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NOTES ON EXPANSIVE MAPPINGS AND A PARTIAL ANSWER TO NIRENBERG'S PROBLEM

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Abstract. In this article, two minor flaws made in the preceding paper [19] are corrected. And then several remarks concerning expansive mappings and their connection with monotone operators are observed. These observations are then used for generalizing the Minty-Browder theorem [9, 2] and further answer to Nirenberg's open question. Along this lines, a strengthened Nirenberg's problem is formulated and a surjective result for expansive mappings is obtained. Finally, two interesting conjectures concerning "reverse" Schauder fixed point theorem are raised.

1. Introduction

Recently, a fixed point theorem concerning expansive mappings has been established in [19]. Using this result, coupled with a standard compact analysis, the authors then obtained some new expansive-type Krasnoselskii fixed point theorems. We now observe, however, that there appear two minor errors in the proofs of Theorems 2.1 and 2.3, even though the conclusions of the theorems are true.

For our purposes and for completeness, we use this opportunity to correct the proof and give an example to justify our assertion. Then we observe several remarks about expansive mappings and their connection with monotone operators. Later on, upon those aforementioned observations, we discover that the captured result may help to extend Minty-Browder's celebrated theorem [9, 2] concerning strongly monotone operators, and hence it gives a further and partial answer to Nirenberg's open question [11], which reads as follows:

> Problem (P): Let H be a Hilbert space and let $T : H \to H$ be a continuous map, which is expansive; i.e.,
> $$\|Tx - Ty\| \geq \|x - y\| \quad \text{for all } x, y \in H,$$
> and $T\theta = \theta$. Suppose T maps a neighborhood of the origin onto a neighborhood of the origin. Does T maps H onto H?

The Problem (P) could be generalized to the case where the space investigated is a Banach space or more generally a topological vector space. In other words, the problem is, whether these conditions guarantee the solvability of the equation

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$Tx = p$ for every $p \in H$? Thus if we consider hT instead of T, without loss of generality, we may assume that T is expansive with constant $h > 1$; that is,

$$\|Tx - Ty\| \geq h\|x - y\| \quad \text{for all } x, y \in H.$$

Up to now, to the best of our knowledge, the problem (P) is still not fully resolved. But there are several partial affirmative answers to the problem (P) under some additional hypotheses:

- H is a finite dimensional (known as Domain Invariance Theorem [1, 14]) Euclidean space.
- $T = I - C$ where C is a compact operator or a contraction or more generally a k-set-contraction [8, 12].
- Let $T : E \to F$ be an expansive mapping with constant $h > 0$, where E is a real Banach space and F is a (reflexive) Banach space, T is Fréchet-differentiable and
$$\limsup_{x \to x_0} \|T'(x) - T'(x_0)\| < h \quad \text{for all } x_0 \in E.$$
Then T maps E onto F [5].
- H is a Hilbert space and T is strongly monotone; i.e., there exists an $s > 0$ such that [2, 9]
$$\operatorname{Re}\langle Tx - Ty, x - y\rangle \geq s\|x - y\|^2 \quad \text{for all } x, y \in H.$$

There are also several counterexamples to the Problem (P), see [10, 15, 16] for instance. Our result gives a further positive answer to the Problem (P) and shows that the above inequality can be relaxed by the following one:

$$\operatorname{Re}\langle Tx - Ty, x - y\rangle \geq -c\|x - y\|^2 \quad \text{for all } x, y \in H,$$

where $T : H \to H$ is an expansive mapping with constant $h > 0$, and for some $c < h$. Along this line, a strengthened Nirenberg 's problem is also formulated.

The framework of the rest of this article is outlined as follows. In the next section, the two minor errors are pointed out and two corrected proofs of the aforementioned theorems are presented. Then some remarks and examples concerning expansive mappings and several connections between expansive mappings and monotone operators are observed. The captured results are then used to study Nirenberg's problem and to study a generalization of Minty-Browder's theorem. Along this line, a strengthened Nirenberg's problem is also formulated and a surjective result for expansive mappings is obtained. In the last section, an "expansive set" type fixed point problem is reformulated and two interesting conjectures concerning "reverse" Schauder fixed point theorem are posed.

2. Expansive mappings and connections with strongly monotone operators

Let (X, d) be a metric space and let M be a subset of X. A mapping $T : M \to X$ is said to be expansive with a constant h if there exists a constant $h > 0$ such that

$$d(Tx, Ty) \geq hd(x, y) \quad \text{for all } x, y \in M. \tag{2.1}$$

Expansive mappings, by definition, satisfy the condition (2.1) with a constant $h > 1$. The following result was obtained in [19], but it appears that its proof contains a minor flaw. For the sake of completeness and our further purposes, we restate the theorem and provide a correct proof for it, even though it is a consequence of the Banach Contraction Mapping Principle.

Theorem 2.1. *Let M be a closed subset of a complete metric space X. Assume that $T : M \to X$ is an expansive mapping and that $T(M) \supset M$. Then there exists a unique point $x^* \in M$ such that $Tx^* = x^*$.*

Proof. It follows from (2.1) that the inverse of $T : M \to T(M)$ exists, and

$$d(T^{-1}x, T^{-1}y) \leq \frac{1}{h} d(x, y) \quad \text{for all } x, y \in T(M),$$

which, in view of the fact that $M \subset T(M)$, shows, in particular, that $T^{-1}|_M : M \to M$ is a contraction, where $T^{-1}|_M$ denotes the restriction of the mapping T^{-1} to the set M. Since, M is a closed subset of a complete metric space, then in view of the Banach Contraction Mapping Principle there exists an $x^* \in M$ such that $T^{-1}x^* = x^*$. Clearly, x^* is also a fixed point of T.

Uniqueness of a fixed point is even easier: If $x = Tx, y = Ty$ for some $x, y \in M$, then

$$d(x, y) = d(Tx, Ty) \geq hd(x, y), \quad \text{i.e., } x = y \quad (h > 1).$$

This completes the proof. □

In [19], because $T^{-1} : T(M) \to M$ was continuous, we claimed that the image $T(M)$ was closed. However, this is not true in general, since an expansive mapping may not be continuous, and hence $T(M)$ may not be closed. The following is a simple counterexample:

Example 2.2. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} 2x - 1, & \text{if } x \leq 0, \\ 2x + 1, & \text{if } x > 0. \end{cases}$$

Then T is expansive with constant $h = 2$, and $T(\mathbb{R})(= (-\infty, -1] \cup (1, +\infty))$ is neither closed nor open in $\mathbb{R}$. Observe also that T has no fixed points in $\mathbb{R}$, and thus no closed "expansive set" under T (a set M satisfying $T(M) \supset M$) can be found by Theorem 2.1. But "contractive set" under T (a set M satisfying $T(M) \subset M$) does exist, and $M = (-\infty, -1]$ is a such one.

The same error occurs in the proof of the following variant of Krasnoselskii fixed point theorem. This minor error can be corrected in the similar way as above.

Theorem 2.3. *Let K be a nonempty closed convex subset of a Banach space E. Suppose that T and S map K into E such that*

(i) *S is continuous, $S(K)$ resides in a compact subset of E;*
(ii) *T is an expansive mapping;*
(iii) *any $z \in S(K)$ implies $K + z \subset T(K) \subset K$, where $K + z = \{y + z \mid y \in K\}$.*

Then there exists a point $x^* \in K$ with $T \circ (I - S)x^* = x^*$.

Proof. It is obvious that $T^{-1} : T(K) \to K$ is a contraction. For any fixed $z \in S(K)$, define $T_z^{-1} : T(K) \to E$ by $T_z^{-1}x = T^{-1}x + z$. Then $T_z^{-1}|_{K+z} : K + z \to K + z$, since $K + z \subset T(K)$. Therefore, the equation

$$T^{-1}x + z = x$$

has a unique solution $x = \tau(z) \in K + z \subset T(K)$. The rest of the proof is identical with that of [19, Theorem 2.3]. □

We now see, in general, that the expansiveness of T does not necessarily imply the closedness of the range of T. However, for closed linear operators, it is so; and we have the following standard result, which is often proved in functional analysis courses.

Proposition 2.4. *Let $T : D(T) \subset E \to E$ be a closed linear operator, where E is a normed linear space. Then the range of T, $R(T)$, is closed if and only if there exists an $h > 0$ such that*

$$\|Tx\| \geq hd(x, \ker T), \quad \forall x \in D(T),$$

where $\ker T$ is the kernel of T and

$$d(x, \ker T) = \inf\{\|x - z\| : z \in \ker T\}.$$

We now recall some well-known notions that will be used in the sequel, see for instance Zeidler [20]. Let H be a Hilbert space and $A : H \to H$ an operator. Then

(i) A is called monotone if $\mathrm{Re}\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in H$.
(ii) A is called strictly monotone if $\mathrm{Re}\langle Ax - Ay, x - y\rangle > 0$ for all $x, y \in H$ with $x \neq y$.
(iii) A is called strongly monotone if there is an $s > 0$ such that

$$\mathrm{Re}\langle Ax - Ay, x - y\rangle \geq s\|x - y\|^2 \quad \text{for all } x, y \in H.$$

(iv) A is called coercive if $\lim_{\|x\|\to+\infty} \mathrm{Re}\langle Ax, x\rangle/\|x\| = +\infty$.
(v) A is called weakly coercive if $\lim_{\|x\|\to+\infty} \|Ax\| = +\infty$.

The following straightforward implications hold: A is strongly monotone $\Rightarrow$ A is strictly monotone $\Rightarrow$ A is monotone.

In what follows, the symbols $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ will usually denote Banach spaces, and $(H, \langle,\rangle)$ will denote a Hilbert space. To avoid misunderstanding, we provide the definition of a compact and a bounded mapping, since in the literature different authors understand these notions differently (cf. [6, p. 112]). A mapping $T : D(T) \subset E \to F$ is bounded if it maps bounded subsets of $D(T)$ into bounded subsets of F, and is compact if it maps bounded subsets of $D(T)$ into relatively compact subsets of F. And we denote the range of T by $R(T)$ if T is linear and by $T(D(T))$ if T is (possibly) nonlinear. Some observations on expansive mappings and their connections with strongly monotone operators are summarized in the following proposition.

Proposition 2.5.
(1) *Let $T : E \to F$ be an expansive mapping with constant $h > 0$. Suppose that the dimension of E is infinite. Then T is never compact; if in addition $h > 1$, then neither T nor $I - T$ is compact.*
(2) *Let $T : E \to E$ be an expansive mapping with constant $h > 0$. Then T is weakly coercive.*
(3) *Let $T : H \to H$ be a strongly monotone operator with constant $s > 0$. Then T is an expansive mapping with constant $s > 0$. Conversely, an expansive mapping may not be strongly monotone.*
(4) *An expansive mapping may be an unbounded operator; i.e., it may transform bounded sets into unbounded ones. Also such an operator may not be continuous.*
(5) *Let $T : D(T) \subset E \to F$ be an expansive mapping with constant $h > 0$. Then T may not be compact even though F is finite dimensional.*

(6) *Let $T : E \to F$ be an expansive mapping with constant $h > 0$. If T is Fréchet-differentiable, then the derivative of T at x, $T'(x)$, is also expansive, and*

$$\|T'(x)u - T'(x)v\|_F \geq h\|u - v\|_E \quad \text{for all } u, v \in E. \tag{2.2}$$

Moreover, the range of $T'(x)$, $R(T'(x))$, is closed in F.

Proof. (1) It suffices to prove the case of $h > 1$, the case of $h > 0$ follows in the same fashion. From the expansiveness of T, we have

$$\|(I - T)x - (I - T)y\|_F \geq (h - 1)\|x - y\|_E \quad \text{for all } x, y \in E. \tag{2.3}$$

It follows from (2.2) that

$$\|(I - T)^{-1}x - (I - T)^{-1}y\|_E \leq \frac{1}{h - 1}\|x - y\|_F \quad \text{for all } x, y \in (I - T)(E).$$

This illustrates that $(I - T)^{-1} : (I - T)(E) \to E$ is continuous and bounded. Therefore, if $I - T$ were compact, we would get that $I = (I - T)^{-1}(I - T) : E \to E$ is compact. This is impossible since E is infinite dimensional.

(2) The desired result follows from the inequality

$$\|Tx\| \geq h\|x\| - \|T\theta\| \quad \text{for all } x \in E.$$

(3) As $T : H \to H$ is strongly monotone,

$$\mathrm{Re}\langle Tx - Ty, x - y\rangle \geq s\|x - y\|^2 \quad \text{for all } x, y \in H.$$

The Cauchy-Schwarz inequality gives

$$\|Tx - Ty\|\|x - y\| \geq |\,\mathrm{Re}\langle Tx - Ty, x - y\rangle| \geq s\|x - y\|^2,$$

which shows that $T : E \to E$ is expansive with constant $s > 0$. The converse statement is apparent. However, a nontrivial specific instance showing this point is provided at the end of this section.

(4) We shall construct a concrete example to justify the assertion. Indeed, for a fixed natural number $N \geq 2$ and $B = \{x \in \mathbb{R}^N : \|x\| \leq 1\}$, let us define $T : B \to \mathbb{R}^N$ in the following way. Choose $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, $\|x_n\| = 1$, $x_n \neq x_m$ for $n \neq m$, and $x_n \to x_0$. For $h > 1$, define

$$Tx = \begin{cases} hx, & \text{if } x \in B,\ x \neq x_m, \text{ for } m = 1, 2, \dots, \\ hx_n + nx_n, & \text{if } x = x_n, \text{ for some } n = 1, 2, \dots. \end{cases}$$

Clearly, $\|Tx_n\| = h + n$ for $n \in \mathbb{N}$, and so $T(B)$ is unbounded. It then follows from $x_n \to x_0$ that T is not continuous.

We now claim that T is strongly monotone by proving the inequality

$$\langle Tx - Ty, x - y\rangle \geq h\|x - y\|^2 \quad \text{for all } x, y \in B. \tag{2.4}$$

We show the above inequality by considering three distinct cases.

Case I: $x \in B, x \neq x_m$, for all $m = 1, 2, \dots$ and $y \in B, y \neq x_m$, for all $m = 1, 2, \dots$. In this case the inequality (2.4) follows obviously by definition.

Case II: $x \in B, x \neq x_m$, for all $m = 1, 2, \dots$ and $y = x_n$, for some $n = 1, 2, \dots$. From the definition, we obtain

$$\langle Tx - Tx_n, x - x_n\rangle = h\|x - x_n\|^2 - n\langle x_n, x\rangle + n\|x_n\|^2 \geq h\|x - x_n\|^2,$$

since $\|x_n\| = 1$ and $|\langle x_n, x\rangle| \leq 1$ for all $x \in B$. This proves the inequality (2.4).

Case III: $x = x_n$ for some $n = 1, 2, \dots$ and $y = x_m$ for some $m = 1, 2, \dots$. Then one has

$$\langle Tx_n - Tx_m, x_n - x_m\rangle = h\|x_n - x_m\|^2 + (n+m)[1 - \langle x_n, x_m\rangle] \geq h\|x_n - x_m\|^2.$$

To sum up, we obtain that $T : B \to \mathbb{R}^N$ is strongly monotone with constant $h > 1$ and therefore is expansive by (3). It is worth noting that it is not an easy matter to show directly that T is expansive.

(5) This is an immediate consequence of (4).

(6) In fact, from the assumption, we have

$$T(x+u) = Tx + T'(x)u + R_x(u),$$

where $T'(x)$ is a bounded linear operator in u and $\|R_x(u)\|_F = o(\|u\|_E)$ as $\|u\|_E \to 0$. Putting $u = \epsilon y$, for ϵ sufficiently small, one has from the expansiveness of T that

$$\|T'(x)y + o(1)\|_F \geq h\|y\|_E.$$

Letting $\epsilon \to 0$, we find that $T'(x)$ fulfills the inequality (2.1). Therefore, $T'(x)$ is injective and has a Lipschitizian continuous inverse with the Lipschitz constant h^{-1}; i.e.,

$$\|[T'(x)]^{-1}v\|_E \leq \frac{1}{h}\|v\|_F \text{ for all } v \in R(T'(x)).$$

Let now $T'(x)u_n = v_n \to v$. Then one has $u_n = [T'(x)]^{-1}v_n$ and

$$\|u_n - u_m\|_E \leq \frac{1}{h}\|v_n - v_m\|_F \to 0, \text{ as } m, n \to +\infty,$$

so that there exists a $u \in E$ such that $u_n \to u$. Accordingly, the continuity of $T'(x)$ implies $v = T'(x)u \in R(T'(x))$. This shows that $R(T'(x))$ is closed. $\square$

Let us explore some properties of surjectivity for mappings of the form $I - T$, where T is a contraction or an expansion.

Lemma 2.6. *Let $T : E \to E$ be either a contraction or an expansion with constant $h > 1$. If $T(E) = E$ then $(I - T)(E) = E$. Furthermore, if T is an expansion with constant $h > 2$, then $(I - T)(E) = E$ implies $T(E) = E$.*

Proof. The conclusion for the case that T is a contraction follows from the contraction principle. In the case when T is an expansion. So let us assume that T is an expansion and for $y \in E$ define $T_y : E \to E$ by $T_y x = Tx + y$. Then T_y maps E onto itself and so the result follows from Theorem 2.1. Finally, for $h > 2$, let $S = I - T$. Then

$$\|Sx - Sy\| \geq \|Tx - Ty\| - \|x - y\| \geq (h-1)\|x - y\| \quad \text{for all } x, y \in E.$$

Thus, S is expansive since $h - 1 > 1$. The preceding arguments yield the desired result. $\square$

A well-known and fundamental result in the theory of monotone operators (see [2, 9, 20] for instance) is collected in the following lemma.

Lemma 2.7. *Let H be a Hilbert space and let $T : H \to H$ be continuous, monotone and weakly coercive. Then $T(H) = H$. If, furthermore, T is strictly monotone, then for any $p \in H$ the equation $Tx = p$ has a unique solution.*

We are now prepared to formulate the main results of this article.

Theorem 2.8. *Let $T : H \to H$ be a continuous expansive mapping with constant $h > 0$. If either T or $-T$ is monotone, then, for each $u \in H$ the equation $Tx = u$ has a unique solution in H, and the solution x depends continuously on u. More precisely, if $Tx = u$ and $Ty = v$, then*

$$\|x - y\| \le \frac{1}{h}\|u - v\|. \tag{2.5}$$

If, in addition, the constant h is assumed to be greater than one, then both T and $I - T$ are global homeomorphisms on H (and T has a unique fixed point in H).

Proof. It follows from Proposition 2.5 (2) that T is weakly coercive. Suppose that T is monotone. Then Lemma 2.7 gives that, for each $u \in H$ the equation $Tx = u$ has a solution. The uniqueness of the solution and (2.5) are direct consequences of the expansiveness of T.

If additionally $h > 1$, then the preceding discussion and Lemma 2.6 yield the desired result. In the case when $-T$ is monotone, the proof stays almost unchanged. $\square$

With the aid of Theorem 2.1, we can now formulate a generalization of Theorem 2.8 where T is monotone. Here T is allowed to be somewhat "nonmonotone".

Theorem 2.9. *Let $T : H \to H$ be a continuous expansive mapping with constant $h > 0$. If there exists a real number $c < \frac{\sqrt{2}}{2}h$ such that*

$$\operatorname{Re}\langle Tx - Ty, x - y\rangle \ge -c\|x - y\|^2 \quad \text{for all } x, y \in H, \tag{2.6}$$

then, for each $u \in H$ the equation $Tx = u$ has a unique solution in H, and the solution x depends continuously on u. More precisely, if $Tx = u$ and $Ty = v$, then

$$\|x - y\| \le \frac{1}{h}\|u - v\|. \tag{2.7}$$

If, in addition, the constant h is assumed to be greater than one, then both T and $I - T$ are global homeomorphisms on H.

Proof. The case $c \le 0$ follows directly from Theorems 2.8. We just need to consider the case $c > 0$. Since $c < h/\sqrt{2}$ there exists a small $\epsilon > 0$ so that $2c^2 + 2c\epsilon < h^2$. For such fixed ϵ and for $x, y \in H$, it follows from (2.6) that

$$\begin{aligned} &\operatorname{Re}\langle Tx + (c+\epsilon)x - Ty - (c+\epsilon)y, x - y\rangle \\ &= \operatorname{Re}\langle Tx - Ty, x - y\rangle + c\|x - y\|^2 + \epsilon\|x - y\|^2 \\ &\ge \epsilon\|x - y\|^2. \end{aligned}$$

This says that $T + (c+\epsilon)I$ is strongly monotone and hence monotone and coercive. In view of Lemma 2.7, we know $(T + (c+\epsilon)I)(H) = H$. Letting $S = T + (c+\epsilon)I$, and taking into account that T is expansive with $h > 0$, we deduce again from (2.6) that, for all $x, y \in H$,

$$\begin{aligned} \|Sx - Sy\|^2 &= \langle (Tx - Ty) + (c+\epsilon)(x - y), (Tx - Ty) + (c+\epsilon)(x - y)\rangle \\ &= \|Tx - Ty\|^2 + 2(c+\epsilon)\operatorname{Re}\langle Tx - Ty, x - y\rangle + (c+\epsilon)^2\|x - y\|^2 \\ &\ge (h^2 - c^2 + \epsilon^2)\|x - y\|^2. \end{aligned} \tag{2.8}$$

Now, the equation $Tx = p$ for each $p \in H$ is equivalent to

$$x = S_\epsilon x - \frac{1}{c+\epsilon}p, \tag{2.9}$$

where $S_\epsilon = \frac{1}{c+\epsilon} S$. In view of (2.8) and the definition of S_ϵ, we have

$$\|S_\epsilon x - S_\epsilon y\| \geq \frac{\sqrt{h^2 - c^2 + \epsilon^2}}{c+\epsilon} \|x - y\| \quad \text{for all } x, y \in H.$$

The choice of ϵ implies

$$\frac{\sqrt{h^2 - c^2 + \epsilon^2}}{c+\epsilon} > 1. \tag{2.10}$$

This, together with $S_\epsilon(H) = H$, proves that (2.9) has a unique solution by Theorem 2.1. This in turn shows that $T(H) = H$. The rest of the proof is the same as that of Theorem 2.8. $\square$

Remark 2.10. If $-T$ satisfies the inequality (2.6), then the conclusion still holds. Moreover, if T is a strongly monotone operator, then it is expansive and (2.6) trivially holds. Therefore, Theorem 2.9 extends Minty-Browder's [9, 2] strongly monotone case to "nonmonotone" case. And it is also a further partial answer to Nirenberg's problem (P). Finally, the continuity assumption on T cannot be dropped. Otherwise, Example 2.2 gives a simple counterexample.

It is quite natural to ask whether the constant c in Theorem 2.9 can be greater than $\frac{h}{\sqrt{2}}$ or even approach h. Observe that the left-hand side of (2.10) as a function of ϵ is decreasing on $(0, (h^2 - c^2)/c]$ and is increasing to 1 on $((h^2 - c^2)/c, +\infty)$. This suggests that the continuity (fixed point) method may not help us achieve our purpose; whereas, by making use of the theory of monotone operators and some delicate techniques, we will show that this goal can be achieved. Before proceeding further, the following well-known notions and facts are needed, see for example again [20].

Let $A : D(A) \subset H \to H$ be an operator. Then:

(i) A is said to be maximal monotone if A is monotone and

$$\mathrm{Re}\langle b - Ay, x - y\rangle \geq 0 \quad \text{for all } y \in D(A)$$

implies $Ax = b$; i.e., A has no proper monotone extension.

(ii) A is said to be accretive if $(I + \mu A) : D(A) \to H$ is injective and $(I + \mu A)^{-1}$ is non-expansive for all $\mu > 0$.

(iii) A is said to be maximal accretive (m-accretive) if A is accretive and $(I + \mu A)^{-1}$ exists on H for all $\mu > 0$.

Lemma 2.11 ([9, 20])**.** *Let $A : D(A) \subset H \to H$ be an operator. Then the following three properties of A are mutually equivalent:*

(i) *A is monotone and $(\lambda I + A)(D(A)) = H$ for some (or, equivalently, all) $\lambda > 0$.*

(ii) *A is maximal accretive.*

(iii) *A is maximal monotone.*

We now show that the constant c in Theorem 2.9 can actually approach h. The proof is different from that of Theorem 2.9.

Theorem 2.12. *Let $T : H \to H$ be a continuous expansive mapping with constant $h > 0$. If there exists a real number $c < h$ such that the inequality (2.6) holds, then the conclusions of Theorem 2.9 are valid.*

Proof. The result in the case of $c \le 0$ is included in the previous theorems. Therefore, without loss of generality, we may assume that $c > 0$. The proof falls into three steps.

Step 1. To show that $T + \lambda I$ is surjective for all $\lambda > c$ and $T(H)$ is closed. For any $\epsilon > 0$ and for all $x, y \in H$, one derives from (2.6) that

$$\mathrm{Re}\langle Tx - Ty + (c+\epsilon)(x-y), x-y\rangle \ge \epsilon \|x-y\|^2.$$

Therefore, $T + (c+\epsilon)I$ is strongly monotone and thus monotone and coercive. By Lemma 2.7, we obtain $(T+(c+\epsilon)I)(H) = H$, which means that $T+\lambda I$ is surjective for all $\lambda > c$.

Next we show that $T(H)$ is closed. To this end, let $y_n = Tx_n \to y$. Then it follows from the expansiveness of T that

$$\|x_n - x_m\| \le \frac{1}{h}\|Tx_n - Tx_m\|,$$

which implies $x_n \to x$ for some $x \in H$, and hence $y = Tx \in T(H)$ since T is continuous.

Step 2. To show that the operator $S = \mu I + T^{-1} : T(H) \to H$ is maximal monotone, where μ satisfies $1/h < \mu < 1/c$.

Again, since T is expansive, we know that $T^{-1} : T(H) \to H$ is well-defined, and for every $x, y \in T(H)$,

$$\|T^{-1}x - T^{-1}y\| \le \frac{1}{h}\|x-y\|. \tag{2.11}$$

This implies that S is continuous. By (2.6) and (2.11), one may derive that for all $x, y \in T(H)$,

$$\begin{aligned}\mathrm{Re}\langle Sx - Sy, x-y\rangle &= \mu\|x-y\|^2 + \mathrm{Re}\langle T^{-1}x - T^{-1}y, x-y\rangle \\ &\ge (\mu - \frac{c}{h^2})\|x-y\|^2.\end{aligned}$$

Note that since $c < h$ and $\mu > 1/h$ we get that $\mu - \frac{c}{h^2} > 0$ and so S is (strongly) monotone. In view of Lemma 2.11, it is sufficient to show that $(S+\lambda I)(T(H)) = H$ for some (or, equivalently, all) $\lambda > 0$. This is the same as saying $((\mu+\lambda)I + T^{-1})(T(H)) = H$. It is easily seen that the equation $(\mu+\lambda)y + T^{-1}y = p$, $y \in T(H)$, is equivalent to the equation

$$Tx + \frac{1}{\mu+\lambda}x = \frac{1}{\mu+\lambda}p, \quad x \in H, \text{ for every given } p \in H. \tag{2.12}$$

Note also that $\mu < 1/c$, thus, we can choose $\lambda > 0$ sufficiently small so that $\frac{1}{\mu+\lambda} > c$, and consequently the equation (2.12) has a solution due to Step 1. This proves that S is a maximal monotone operator.

Step 3. We complete the proof by showing that T is surjective onto H. From Step 1 we have known that $T(H)$ is closed. Here we want to prove that $T(H)$ is actually the whole space H. Based on the well-known Kirszbraun-Valentine (cf. [7, 14, 17, 18]) theorem, it follows from (2.11) that the mapping T^{-1} admits an extension $\overline{T}$ defined on the whole space H, which is $1/h$-Lipschitizan on H. Apparently, the mapping $\overline{S} = \mu I + \overline{T}$ is continuous. Moreover, for any $x, y \in H$, the Cauchy-Schwarz inequality once more yields

$$\mathrm{Re}\langle \overline{S}x - \overline{S}y, x-y\rangle = \mu\|x-y\|^2 + \mathrm{Re}\langle \overline{T}x - \overline{T}y, x-y\rangle \ge (\mu - \frac{1}{h})\|x-y\|^2$$

for all $x, y \in H$. This indicates that $\overline{S}$ is strongly monotone, and so is $\overline{S} + \lambda I$ for all $\lambda > 0$. Then by Lemma 2.7 $(\overline{S} + \lambda I)(H) = H$. Thus, according to Lemma 2.11, $\overline{S} : H \to H$ is maximal monotone. Notice that $\overline{T}$ coincides with T^{-1} on $T(H)$, and hence $\overline{S}$ is a monotone extension of S. Because $\overline{S}$ is an extension of S, we will reach a contradiction to the maximal monotonicity of S, unless $D(S) = T(H) = H$.

The rest of the proof is the same as that of Theorem 2.8. □

From the proof of Theorem 2.12, we see that the assumption that $c < h$ is essential. But if we let for simplicity $T = -hI$, then the inequality (2.6) holds as an equality for $c = h$ and trivially T is a global homeomorphism on H. We are thus led naturally to the following question: Is the critical value $c = h$ also a sufficient condition for T being a global homeomorphism on H? More specifically, we are led to a strengthened Nirenberg 's problem:

Problem 1'. Let $T : H \to H$ be a continuous expansive mapping with constant $h > 0$. Assume that either T or $-T$ satisfies

$$\operatorname{Re}\langle Tx - Ty, x - y\rangle \ge -h\|x - y\|^2 \quad \text{for all } x, y \in H. \tag{2.13}$$

Is T a surjective map and so a global homeomorphism on H?

From the discussion of Nirenberg's problem (P) (see Section 1) we know that the Problem 1' above is true for $H = \mathbb{R}^n$ even without the requirement (2.13) and that it could be false in infinite dimensional Hilbert spaces without the requirement (2.13). This problem could be generalized to the case where the space considered is a Banach space or more generally a topological vector space.

Problem 1". Let $T : E \to E^*$ be a continuous expansive mapping with constant $h > 0$, where E^* is the dual space of E. Assume that either T or $-T$ satisfies

$$\operatorname{Re}\langle Tx - Ty, x - y\rangle \ge -h\|x - y\|^2 \quad \text{for all } x, y \in E. \tag{2.14}$$

Is T a surjective map onto E^*?

The first fact that the validity of (2.13) is in general not a consequence of the expansiveness of T is demonstrated at the end of this section.

Next we show that Problem 1' is in general not true. To see this, consider the linear operator T defined on l^2 by $Tx = \sigma x$, where l^2 is the real Hilbert space of square-summable sequences and σ is the forward shift map; i.e., $\sigma(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Then it follows readily that $\|Tx\| = \|x\|$ for all $x \in L^2(\mathbb{N})$ and that the range of T has empty interior. Moreover, by the Cauchy-Schwarz inequality,

$$\langle Tx, x\rangle + \|x\|^2 = \sum_{n=1}^{\infty} x_n x_{n+1} + \|x\|^2 \ge \|x\|^2 - \|x\|(\|x\|^2 - |x_1|^2)^{\frac{1}{2}} \ge 0,$$

for all $x = (x_n)_{n=1}^{\infty} \in L^2(\mathbb{N})$. This shows that (2.13) is true, while T is not surjective on $L^2(\mathbb{N})$. Also, for this expansive mapping, we show that there is no $c < 1$ such that (2.6) is true. Indeed, for $c \le 0$, let $x = (1, -1, 0, 0, \dots)$. Then $\langle Tx, x\rangle + c\|x\|^2 = -1 + 2c \le -1$. For $0 < c < 1$, choose $\epsilon > 0$ so that $c + \epsilon < 1$ and then let $x = \{(-1)^{n-1}(c + \epsilon)^n\}_{n=1}^{\infty}$. Then

$$\langle Tx, x\rangle + c\|x\|^2 = -(c + \epsilon)\sum_{n=1}^{\infty}(c + \epsilon)^{2n} + c\sum_{n=1}^{\infty}(c + \epsilon)^{2n} = -\epsilon\sum_{n=1}^{\infty}(c + \epsilon)^{2n} < 0.$$

Therefore, a more mature version of the strengthened Nirenberg's problem 1' reads

Problem 1. Let $T : H \to H$ be a continuous expansive mapping with constant $h > 0$. Assume that the range of T has nonempty interior and either T or $-T$ satisfies

$$\mathrm{Re}\langle Tx - Ty, x - y\rangle \geq -h\|x - y\|^2 \quad \text{for all } x, y \in H. \tag{2.15}$$

Is T a surjective map and so a global homeomorphism on H?

Using the Fredholm alternative for nonlinear operators [3], we shall see that, for a differentiable expansive mapping, its surjectivity is closely related to the null space of the adjoint operator of its derivative.

Theorem 2.13. *Let $T : E \to F$ be an expansive mapping with constant $h > 0$, where E is a Banach space and F is a real Banach space. Let N be a finite subset of E and assume that T has a linear Gâteaux differential $dT(x)$, a bounded linear operator from E to F, at every point $x \in E \setminus N$ and that* $\ker([dT(x)]^*) = \{0\}$ *for each $x \in E \setminus N$, where $[dT(x)]^*$ denotes the adjoint operator of $dT(x)$ and* $\ker([dT(x)]^*)$ *denotes the kernel of $[dT(x)]^*$. Then T is surjective onto F; i.e., $T(E) = F$.*

Proof. From the proof of Theorem 2.12, we know that the range of T, $T(E)$, is closed in F. Then the result follows from [3, Theorem 2]. □

Corollary 2.14. *Let $T : H \to H$ be a Fréchet differentiable expansive mapping with constant $h > 0$, where H is a real Hilbert space. Suppose that either $T'(x)$ is self-adjoint or there exists a constant $k > 0$ such that*

$$|\langle u, T'(x)u\rangle| \geq k \min\{\langle T'(x)u, T'(x)u\rangle, \langle u, u\rangle\} \tag{2.16}$$

for all $x \in H\setminus N$ and $u \in H$, where N is a finite subset of H. Then T is a surjective map onto H and thus a global homeomorphism on H.

Proof. We have by the item 6 of Proposition 2.5 that $T'(x)$ is expansive for all $x \in H$, and

$$\|T'(x)u\| \geq h\|u\| \quad \text{for all } u \in H. \tag{2.17}$$

This immediately indicates that $\ker(T'(x)) = \{0\}$ for all $x \in H \setminus N$. Thus, if $T'(x)$ is self-adjoint then $\ker([T'(x)]^*) = \ker(T'(x)) = \{0\}$ for all $x \in H \setminus N$.

For the second case, it follows once more from (2.16) and (2.17) that, for all $x \in H \setminus N$ and $u \in H$,

$$\begin{aligned} |\langle [T'(x)]^* u, u\rangle| &= |\langle u, T'(x)u\rangle| \\ &\geq k \min\{\langle T'(x)u, T'(x)u\rangle, \langle u, u\rangle\} \\ &\geq k \min\{h^2, 1\}\|u\|^2, \end{aligned}$$

which implies that $\ker([T'(x)]^*) = \{0\}$ for all $x \in H \setminus N$. The corollary now follows from the previous theorem. □

Next, we illustrate Theorem 2.13 and its corollary with the following Fredholm integral equation of the second kind

$$f(x) + \lambda \int_a^b k(x, y)\phi(y)dy = \phi(x), \quad a \leq x \leq b, \tag{2.18}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $f \in L^2(a, b)$ and $k \in L^2((a, b) \times (a, b))$, here $L^2(a, b)$ and $L^2((a, b) \times (a, b))$ stand for the real Hilbert space of real-valued square-integrable functions defined on (a, b) and $(a, b) \times (a, b)$, respectively. Assume that the kernel

k is hermitian; i.e., $k(x,y) = k(y,x)$ for almost all x and y, then by the spectral theorem the integral operator has an eigen-expansion

$$K\phi = \sum_{n=1}^{\infty} \mu_n \langle \phi, \phi_n \rangle \phi_n,$$

where $\{\mu_n\}_{n=1}^{\infty} \subset l^2$ is a sequence of nonzero eigenvalues of K and ϕ_n is a corresponding orthonormal sequence of eigenvectors in $L^2(a,b)$; see [13, p. 109-120]. If there are only finitely many eigenvalues, the above sum is a finite sum. Equation (2.18) now takes the form

$$f + \lambda \sum_{n=1}^{\infty} \mu_n \langle \phi, \phi_n \rangle \phi_n = \phi.$$

Consider a nonlinear equation similar to one above, having the form

$$f + \lambda \sum_{n=1}^{\infty} k_n(\langle \phi, \phi_n \rangle) \phi_n = \phi, \tag{2.19}$$

where $k_n : \mathbb{R} \to \mathbb{R}$ is a sequence of C^1-smooth functions with the properties (i) $k_n(0) = 0$, (ii) $0 < \alpha \le |k_n'(x)|$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ and (iii) there exist $N \in \mathbb{N}$, $\delta > 0$ and β_0 such that $|k_n'(x)| \le \beta_0$ for all $n > N$ and $|x| \le \delta$. From the reasonings below, the orthonormal sequence (ϕ_n) is complete. A simple example of such sequence is given by

$$k_n(x) = \alpha_n x + n(e^{\frac{x}{n}} - 1),$$

where $0 < \alpha \le \alpha_n \le \beta_0 < +\infty$ for all $n \in \mathbb{N}$.

Let us formally set

$$T\phi = \lambda \sum_{n=1}^{\infty} k_n(\langle \phi, \phi_n \rangle) \phi_n - \phi.$$

Then T is well-defined on the whole space $L^2(a,b)$. Indeed, for $\phi \in L^2(a,b)$, note that $\langle \phi, \phi_n \rangle \to 0$ as $n \to +\infty$, we have by (iii) that

$$\begin{aligned} \| \sum_{n=1}^{\infty} k_n(\langle \phi, \phi_n \rangle) \phi_n \|^2 &\le \sum_{n=1}^{\infty} |k_n(\langle \phi, \phi_n \rangle)|^2 \\ &= \sum_{n=1}^{N} |k_n(\langle \phi, \phi_n \rangle)|^2 + \beta_0^2 \sum_{n=N+1}^{\infty} |\langle \phi, \phi_n \rangle|^2 < +\infty, \end{aligned}$$

which implies that $T : L^2(a,b) \to L^2(a,b)$ is well-defined. We now consider two cases: Case 1. $|\lambda|\alpha > 1$ and Case 2. $|\lambda|\beta < 1$, where $|k_n'(x)| \le \beta < +\infty$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Conclusion 1. Suppose that either Case 1 or 2 holds, then $T : L^2(a,b) \to L^2(a,b)$ is expansive with constant $\big||\lambda|\gamma - 1\big|$, where $\gamma = \alpha$ for Case 1 and $\gamma = \beta$ for Case 2.

As a matter of fact, for arbitrary $\phi, \psi \in L^2(a,b)$, in light of the Parseval identity and the mean value theorem, if $|\lambda|\alpha > 1$ we obtain,

$$
\begin{aligned}
\|T\psi - T\phi\|^2 &= \|\lambda \sum_{n=1}^{\infty} [k_n(\langle \psi, \phi_n\rangle) - k_n(\langle \phi, \phi_n\rangle)]\phi_n - (\psi - \phi)\|^2 \\
&= \|\lambda \sum_{n=1}^{\infty} k_n'(\theta_n)\langle \psi - \phi, \phi_n\rangle \phi_n - (\psi - \phi)\|^2 \\
&= \sum_{n=1}^{\infty} [\lambda k_n'(\theta_n) - 1]^2 |\langle \psi - \phi, \phi_n\rangle|^2 \\
&\ge (|\lambda|\alpha - 1)^2 \sum_{n=1}^{\infty} |\langle \psi - \phi, \phi_n\rangle|^2 = (|\lambda|\alpha - 1)^2 \|\psi - \phi\|^2,
\end{aligned}
\tag{2.20}
$$

where $\theta_n \in \mathbb{R}$ is taken according to the mean value theorem. Similarly if $|\lambda|\beta < 1$ we have

$$
\|T\psi - T\phi\|^2 \ge (|\lambda|\beta - 1)^2 \|\psi - \phi\|^2, \tag{2.21}
$$

Therefore, we conclude from (2.20) and (2.21) that

$$
\|T\psi - T\phi\| \ge ||\lambda|\gamma - 1| \|\psi - \phi\|, \tag{2.22}
$$

which proves the conclusion.

In view of the fact that k_n are C^1-smooth, we get that T is differentiable and

$$
T'(\phi)\psi = \lambda \sum_{n=1}^{\infty} k_n'(\langle \phi, \phi_n\rangle)\langle \psi, \phi_n\rangle \phi_n - \psi, \tag{2.23}
$$

which is well-defined for any $\psi \in L^2(a,b)$ again by (iii). Now, for any $u, v \in L^2(a,b)$, we calculate from (2.23) that

$$
\langle T'(\phi)u, v\rangle = \lambda \sum_{n=1}^{\infty} k_n'(\langle \phi, \phi_n\rangle)\langle u, \phi_n\rangle\langle \phi_n, v\rangle - \langle u, v\rangle = \langle u, T'(\phi)v\rangle.
$$

This tells us that $T'(\phi)$ is self-adjoint for all $\phi \in L^2(a,b)$. Applying the first case of Corollary 2.14 we conclude that

Conclusion 2. If either (i), (ii) and (iii) for Case 1 or (i) and (iii) for Case 2 hold, then for each $f \in L^2(a,b)$ the nonlinear equation (2.19) has a unique solution in $L^2(a,b)$.

Conclusion 3. If (i), (ii) and (iii) for Case 1 hold, then $T : L^2(a,b) \to L^2(a,b)$ is expansive with constant $|\lambda|\alpha - 1$, but it may fail to satisfy (2.13), and so it may not be strongly monotone.

The expansiveness has already been established above. By the definition of T, we have

$$
\langle T\psi - T\phi, \psi - \phi\rangle + (|\lambda|\alpha - 1)\|\psi - \phi\|^2 = \sum_{n=1}^{\infty} [\lambda k_n'(\theta_n) + |\lambda|\alpha - 2] |\langle \psi - \phi, \phi_n\rangle|^2,
$$

which may be negative for some ψ and ϕ if k_n''s change signs; for example, let us take $k_n' = (-1)^n$ and $\lambda > 1$. Then $\alpha = 1$ and the above equality becomes

$$
\langle T\psi - T\phi, \psi - \phi\rangle + (\lambda - 1)\|\psi - \phi\|^2 = 2\lambda \sum_{n=1}^{\infty} |\langle \psi - \phi, \phi_{2n}\rangle|^2 - 2\sum_{n=1}^{\infty} |\langle \psi - \phi, \phi_n\rangle|^2,
$$

which will be definitely negative for some ψ and ϕ. The same argument can be made for $-T$. Consequently, T can not satisfy (2.13) in this case.

Finally, we would like to point out that the second case of Corollary 2.14 is fulfilled for Case 2 and

Case 1'. $|\lambda|\alpha > 1$ and either $0 < \alpha \le \inf_n \inf_x k_n'(x)$ or $\sup_n \sup_x k_n'(x) \le -\alpha < 0$. To this end, we compute as above from (2.23) that

$$\langle T'(\phi)\psi, \psi\rangle = \sum_{n=1}^{\infty}[\lambda k_n'(\langle\phi,\phi_n\rangle) - 1]|\langle\psi,\phi_n\rangle|^2 \tag{2.24}$$

and

$$\langle T'(\phi)\psi, T'(\phi)\psi\rangle = \sum_{n=1}^{\infty}[\lambda k_n'(\langle\phi,\phi_n\rangle) - 1]^2|\langle\psi,\phi_n\rangle|^2. \tag{2.25}$$

Under Case 2, we have from (2.24) and (2.25) that

$$\begin{aligned}|\langle T'(\phi)\psi, \psi\rangle| &\ge \frac{1}{1+|\lambda|\beta}\sum_{n=1}^{\infty}[\lambda k_n'(\langle\phi,\phi_n\rangle) - 1]^2|\langle\psi,\phi_n\rangle|^2 \\ &= \frac{1}{1+|\lambda|\beta}\langle T'(\phi)\psi, T'(\phi)\psi\rangle, \quad \forall\psi \in L^2(a,b).\end{aligned}$$

Under Case 1', we have from (2.24) that

$$|\langle T'(\phi)\psi, \psi\rangle| \ge (|\lambda|\alpha - 1)\sum_{n=1}^{\infty}|\langle\psi,\phi_n\rangle|^2 = (|\lambda|\alpha - 1)\|\psi\|^2, \quad \forall\psi \in L^2(a,b),$$

Remark 2.15. Using a surjective result for expansive mappings like Theorem 2.1, the integral equation (2.19) was studied in [4]. The result there was for a given function $f \in L^2(a,b)$ such that $\sum_{n=1}^{\infty}|k_n^{-1}(\langle f,\phi_n\rangle)|^2 < +\infty$, and k_n fulfills (i) and (ii) for each n.

3. A further reflection on "expansive set" fixed point problems

Let X be a topological space and M a subset of X, and let $T : M \to X$ be a mapping. In the preceding paper [19], we illustrated by example that the condition $T(M) \supseteq M$ can not insure the existence of a fixed point of T in M. Observe that the set M there was only assumed closed but not convex. Hence, it did not completely negate the corresponding conclusion for the case $X = \mathbb{R}$ in a higher dimensional space. In the setting $X = \mathbb{R}$, the following result holds:

> Let $[a,b] \subset \mathbb{R}$ and f be a real-valued continuous function on $\mathbb{R}$, then f has a fixed point in $[a,b]$ provided $f([a,b]) \supset [a,b]$.

As a matter of fact, let us modify the example there as follows: let $\theta \in (0, 2\pi)$ be a given constant, and define

$$T(x,y,z) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, \tan z), \quad (x,y,z) \in M,$$

where

$$M = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 \le 1, -\frac{\pi}{4} \le z \le \frac{\pi}{4}\}$$

is a compact and convex set. Then $T : M \to \mathbb{R}^3$ is a continuous but not an expansive mapping, $T(M) \supsetneqq M$ and T has a trivial fixed point $(0,0,0)$ in M. This shows that Theorem 2.1 is just a sufficient condition to ensure the existence of a fixed point of T in M. However, the conclusion of Theorem 2.1 holds regardless of the convexity of M and of the continuity of T. Given that the set M is convex and that $T(M) \supset M$, then what additionally mild conditions imposed on T and M can guarantee that T has at least one fixed point in M? By the well-known Schauder fixed point theorem, it is easily seen that the conditions ensuring the continuity of T^{-1} plus the conditions that M is both compact and convex can induce the desired result. Obviously, the conditions concerning T^{-1} are usually unpopular.

Based on the above observations, one may conjecture the following interesting "reverse" Schauder fixed point problem:

> Let X be a topological space, M a compact and convex subset of X, and $T : M \to X$ a continuous mapping. Does T have a fixed point in M, provided $T(M) \supsetneqq M$?

However, it turns out that this conjecture is an immature one, since it can be easily negated by the following simple example: Let $X = \mathbb{C}, M = [-2, 2] \subset \mathbb{R} \subset X$ and $T : M \to X$,

$$Tx = \begin{cases} 2x + 4, & \text{if } -2 \le x < -1, \\ 2e^{\frac{\pi}{2} i(1+x)}, & \text{if } -1 \le x \le 1, \\ 2x - 4, & \text{if } 1 < x \le 2. \end{cases}$$

Notice that in this counterexample the interior of the compact convex subset M in X is empty; on the other hand, that X is the complex plane. Hence, two versions of a more mature conjecture may be formulated as follows:

> **Problem 2.** Let X be a topological space, M a compact and convex subset of X with $T : M \to X$ a (continuous) mapping. Does T have a fixed point in M, provided $T(M) \supsetneqq M$?
>
> **Problem 3.** Let X be a topological space, M a compact and convex subset of X, and $T : M \to X$ a (continuous) mapping. Does T have a fixed point in M, provided $T(M) \supsetneqq M$?

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