constants depend on $E_u(0)$.

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