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AN APPROXIMATION PROPERTY OF GAUSSIAN FUNCTIONS

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Abstract. Using the power series method, we solve the inhomogeneous linear first order differential equation

$$y'(x) + \lambda(x-\mu)y(x) = \sum_{m=0}^{\infty} a_m (x-\mu)^m,$$

and prove an approximation property of Gaussian functions.

1. Introduction

Let Y and I be a normed space and an open subinterval of $\mathbb{R}$, respectively. If for any function $f : I \to Y$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \le \varepsilon$$

for all $x \in I$ and for some $\varepsilon \ge 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0$$

such that $\|f(x) - f_0(x)\| \le K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ depends on ε only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain I is not the whole space $\mathbb{R}$). We may apply these terminologies for other differential equations. For a more detailed definition of the Hyers-Ulam stability, refer to [2, 3, 5].

Obłoza seems to be the first author who investigated the Hyers-Ulam stability of linear differential equations (see [9, 10]). Here, we introduce a result of Alsina and Ger (see [1]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \le \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \le 3\varepsilon$ for any $x \in I$. This result of Alsina and Ger was generalized by Takahasi, Miura and Miyajima: They proved in [12] that the Hyers-Ulam stability holds for the Banach space valued differential equation $y'(x) = \lambda y(x)$ (see also [7, 8, 11]).

Using the conventional power series method, the first author investigated the general solution of the inhomogeneous linear first order differential equations of the

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form,

$$y'(x) - \lambda y(x) = \sum_{m=0}^{\infty} a_m (x-c)^m,$$

where λ is a complex number and the convergence radius of the power series is positive. This result was applied for proving an approximation property of exponential functions in a neighborhood of c (see [4]).

Throughout this paper, we assume that ρ is a positive real number or infinity. In §2 of this paper, using an idea from [4], we will investigate the general solution of the inhomogeneous linear differential equation of the first order,

$$y'(x) + \lambda(x-\mu)y(x) = \sum_{m=0}^{\infty} a_m (x-\mu)^m, \tag{1.1}$$

where the coefficients a_m of the power series are given such that the radius of convergence is at least ρ. Moreover, we prove the (local) Hyers-Ulam stability of linear first order differential equation (2.1) in a class of special analytic functions.

2. General Solution of (1.1)

The linear first order differential equation

$$y'(x) + \lambda(x-\mu)y(x) = 0 \tag{2.1}$$

has a general solution of the form $y(x) = c\exp\big\{-\frac{\lambda}{2}(x-\mu)^2\big\}$, which is called a Gaussian function. We recall that ρ is a positive real number or infinity.

Theorem 2.1. *Let $\lambda \neq 0$ and μ be a complex number and a real number, respectively. Assume that the radius of convergence of power series $\sum_{m=0}^{\infty} a_m(x-\mu)^m$ is at least ρ. Every solution $y : (\mu-\rho, \mu+\rho) \to \mathbb{C}$ of the inhomogeneous differential equation (1.1) can be expressed as*

$$y(x) = y_h(x) + \sum_{m=0}^{\infty} c_m (x-\mu)^m, \tag{2.2}$$

where the coefficients c_m are given by

$$c_{2m} = \sum_{i=0}^{m-1} (-1)^i \frac{a_{2m-1-2i}}{\lambda} \prod_{k=0}^{i} \frac{\lambda}{2m-2k} + (-1)^m c_0 \prod_{k=0}^{m-1} \frac{\lambda}{2m-2k}, \tag{2.3}$$

$$c_{2m+1} = \sum_{i=0}^{m-1} (-1)^i \frac{a_{2m-2i}}{\lambda} \prod_{k=0}^{i} \frac{\lambda}{2m+1-2k} + (-1)^m c_1 \prod_{k=0}^{m-1} \frac{\lambda}{2m+1-2k} \tag{2.4}$$

for each $m \in \mathbb{N}_0$, and $y_h(x)$ is a solution of the corresponding homogeneous differential equation (2.1).

Proof. Since each solution of (1.1) can be expressed as a power series in $x-\mu$, we put $y(x) = \sum_{m=0}^{\infty} c_m(x-\mu)^m$ in (1.1) to obtain

$$\begin{aligned} y'(x) + \lambda(x-\mu)y(x) &= c_1 + \sum_{m=0}^{\infty} (m+2)c_{m+2}(x-\mu)^{m+1} + \sum_{m=0}^{\infty} \lambda c_m (x-\mu)^{m+1} \\ &= c_1 + \sum_{m=0}^{\infty} \big[(m+2)c_{m+2} + \lambda c_m\big](x-\mu)^{m+1} \end{aligned}$$

$$= a_0 + \sum_{m=0}^{\infty} a_{m+1}(x-\mu)^{m+1},$$

from which we obtain the following recurrence formula

$$\begin{gathered} c_1 = a_0, \\ (m+2)c_{m+2} + \lambda c_m = a_{m+1} \quad (m \in \mathbb{N}_0). \end{gathered} \tag{2.5}$$

We will now prove the formula (2.3) for any $m \in \mathbb{N}_0$: If we set $m = 0$ in (2.3), then we get $c_0 = c_0$ which is true. We assume that the formula (2.3) is true for some $m \in \mathbb{N}_0$. Then, it follows from (2.5) and the induction hypothesis that

$$\begin{aligned}
&c_{2m+2} \\
&= \frac{a_{2m+1}}{2m+2} - \frac{\lambda}{2m+2} c_{2m} \\
&= \frac{a_{2m+1}}{2m+2} - \frac{\lambda}{2m+2} \Big[\sum_{i=0}^{m-1} (-1)^i \frac{a_{2m-1-2i}}{\lambda} \prod_{k=0}^{i} \frac{\lambda}{2m-2k} + (-1)^m c_0 \prod_{k=0}^{m-1} \frac{\lambda}{2m-2k} \Big] \\
&= \frac{a_{2m+1}}{2m+2} + \sum_{i=0}^{m-1} (-1)^{i+1} \frac{a_{2m-1-2i}}{\lambda} \prod_{k=-1}^{i} \frac{\lambda}{2m-2k} + (-1)^{m+1} c_0 \prod_{k=-1}^{m-1} \frac{\lambda}{2m-2k} \\
&= \frac{a_{2m+1}}{2m+2} + \sum_{i=0}^{m-1} (-1)^{i+1} \frac{a_{2m-1-2i}}{\lambda} \prod_{k=0}^{i+1} \frac{\lambda}{2m+2-2k} + (-1)^{m+1} c_0 \prod_{k=0}^{m} \frac{\lambda}{2m+2-2k} \\
&= \frac{a_{2m+1}}{2m+2} + \sum_{i=1}^{m} (-1)^{i} \frac{a_{2m+1-2i}}{\lambda} \prod_{k=0}^{i} \frac{\lambda}{2(m+1)-2k} + (-1)^{m+1} c_0 \prod_{k=0}^{m} \frac{\lambda}{2(m+1)-2k} \\
&= \sum_{i=0}^{m} (-1)^{i} \frac{a_{2m+1-2i}}{\lambda} \prod_{k=0}^{i} \frac{\lambda}{2(m+1)-2k} + (-1)^{m+1} c_0 \prod_{k=0}^{m} \frac{\lambda}{2(m+1)-2k},
\end{aligned}$$

which can be obtained provided we replace m in (2.3) with $m+1$. Hence, we conclude that the formula (2.3) is true for all $m \in \mathbb{N}_0$. Similarly, we can also prove the validity of (2.4) for all $m \in \mathbb{N}_0$.

Indeed, in view of (2.5), $y_p(x) = \sum_{m=0}^{\infty} c_m(x-\mu)^m$ is a solution of the inhomogeneous linear differential equation (1.1). Since every solution of Eq. (1.1) is a sum of a solution $y_h(x)$ of the corresponding homogeneous equation and a particular solution $y_p(x)$ of the inhomogeneous equation, it can be expressed by (2.2).

The formulas (2.3) and (2.4) can be merged in a new one:

$$c_m = \sum_{i=0}^{[m/2]-1} (-1)^i \frac{a_{m-1-2i}}{\lambda} \prod_{k=0}^{i} \frac{\lambda}{m-2k} + (-1)^{[m/2]} c_{0,1} \prod_{k=0}^{[m/2]-1} \frac{\lambda}{m-2k} \tag{2.6}$$

for all $m \in \mathbb{N}_0$, where $c_{0,1} = c_0$ for m even, $c_{0,1} = c_1$ for m odd, and $[m/2]$ denotes the largest integer not exceeding $m/2$. Let us define

$$C := \max \Big\{ \frac{1}{|\lambda|} \prod_{k=0}^{i} \frac{|\lambda|}{m-2k} \mid m \in \mathbb{N}_0;\ i \in \{0, 1, \ldots, [m/2]-1\} \Big\}.$$

For any $\varepsilon > 0$, we can choose an (sufficiently large) integer m_ε such that

$$\prod_{k=0}^{[m/2]-1} \frac{|\lambda|}{m-2k} \le \varepsilon$$

for all integers $m \ge m_\varepsilon$. Thus, in view of (2.6), there exists a constant $D > 0$ such that

$$|c_m| \le (C + D) \sum_{i=0}^{m-1} |a_i| \tag{2.7}$$

for all sufficiently large integers m. (Since the inhomogeneous term $\sum_{m=0}^{\infty} a_m(x - \mu)^m$ has to be nonzero for some $x \in (\mu - \rho, \mu + \rho)$, there exists an $m_0 \in \mathbb{N}_0$ such that $a_{m_0} \neq 0$ and hence, $\sum_{i=0}^{m-1} |a_i| > 0$ for all sufficiently large integer m.)

Finally, it follows from (2.7) and [6, Problem 8.8.1 (p)] that

$$\begin{aligned}
\limsup_{m\to\infty} |c_m|^{1/m} &= \limsup_{m\to\infty} \Big(\frac{1}{m}|c_m|\Big)^{1/m} \\
&\le \limsup_{m\to\infty} \Big(\frac{C+D}{m} \sum_{i=0}^{m-1} |a_i|\Big)^{1/m} \\
&\le \limsup_{m\to\infty} |a_m|^{1/m}.
\end{aligned}$$

By use of the Cauchy-Hadamard theorem (see [6, Theorem 8.8.2]), the radius of convergence of the power series for $y_p(x)$ is at least ρ. Therefore, $y(x)$ in Eq. (2.2) is well defined on $(\mu - \rho, \mu + \rho)$. □

Remark 2.2. We notice that Theorem 2.1 is true if we set $c_0 = 0$.

3. Local Hyers-Ulam stability of (2.1)

Let ρ be a positive real number or the infinity. We denote by $\widetilde{C}$ the set of all functions $f : (\mu - \rho, \mu + \rho) \to \mathbb{C}$ with the following properties:

(a) $f(x)$ is expressible by a power series $\sum_{m=0}^{\infty} b_m(x - \mu)^m$ whose radius of convergence is at least ρ;

(b) There exists a constant $K \ge 0$ such that

$$\sum_{m=0}^{\infty} |a_m(x - \mu)^m| \le K \Big| \sum_{m=0}^{\infty} a_m(x - \mu)^m \Big|$$

for all $x \in (\mu - \rho, \mu + \rho)$, where $a_0 = b_1$ and $a_m = (m + 1)b_{m+1} + \lambda b_{m-1}$ for any $m \in \mathbb{N}$.

If we define

$$(y_1 + y_2)(x) = y_1(x) + y_2(x) \quad \text{and} \quad (\lambda y_1)(x) = \lambda y_1(x)$$

for all $y_1, y_2 \in \widetilde{C}$ and $\lambda \in \mathbb{C}$, then $\widetilde{C}$ is a vector space over complex numbers. We remark that the set $\widetilde{C}$ is large enough to be a vector space.

We investigate an approximation property of Gaussian functions. More precisely, we prove the (local) Hyers-Ulam stability of the linear first order differential equation (2.1) for the functions in $\widetilde{C}$.

Theorem 3.1. *Let $\lambda \neq 0$ and μ be a complex number and a real number, respectively. If a function $y \in \widetilde{C}$ satisfies the differential inequality*

$$\big|y'(x) + \lambda(x - \mu)y(x)\big| \le \varepsilon \tag{3.1}$$

for all $x \in (\mu-\rho, \mu+\rho)$ *and for some* $\varepsilon \ge 0$*, then there exists a solution* $y_h : (\mu-\rho, \mu+\rho) \to \mathbb{C}$ *of the differential equation* (2.1) *such that*

$$\big|y(x) - y_h(x)\big| \le \Big(|b_1| \exp\big\{\frac{|\lambda|}{2}(x-\mu)^2\big\} + \frac{K\varepsilon}{2} \frac{\exp\big\{\frac{|\lambda|}{2}(x-\mu)^2\big\} - 1}{\frac{|\lambda|}{2}(x-\mu)^2}\Big)|x-\mu|$$

for any $x \in (\mu-\rho, \mu+\rho)$*. In particular, it holds that* $y_h \in \widetilde{C}$*.*

Proof. Since y belongs to $\widetilde{C}$, $y(x)$ can be expressed by $y(x) = \sum_{m=0}^{\infty} b_m (x-\mu)^m$ and it follows from (a) and (b) that

$$\begin{aligned} &y'(x) + \lambda(x-\mu)y(x) \\ &= b_1 + \sum_{m=0}^{\infty} (m+2)b_{m+2}(x-\mu)^{m+1} + \sum_{m=0}^{\infty} \lambda b_m (x-\mu)^{m+1} \\ &= b_1 + \sum_{m=0}^{\infty} \big[(m+2)b_{m+2} + \lambda b_m\big](x-\mu)^{m+1} \\ &= \sum_{m=0}^{\infty} a_m (x-\mu)^m \end{aligned} \tag{3.2}$$

for all $x \in (\mu-\rho, \mu+\rho)$. By considering (3.1) and (3.2), we have

$$\Big|\sum_{m=0}^{\infty} a_m (x-\mu)^m\Big| \le \varepsilon$$

for any $x \in (\mu-\rho, \mu+\rho)$. This inequality, together with (b), yields

$$\sum_{m=0}^{\infty} \big|a_m (x-\mu)^m\big| \le K \Big|\sum_{m=0}^{\infty} a_m (x-\mu)^m\Big| \le K\varepsilon \tag{3.3}$$

for all $x \in (\mu-\rho, \mu+\rho)$.

Now, it follows from Theorem 2.1, (2.6), (3.2), and (3.3) that there exists a solution $y_h : (\mu-\rho, \mu+\rho) \to \mathbb{C}$ of the differential equation (2.1) such that

$$\begin{aligned} &\big|y(x) - y_h(x)\big| \\ &\le \sum_{m=0}^{\infty} |c_m||x-\mu|^m \ \le \ |c_0| + |c_1||x-\mu| + \sum_{m=2}^{\infty} |c_m||x-\mu|^m \\ &\le |c_0| + |c_1||x-\mu| + \sum_{m=2}^{\infty} \sum_{i=0}^{[m/2]-1} \frac{|a_{m-2i-1}(x-\mu)^{m-2i-1}|}{|\lambda(x-\mu)|} \prod_{k=0}^{i} \frac{|\lambda(x-\mu)^2|}{m-2k} \\ &\quad + \sum_{m=2}^{\infty} |c_{0,1}||x-\mu|^{m-2[m/2]} \prod_{k=0}^{[m/2]-1} \frac{|\lambda(x-\mu)^2|}{m-2k} \\ &\le |c_0| + |c_1||x-\mu| + \sum_{m=2}^{\infty} \frac{|a_{m-1}(x-\mu)^{m-1}|}{|\lambda(x-\mu)|} \frac{|\lambda(x-\mu)^2|}{m} \\ &\quad + \sum_{m=4}^{\infty} \frac{|a_{m-3}(x-\mu)^{m-3}|}{|\lambda(x-\mu)|} \frac{|\lambda(x-\mu)^2|}{m} \frac{|\lambda(x-\mu)^2|}{m-2} \\ &\quad + \sum_{m=6}^{\infty} \frac{|a_{m-5}(x-\mu)^{m-5}|}{|\lambda(x-\mu)|} \frac{|\lambda(x-\mu)^2|}{m} \frac{|\lambda(x-\mu)^2|}{m-2} \frac{|\lambda(x-\mu)^2|}{m-4} + \dots \end{aligned}$$

$$
\begin{aligned}
&+ |c_0| \frac{|\lambda(x-\mu)^2|}{2} + |c_1||x-\mu| \frac{|\lambda(x-\mu)^2|}{3} + |c_0| \frac{|\lambda(x-\mu)^2|}{4} \frac{|\lambda(x-\mu)^2|}{2} \\
&+ |c_1||x-\mu| \frac{|\lambda(x-\mu)^2|}{5} \frac{|\lambda(x-\mu)^2|}{3} + |c_0| \frac{|\lambda(x-\mu)^2|}{6} \frac{|\lambda(x-\mu)^2|}{4} \frac{|\lambda(x-\mu)^2|}{2} \\
&+ |c_1||x-\mu| \frac{|\lambda(x-\mu)^2|}{7} \frac{|\lambda(x-\mu)^2|}{5} \frac{|\lambda(x-\mu)^2|}{3} + \cdots \\
\le{}& K\varepsilon \Big(\frac{|x-\mu|}{2} + \frac{|\lambda(x-\mu)^3|}{4\cdot 2} + \frac{|\lambda^2(x-\mu)^5|}{6\cdot 4\cdot 2} + \cdots \Big) \\
&+ |c_0| \Big(1 + \frac{|\lambda(x-\mu)^2|}{2} + \frac{|\lambda(x-\mu)^2|^2}{4\cdot 2} + \frac{|\lambda(x-\mu)^2|^3}{6\cdot 4\cdot 2} + \cdots \Big) \\
&+ |c_1||x-\mu| \Big(1 + \frac{|\lambda(x-\mu)^2|}{3} + \frac{|\lambda(x-\mu)^2|^2}{5\cdot 3} + \frac{|\lambda(x-\mu)^2|^3}{7\cdot 5\cdot 3} + \dots \Big)
\end{aligned}
$$

for all $x \in (\mu-\rho, \mu+\rho)$, where $c_{0,1} = c_0$ for m even, $c_{0,1} = c_1$ for m odd.

In view of (2.5), Remark 2.2, and (b), we know that $y_p(x) = b_1(x-\mu) + \sum_{m=2}^{\infty} c_m (x-\mu)^m$ is a particular solution of the inhomogeneous differential equation (1.1), i.e., we can set $c_0 = 0$ and $c_1 = b_1$ in Theorem 2.1. Hence, we obtain

$$
\begin{aligned}
&\big| y(x) - y_h(x) \big| \\
&\le |c_0| + |c_1||x-\mu| + \Big(\frac{K\varepsilon}{|\lambda(x-\mu)|} + |c_0| + |c_1||x-\mu| \Big) \sum_{i=1}^{\infty} \frac{|\lambda(x-\mu)^2|^i}{2^i i!} \\
&= |b_1||x-\mu| + \Big(\frac{K\varepsilon}{|\lambda(x-\mu)|} + |b_1||x-\mu| \Big) \sum_{i=1}^{\infty} \frac{1}{i!} \Big| \frac{\lambda}{2}(x-\mu)^2 \Big|^i \\
&= \Big(|b_1| \exp\Big\{ \frac{|\lambda|}{2}(x-\mu)^2 \Big\} + \frac{K\varepsilon}{2} \frac{\exp\Big\{ \frac{|\lambda|}{2}(x-\mu)^2 \Big\} - 1}{\frac{|\lambda|}{2}(x-\mu)^2} \Big) |x-\mu|
\end{aligned}
$$

for any $x \in (\mu-\rho, \mu+\rho)$.

As we already remarked, there exists a real number c such that

$$
y_h(x) = c \exp\big\{ -\frac{\lambda}{2}(x-\mu)^2 \big\}.
$$

Hence, $y_h(x)$ has a power series expansion in $x-\mu$, namely,

$$
y_h(x) = \sum_{m=0}^{\infty} b_m^*(x-\mu)^m, \tag{3.4}
$$

where

$$
b_{2m}^* = (-1)^m \frac{c}{m!} \Big(\frac{\lambda}{2} \Big)^m \quad \text{and} \quad b_{2m+1}^* = 0
$$

for all $m \in \mathbb{N}_0$. The radius of convergence of the power series (3.4) is infinity.

It follows from (b) that $a_0^* = b_1^* = 0$ and

$$
a_{2m}^* = (2m+1)b_{2m+1}^* + \lambda b_{2m-1}^* = 0
$$

for every $m \in \mathbb{N}$. Moreover, we have

$$
\begin{aligned}
a_{2m+1}^* &= (2m+2)b_{2m+2}^* + \lambda b_{2m}^* \\
&= (2m+2)(-1)^{m+1} \frac{c}{(m+1)!} \Big(\frac{\lambda}{2} \Big)^{m+1} + \lambda(-1)^m \frac{c}{m!} \Big(\frac{\lambda}{2} \Big)^m = 0
\end{aligned}
$$

for all $m \in \mathbb{N}_0$, i.e., $a_m^* = 0$ for all $m \in \mathbb{N}_0$. Therefore, $y_h(x) = c\exp\big\{-\frac{\lambda}{2}(x-\mu)^2\big\}$ satisfies both conditions (a) and (b). That is, y_h belongs to $\widetilde{C}$. □

According to the previous theorem, each approximate solution of the differential equation (2.1) can be well approximated by a Gaussian function in a (small) neighborhood of μ. More precisely, by applying l'Hospital's rule, we can easily prove the following corollary.

Corollary 3.2. *Let $\lambda \neq 0$ and μ be a complex number and a real number, respectively. If a function $y \in \widetilde{C}$ satisfies the differential inequality* (3.1) *for all $x \in (\mu-\rho, \mu+\rho)$ and for some $\varepsilon \geq 0$, then there exists a complex number c such that*
$$\Big|y(x) - c\exp\Big\{-\frac{\lambda}{2}(x-\mu)^2\Big\}\Big| = O\big(|x-\mu|\big) \quad \text{as} \quad x \to \mu,$$
where $O(\cdot)$ denotes the Landau symbol (*big-O*).

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