Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 05, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR CRITICAL HÉNON EQUATIONS IN HYPERBOLIC SPACES

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Abstract. In this article, we use variational methods to prove that for a suitable value of λ, the problem

$$-\Delta_{\mathbb{B}^N} u = (d(x))^\alpha |u|^{2^*-2}u + \lambda u, \quad u \geq 0, \quad u \in H_0^1(\Omega')$$

possesses at least one non-trivial solution u as $\alpha \to 0^+$, where Ω' is a bounded domain in Hyperbolic space $\mathbb{B}^N$, $d(x) = d_{\mathbb{B}^N}(0,x)$. $\Delta_{\mathbb{B}^N}$ denotes the Laplace-Beltrami operator on $\mathbb{B}^N$, $N \geq 4$, $2^* = 2N/(N-2)$.

1. Introduction and statement of main result

In this article, we study the existence of non-trivial solution for the problem

$$-\Delta_{\mathbb{B}^N} u = (d(x))^\alpha |u|^{2^*-2}u + \lambda u, \quad u \geq 0, u \in H_0^1(\Omega') \tag{1.1}$$

where $N \geq 4$,

$$\frac{N(N-2)}{4} < \lambda < \lambda_1, \quad 2^* = \frac{2N}{N-2},$$

$d(x) = d_{\mathbb{B}^N}(0,x)$. Here $\Delta_{\mathbb{B}^N}$ denotes the Laplace Beltrami operator on $\mathbb{B}^N$. We denote by λ_1 is the first eigenvalue of the Laplace-Beltrami operator with Dirichlet boundary conditions. The domain Ω' is a bounded domain with an interior sphere condition, $0 \in \Omega' \subset \mathbb{B}^N$, $\Omega \subset B_1(0)$ and $\bar{\Omega} \cap \partial B_1(0) \neq 0$, where $B_1(0) \subset \mathbb{B}^N$ is the geodesic ball with radius 1.

When posed in the Euclidean space $\mathbb{R}^N$, problem (1.1) is a generalization of the celebrated Brezis-Nirenberg problem

$$-\Delta u = |x|^\alpha |u|^{2^*-2}u + \lambda u, \quad u \geq 0, u \in H_0^1(\Omega), \tag{1.2}$$

see [1, 8, 9, 12, 13] for more general and recent existence results. In spaces of constant curvature it has been studied by Bandle, Brillard and Flucher [3]. The special case of S^3 has been treated in [2].

When $\alpha \neq 0$ and $\lambda = 0$, problem (1.2) is known as the Hénon equation

$$-\Delta u = |x|^\alpha |u|^{p-2}u, \quad u \geq 0, u \in H_0^1(\Omega) \tag{1.3}$$

and the study goes to Hénon [14], Ni [17], Smets [19], Cao-Peng [10] and others. Attention was focused on the existence and multiplicity of nonradial solutions

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2000 *Mathematics Subject Classification.* 58J05, 35J60.
Key words and phrases. Hénon equations; mountain pass theorem; critical growth; hyperbolic space.

Submitted June 13, 2012. Published January 8, 2013.

for critical, supercritical and slightly subcritical growth, symmetry properties and asymptotic behavior of ground states (for $p \to \frac{2N}{N-2}$, or $\alpha \to \infty$). We refer to [4, 6, 7, 11, 15] for more information. As far as we know, the Brezis-Nirenberg problem for the critical Hénon equation has been studied only in [16], where the authors prove that there always exists a solution to (1.3), provided α is small enough.

In the hyperbolic space, the existence of Brezis-Nirenberg problem for the critical equation

$$-\Delta_{\mathbb{B}^N} u = |u|^{2^*-2}u + \lambda u, \quad u \geq 0, u \in H_0^1(\Omega) \tag{1.4}$$

has been studied in [20] and the results are very similar to the results in the Euclidean case. However, for problem (1.1), there exists some difference from Euclidean space. Firstly, the weight function $d(x)$ depends on the Riemannian distance r from a pole o. Secondly, there is a lack of compactness due to the fact that the Sobolev imbedding $H_0^1(\Omega') \hookrightarrow L^{2^*}(\Omega')$ is noncompact, so the functional of problem (1.1) cannot satisfy the $(PS)_c$ condition for all $c > 0$. In generally, to prove the functional of problem (1.1) satisfying the local $(PS)_c$ condition, we need to use the unique positive solution of the problem

$$-\Delta_{\mathbb{B}^N} u = u^{2^*-1} \quad \text{in } \mathbb{B}^N. \tag{1.5}$$

to control the energy of the functional. However, Mancini and Sandeep [5] proved that (1.5) did not have any positive solutions. Thirdly, when we study the critical elliptic problem

$$-\Delta_{\mathbb{B}^N} u = Q(x)u^{2^*-1} + \lambda u,\ x \in \Omega', \quad u = 0,\ x \in \partial\Omega', \tag{1.6}$$

it is necessary that the function $Q(x)$ have the maximum in Ω'. But the weight function $d(x)^\alpha$ of problem (1.1) has the maximum on $\partial\Omega'$. So we have the difficulty to control the energy. Our main result is as follows.

Theorem 1.1. *There exists $\bar{\alpha} > 0$, such that when $0 < \alpha < \bar{\alpha}$, problem (1.1) has at least one non-trivial positive solution.*

The proof of this result will be given in Section 3. In section 2, we give some basic facts about hyperbolic space and prove that the functional of problem (1.1) satisfies the local $(PS)_c$ condition.

2. Preliminaries

A hyperbolic space, denoted by $\mathbb{H}^N$, is a complete simple connected Riemannian manifold which has constant sectional curvature equal to -1. There are several models for hyperbolic space, and we will use the Poincaré ball model

$$\mathbb{B}^N = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^N : |x| < 1\}$$

endowed with Riemannian metric $g_{ij} = (p(x))^2\delta_{ij}$ where $p(x) = \frac{2}{1-|x|^2}$. We denote the hyperbolic volume by $dV_{\mathbb{B}^N}$ and is given by $dV_{\mathbb{B}^N} = (p(x))^N\,dx$. The hyperbolic gradient and the Laplace Beltrami operator are:

$$\Delta_{\mathbb{B}^N} = (p(x))^{-N}\operatorname{div}((p(x))^{N-2}\nabla u)), \quad \nabla_{\mathbb{B}^N} u = \frac{\nabla u}{p(x)}$$

where ∇ and div denotes the Euclidean gradient and divergence in $\mathbb{R}^N$, respectively.

The hyperbolic distance $d_{\mathbb{B}^N}(x,y)$ between $x,y\in\mathbb{B}^N$ in the Poincaré ball model is
$$d_{\mathbb{B}^N}(x,y)=\operatorname{arccosh}\Big(1+\frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\Big).$$
From this we immediately obtain that for $x\in\mathbb{B}^N$,
$$d(x)=d_{\mathbb{B}^N}(0,x)=\log\Big(\frac{1+|x|}{1-|x|}\Big).$$
Let us denote the energy functional corresponding to (1.1) by
$$I(u)=\frac{1}{2}\int_{\Omega'}(|\nabla_{\mathbb{B}^N}u|^2-\lambda u^2)dV_{\mathbb{B}^N}-\frac{1}{2^*}\int_{\Omega'}|d(x)|^{\alpha}(u^+)^{2^*}dV_{\mathbb{B}^N} \tag{2.1}$$
defined on $H_0^1(\Omega')$. If $\lambda<\lambda_1$, we know that
$$\|u\|_\lambda:=\Big[\int_{\Omega'}(|\nabla_{\mathbb{B}^N}u|^2-\lambda u^2)dV_{\mathbb{B}^N}\Big]^{1/2}$$
is a norm equivalent to the $H_0^1(\Omega')$ norm, and it is known that critical points of the functional $I\in C^1(H_0^1(\Omega'),\mathbb{R})$ correspond to solutions of (1.1). If u is a nontrivial solution of (1.1), we define
$$v(x)=\Big(\frac{2}{1-|x|^2}\Big)^{(N-2)/2}u$$
which is a nontrivial solution of the Euclidean equation
$$-\Delta v+\frac{N(N-2)}{4}p^2v=\Big(\ln\frac{1+|x|}{1-|x|}\Big)^{\alpha}|v|^{2^*-2}v+\lambda p^2v,\quad x\in\Omega;\quad v\geq 0,\ v\in H_0^1(\Omega), \tag{2.2}$$
where $\Omega\subset\mathbb{R}^N$ is the stereographic projection of Ω' into $\mathbb{R}^N$, and $\bar\Omega\cap\partial B_{\frac{e-1}{e+1}}(0)\neq\emptyset$, $B_{\frac{e-1}{e+1}}(0)$ is a ball in the Euclidean space. Let us define the energy functional corresponding to (2.2) by
$$\begin{aligned}J(v)=&\frac{1}{2}\int_{\Omega}|\nabla v|^2-\Big(\lambda-\frac{N(N-2)}{4}\Big)\Big(\frac{2}{1-|x|^2}\Big)^2v^2\,dx\\&-\frac{1}{2^*}\int_{\Omega}\Big|\ln\frac{1+|x|}{1-|x|}\Big|^{\alpha}(v^+)^{2^*}\,dx.\end{aligned} \tag{2.3}$$
Thus for any $u\in H^1(\Omega)$ if $\tilde u$ is defined as $\tilde u=(\frac{2}{1-|x|^2})^{(N-2)/2}u$, then $I(u)=J(\tilde u)$. Moreover $\langle I'(u),v\rangle=\langle J'(\tilde u),\tilde v\rangle$ where $\tilde v$ is defined in the same way.

Now, we want to prove that the functional I satisfies the $(PS)_c$ condition. It is well known that the best Sobolev constant
$$S=\inf\Big\{\int_{\mathbb{R}^N}|\nabla u|^2\,dx:u\in\mathcal{D}^{1,2}(\mathbb{R}^N),\int_{\mathbb{R}^N}|u|^{2^*}\,dx=1\Big\}$$
is attained by the function
$$U(x)=\frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}},$$
which is a solution of the problem
$$-\Delta u=|u|^{2^*-2}u,\quad x\in\mathbb{R}^N \tag{2.4}$$
with $\int_{\mathbb{R}^N}|\nabla U|^2=\int_{\mathbb{R}^N}U^{2^*}dx=S^{N/2}$.

Lemma 2.1. *For all $c \in (0, S^{N/2}/N)$, the function $I(u)$ satisfies the $(PS)_c$ condition.*

Proof. Suppose $c \in (0, S^{N/2}/N), \{u_n\} \subset H_0^1(\Omega')$ is the $(PS)_c$ sequence of the function $I(u)$, then $I(u_n) \to c$ as $n \to \infty, I'(u_n) \to 0$ as $n \to \infty$. We have that

$$\begin{aligned}
c + 1 + \|u_n\|_{\mathbb{B}^N} &\geq I(u_n) - \frac{1}{2^*}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{2}\int_{\Omega'}(|\nabla_{\mathbb{B}^N} u_n|^2 - \lambda u_n^2)dV_{\mathbb{B}^N} - \frac{1}{2^*}\int_{\Omega'} d(x)^\alpha (u_n^+)^{2^*} dV_{\mathbb{B}^N} \\
&\quad - \frac{1}{2^*}\int_{\Omega'}(|\nabla_{\mathbb{B}^N} u_n|^2 - \lambda u_n^2)dV_{\mathbb{B}^N} + \frac{1}{2^*}\int_{\Omega'} d(x)^\alpha (u_n^+)^{2^*} dV_{\mathbb{B}^N} \\
&= (\frac{1}{2} - \frac{1}{2^*})\int_{\Omega}(|\nabla_{\mathbb{B}^N} u_n|^2 - \lambda u_n^2)dV_{\mathbb{B}^N} \\
&= (\frac{1}{2} - \frac{1}{2^*})\|u_n\|^2_{\mathbb{B}^N}.
\end{aligned}$$

It follows that $\|u_n\|$ is bounded in $H_0^1(\Omega')$. It implies that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{for } x \in \ H_0^1(\Omega'), \\
u_n \to u \quad \text{for } x \in L^p(\Omega'),\ 2 < p < 2^* \\
u_n \to u \quad \text{a.e. on } \Omega',
\end{gathered} \tag{2.5}$$

From $\{u_n^+\}$ being bounded in $L^{2^*}(\Omega')$, it follows that $\{(u_n^+)^{2^*-1}\}$ is bounded in $L^{\frac{2N}{N+2}}(\Omega')$. It follows that $\{d(x)^\alpha (u_n^+)^{2^*-1}\}$ is bounded in $L^{\frac{2N}{N+2}}(\Omega')$ and

$$d(x)^\alpha (u_n^+)^{2^*-1} \rightharpoonup d(x)^\alpha (u^+)^{2^*-1}, \quad \text{in } L^{\frac{2N}{N+2}}(\Omega').$$

So u is the solution of problem (1.1) and

$$I(u) = \frac{1}{2}\|u\|^2_{\mathbb{B}^N} - \frac{1}{2^*}\int_{\Omega'} d(x)^\alpha (u^+)^{2^*} dV_{\mathbb{B}^N} = \frac{1}{N}\int_{\Omega'} d(x)^\alpha (u^+)^{2^*} dV_{\mathbb{B}^N} \geq 0.$$

Let us define $v_n = u_n - u$. The Brézis-Lieb Lemma leads to

$$|u_n|^{2^*}_{2^*} = |u_n - u|^{2^*}_{2^*} + |u|^{2^*}_{2^*} + o(1),$$

and

$$\int_{\Omega'} d(x)^\alpha (u_n^+)^{2^*} dV_{\mathbb{B}^N} = \int_{\Omega'} d(x)^\alpha [(u_n - u)^+]^{2^*} dV_{\mathbb{B}^N} + \int_{\Omega'} d(x)^\alpha (u^+)^{2^*} dV_{\mathbb{B}^N} + o(1).$$

So we have

$$\begin{aligned}
I(u_n) &= \frac{1}{2}\|u_n\|^2_{\mathbb{B}^N} - \frac{1}{2^*}\int_{\Omega'} d(x)^\alpha (u_n^+)^{2^*} dV_{\mathbb{B}^N} \\
&= I(u) + \frac{1}{2}\|v_n\|^2_{\mathbb{B}^N} - \frac{1}{2^*}\int_{\Omega} d(x)^\alpha (v_n^+)^{2^*} dV_{\mathbb{B}^N} + o(1) \to c.
\end{aligned}$$

Since $\|u_n\|$ is bounded in $H_0^1(\Omega')$, and $\langle I'(u_n), u_n\rangle \to 0$, we obtain

$$\|v_n\|^2_{\mathbb{B}^N} - \int_{\Omega'} d(x)^\alpha (v_n^+)^{2^*} dV_{\mathbb{B}^N} = \langle I'(u_n), u_n\rangle - \langle I'(u), u\rangle \to -\langle I'(u), u\rangle = 0.$$

It implies that

$$\|v_n\|^2_{\mathbb{B}^N} - \int_{\Omega'} d(x)^\alpha (v_n^+)^{2^*} dV_{\mathbb{B}^N} \to 0,$$

So we can assume that

$$\|v_n\|_{\mathbb{B}^N}^2 \to b, \quad \int_{\Omega'} d(x)^\alpha (v_n^+)^{2^*} dV_{\mathbb{B}^N} \to b.$$

Since $u_n \to u$, in $L^2(\Omega')$, it follows that $v_n = u_n - u \to 0$ in $L^2(\Omega')$, and

$$\int_{\Omega'} |\nabla_{\mathbb{B}^N} v_n|^2 \; dV_{\mathbb{B}^N} \to b.$$

We know that when we consider the above calculation in the Hyperbolic space, the Sobolev inequality is not satisfied; so we should transfer it into the Euclidean space. Let us define:

$$v_n(x) := p^{-\frac{N-2}{2}} w_n(x), \quad p = \frac{2}{1-|x|^2}.$$

Then

$$\begin{aligned} \int_{\Omega'} |\nabla_{\mathbb{B}^N} v_n|^2 dV_{\mathbb{B}^N} &= \int_{\Omega} |\nabla (p^{-\frac{N-2}{2}} w_n)|^2 p^{N-2} dx \\ &= \int_{\Omega} |\nabla w_n|^2 dx + \frac{N(N-2)}{4} \int_{\Omega} p^2 w_n^2 dx \end{aligned}$$

and

$$\int_{\Omega'} d(x)^\alpha (v_n^+)^{2^*} dV_{\mathbb{B}^N} = \int_{\Omega} |\ln \frac{1+|x|}{1-|x|}|^\alpha (w_n^+)^{2^*} dx \to b \quad \text{as } n \to \infty.$$

From $v_n \to 0$ in $L^2(\Omega')$, and

$$2 \le p(x) \le \frac{(e+1)^2}{2e},$$

we have that $\int_\Omega p^2 w_n^2 dx \to 0$ in $L^2(\Omega)$. So

$$\int_{\Omega} |\nabla w_n|^2 dx \to b, \quad \text{as } n \to \infty.$$

By the Sobolev inequality, we have

$$\frac{\int_\Omega |\nabla w_n|^2 dx}{(\int_\Omega (w_n^+)^{2^*} dx)^{2/2^*}} \ge S.$$

Thus

$$\int_{\Omega} |\nabla w_n|^2 dx \ge S\Big(\int_{\Omega} (w_n^+)^{2^*} dx\Big)^{2/2^*},$$

which implies $b \ge S b^{2/2^*}$. Then either $b = 0$ or $b \ge S^{N/2}$.

If $b \ge S^{N/2}$, by the above, it follows that

$$\frac{S^{N/2}}{N} \le (\frac{1}{2} - \frac{1}{2^*}) b \le c < \frac{S^{N/2}}{N},$$

we know that is a contradiction. So $b = 0$, the lemma is proved . □

Lemma 2.2. *There exist $\bar{\alpha} > 0$, and a nonnegative function $w \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$\int_{\Omega} (|\nabla w|^2 + (\frac{N(N-2)}{4} - \lambda) p^2 w^2) dx \Big/ \Big(\int_{\Omega} (\ln \frac{1+|x|}{1-|x|})^\alpha (w^+)^{2^*} dx\Big)^{2/2^*} < S,$$

for any $0 < \alpha < \bar{\alpha}$.

Proof. When $N \geq 5$, suppose $(\frac{e-1}{e+1}, 0, \dots, 0) \in \Omega$, $x_0 = (\frac{e-1}{e+1} - \frac{\sqrt[4]{\varepsilon}}{2}, 0, \dots, 0) \in R^N$. Fix $\varphi \in C_0^\infty(\Omega)$, such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_{\frac{1}{4}\sqrt[4]{\varepsilon}}(x_0), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}}(x_0), \end{cases} \tag{2.6}$$

$0 \leq \varphi(x) \leq 1$, $|\nabla\varphi(x)| \leq \frac{c}{\sqrt[4]{\varepsilon}}$. Let

$$u_\varepsilon(x) = \varphi(x)U_\varepsilon(x), \quad \tilde{u_\varepsilon}(x) = p^{-\frac{N-2}{2}} u_\varepsilon(x),$$

where

$$p = \frac{2}{1-|x|^2}, \quad U_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2 + |x-x_0|^2]^{(N-2)/2}}.$$

First we prove that

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx + o(\varepsilon^{\frac{3N-6}{4}}) = S^{N/2} + o(\varepsilon^{\frac{3N-6}{4}}). \tag{2.7}$$

Indeed, since

$$\begin{aligned}
\int_{\Omega'} |\nabla u_\varepsilon|^2 dx &= \int_{\Omega'} |\varphi(x) \cdot \nabla U_\varepsilon(x) + \nabla\varphi(x) \cdot U_\varepsilon(x)|^2 dx \\
&= \int_{B_{\frac{1}{4}\sqrt[4]{\varepsilon}}(x_0)} |\nabla U_\varepsilon(x)|^2 dx \\
&\quad + \int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} |\varphi(x) \cdot \nabla U_\varepsilon(x) + \nabla\varphi(x) \cdot U_\varepsilon(x)|^2 dx,
\end{aligned}$$

we have

$$\begin{aligned}
&\Big| \int_\Omega |\nabla u_\varepsilon|^2 dx - \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx \Big| \\
&\leq | \int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} |\varphi \cdot \nabla U_\varepsilon + \nabla\varphi \cdot U_\varepsilon|^2 dx \\
&\quad + \Big| \int_{\mathbb{R}^N \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} |\nabla U_\varepsilon|^2 dx \Big| \\
&\quad + \int_{\mathbb{R}^N \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} |\nabla U_\varepsilon|^2 dx \\
&\leq 2 \int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} |\nabla U_\varepsilon|^2 dx + \frac{2c}{\sqrt[4]{\varepsilon}} \int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} |U_\varepsilon|^2 dx \\
&\quad + \int_{\mathbb{R}^N \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} |\nabla U_\varepsilon|^2 dx \\
&\leq c \int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} \frac{\varepsilon^{N-2}|x-x_0|^2}{[\varepsilon^2 + |x-x_0|^2]^N} dx \\
&\quad + \frac{c}{\sqrt[4]{\varepsilon}} \int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} \frac{\varepsilon^{N-2}}{[\varepsilon^2 + |x-x_0|^2]^{N-2}} dx \\
&\quad + c \int_{\mathbb{R}^N \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} \frac{\varepsilon^{N-2}|x-x_0|^2}{[\varepsilon^2 + |x-x_0|^2]^N} dx
\end{aligned}$$

$$\le c\varepsilon^{\frac{3N-6}{4}} + c\varepsilon^{\frac{3N-5}{4}} = O(\varepsilon^{\frac{3N-6}{4}}).$$

Therefore (2.7) is proved.

Now we prove that

$$\begin{aligned}\int_\Omega (\ln \frac{1+|x|}{1-|x|})^\alpha |u_\varepsilon|^{2^*}dx &\ge \Big(\ln \frac{1+(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}{1-(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}\Big)^\alpha \int_\Omega |u_\varepsilon|^{2^*}dx\\ &= \Big(\ln \frac{1+(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}{1-(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}\Big)^\alpha \Big[\int_{\mathbb{R}^N} |U_\varepsilon|^{2^*}dx + O(\varepsilon^{3N/4})\Big]\\ &= \Big(\ln \frac{1+(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}{1-(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}\Big)^\alpha \Big[S^{N/2} + O(\varepsilon^{3N/4})\Big].\end{aligned} \tag{2.8}$$

Indeed,

$$\begin{aligned}&\Big|\int_\Omega |u_\varepsilon|^{2^*}dx - \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*}dx\Big|\\ &= \Big|\int_{B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |u_\varepsilon|^{2^*}dx + \int_{B_{\frac12\sqrt[4]{\varepsilon}(x_0)}\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |u_\varepsilon|^{2^*}dx - \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*}dx\Big|\\ &= \Big|\int_{B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |\varphi\cdot U_\varepsilon|^{2^*}dx + \int_{B_{\frac12\sqrt[4]{\varepsilon}(x_0)}\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |\varphi\cdot U_\varepsilon|^{2^*}dx - \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*}dx\Big|\\ &\le \Big|\int_{B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |U_\varepsilon|^{2^*}dx + \int_{B_{\frac12\sqrt[4]{\varepsilon}(x_0)}\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |U_\varepsilon|^{2^*}dx - \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*}dx\Big|\\ &\le \int_{B_{\frac12\sqrt[4]{\varepsilon}(x_0)}\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |U_\varepsilon|^{2^*}dx + \int_{\mathbb{R}^N\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} |U_\varepsilon|^{2^*}dx\\ &= c\int_{B_{\frac12\sqrt[4]{\varepsilon}(x_0)}\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} \frac{\varepsilon^N}{[\varepsilon^2+|x-x_0|^2]^N}dx + c\int_{\mathbb{R}^N\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} \frac{\varepsilon^N}{[\varepsilon^2+|x-x_0|^2]^N}dx\\ &= c\varepsilon^N \int_{\frac{\sqrt[4]{\varepsilon}}{4}}^{\frac12\sqrt[4]{\varepsilon}} \frac{r^{N-1}}{[\varepsilon^2+r^2]^N}dr + c\varepsilon^N\int_{\frac{\sqrt[4]{\varepsilon}}{4}}^{+\infty} \frac{r^{N-1}}{[\varepsilon^2+r^2]^N}dr\\ &\le c\varepsilon^N \int_{\frac{\sqrt[4]{\varepsilon}}{4}}^{\frac12\sqrt[4]{\varepsilon}} r^{-N-1}dr + c\varepsilon^N\int_{\frac{\sqrt[4]{\varepsilon}}{4}}^{+\infty} r^{-N-1}dr\\ &\le c\varepsilon^{3N/4} + c\varepsilon^{3N/4} = O(\varepsilon^{3N/4}).\end{aligned}$$

Now we estimate $\int_\Omega (\frac{N(N-2)}{4}-\lambda)p^2u_\varepsilon^2dx$. We claim that

$$\int_\Omega p^2u_\varepsilon^{2^*}dx \ge c\varepsilon^2 + O(\varepsilon^{\frac{3N-4}{4}}). \tag{2.9}$$

Indeed, since $p(x) = \frac{2}{1-|x|^2} \ge p(0) = 2$,

$$\begin{aligned}\int_\Omega p^2u_\varepsilon^2dx &\ge 4\int_\Omega u_\varepsilon^2dx\\ &= 4\int_\Omega \frac{\varphi^2[N(N-2)\varepsilon^2]^{(N-2)/2}}{[\varepsilon^2+|x-x_0|^2]^{N-2}}dx\end{aligned}$$

$$
\begin{aligned}
&= 4\int_{B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} \frac{[N(N-2)\varepsilon^2]^{(N-2)/2}}{[\varepsilon^2+|x-x_0|^2]^{N-2}}dx\\
&\quad + 4\int_{B_{\frac12\sqrt[4]{\varepsilon}(x_0)}\setminus B_{\frac14\sqrt[4]{\varepsilon}(x_0)}} \frac{\varphi^2[N(N-2)\varepsilon^2]^{(N-2)/2}}{[\varepsilon^2+|x-x_0|^2]^{N-2}}dx\\
&\geq c\varepsilon^2 + O(\varepsilon^{\frac{3N-4}{4}}).
\end{aligned}
$$

By (2.7), (2.8), (2.9), we know that

$$
\begin{aligned}
&\frac{\int_\Omega |\nabla u_\varepsilon|^2 + (\frac{N(N-2)}{4}-\lambda)p^2u_\varepsilon^2dx}{[\int_\Omega (\ln\frac{1+|x|}{1-|x|})^\alpha |u_\varepsilon|^{2^*}dx]^{2/2^*}}\\
&\leq \frac{1}{(\ln\frac{1+(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}{1-(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})})^{2\alpha/2^*}} \frac{S^{N/2}+O(\varepsilon^{\frac{3N-6}{4}})+(\frac{N(N-2)}{4}-\lambda)(c\varepsilon^2+O(\varepsilon^{\frac{3N-4}{4}}))}{[S^{N/2}+O(\varepsilon^{3N/4})]^{2/2^*}}\\
&= \frac{1}{(\ln\frac{1+(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}{1-(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})})^{2\alpha/2^*}}S(\varepsilon).
\end{aligned}
$$

Since

$$
\Big(\ln\frac{1+(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}{1-(\frac{e-1}{e+1}-\sqrt[4]{\varepsilon})}\Big)\to 1 \quad \text{as } \alpha\to 0^+,
$$

there exists $\varepsilon_0>0$(small enough), such that $S(\varepsilon_0)<S$. The case $N\geq 5$ is proved.

When $N=4$, let $x_0=(\frac{e-1}{e+1}-2\beta,0,\dots,0)\in\mathbb{R}^N, \varphi(x)\in C_0^\infty(\Omega)$,

$$
\varphi(x)=\begin{cases}1 & \text{if } x\in\ B_\beta(x_0),\\ 0 & \text{if } x\in\ \mathbb{R}^N\setminus B_{2\beta}(x_0),\end{cases} \tag{2.10}
$$

for all $x\in\mathbb{R}^N$, $\beta>0$, $0\leq\varphi(x)\leq 1, |\nabla\varphi(x)|\leq c$, and let

$$
\begin{aligned}
\tilde u_\varepsilon(x) &= p^{-\frac{N-2}{2}}\varphi(x)\frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{[\varepsilon^2+|x-x_0|^2]^{(N-2)/2}}\\
&= p^{-\frac{N-2}{2}}u_\varepsilon(x)\\
&= \frac1p u_\varepsilon(x)=\frac{1-|x|^2}{2}u_\varepsilon(x),
\end{aligned}
$$

that is to prove that

$$
\frac{\int_\Omega(|\nabla u_\varepsilon|^2+(2-\lambda)p^2u_\varepsilon^2)dx}{[\int_\Omega(\ln\frac{1+|x|}{1-|x|})^\alpha|u_\varepsilon|^{2^*}dx]^{2/2^*}}<S.
$$

Similarly, we can prove that

(i)

$$
\int_\Omega |\nabla u_\varepsilon|^2dx=\int_{\mathbb{R}^N}|\nabla U_\varepsilon|^2dx+O(\varepsilon^2)=S^2+O(\varepsilon^2),
$$

(ii)

$$
\int_\Omega(\ln\frac{1+|x|}{1-|x|})^\alpha|u_\varepsilon|^{2^*}dx
$$

$$\geq \Big(\ln \frac{1+(\frac{e-1}{e+1}-4\beta)}{1-(\frac{e-1}{e+1}-4\beta)}\Big)^{\alpha} \int_{\Omega} |u_\varepsilon|^{2^*}\,dx$$
$$= \Big(\ln \frac{1+(\frac{e-1}{e+1}-4\beta)}{1-(\frac{e-1}{e+1}-4\beta)}\Big)^{\alpha} \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*}\,dx + O(\varepsilon^4) = S^2 + O(\varepsilon^4),$$

(iii)
$$\int_{\Omega} p^2 u_\varepsilon^2 dx \geq c\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2).$$

Thus,

$$\begin{aligned}
&\frac{\int_\Omega(|\nabla u_\varepsilon|^2+(2-\lambda)p^2u_\varepsilon^2)dx}{[\int_\Omega(\ln\frac{1+|x|}{1-|x|})^\alpha|u_\varepsilon|^{2^*}dx]^{2/2^*}}\\
&< \frac{1}{\Big(\ln \frac{1+(\frac{e-1}{e+1}-4\beta)}{1-(\frac{e-1}{e+1}-4\beta)}\Big)^{2\alpha/2^*}} \frac{S^2+O(\varepsilon^2)+(2-\lambda)(c\varepsilon^2|\ln\varepsilon|+O(\varepsilon^2))}{(S^2+O(\varepsilon^4))^{2/2^*}}\\
&= \frac{1}{\Big(\ln \frac{1+(\frac{e-1}{e+1}-4\beta)}{1-(\frac{e-1}{e+1}-4\beta)}\Big)^{2\alpha/2^*}} S(\varepsilon).
\end{aligned}$$

Since $(\ln\frac{1+(\frac{e-1}{e+1}-4\beta)}{1-(\frac{e-1}{e+1}-4\beta)})^\alpha \to 1$ as $\alpha\to 0^+$, then there exists $\varepsilon_0>0$, such that $S(\varepsilon_0)<S$.

Hence, there exists $\bar\alpha>0$, such that when $0<\alpha<\bar\alpha$,

$$\frac{\int_\Omega(|\nabla u_\varepsilon|^2+(\frac{N(N-2)}{4}-\lambda)p^2u_\varepsilon^2)dx}{[\int_\Omega(\ln\frac{1+|x|}{1-|x|})^\alpha|u_\varepsilon|^{2^*}dx]^{2/2^*}} < S.$$

It implies that in the hyperbolic space, we have

$$\frac{\int_{\Omega'}(|\nabla_{\mathbb{B}^N}\tilde u_\varepsilon|^2-\lambda\tilde u_\varepsilon^2)dV_{\mathbb{B}^N}}{[\int_{\Omega'}d(x)^\alpha|\tilde u_\varepsilon|^{2^*}dV_{\mathbb{B}^N}]^{2/2^*}} < S. \qquad \square$$

3. Proof of main results

Proof of Theorem 1.1. Since the solution of (1.1) is the critical point of the function I, it suffices to apply the Mountain Pass theorem with a value $c<\frac{1}{N}S^{N/2}$.

By the Lemma 2.1, we know if $0<\alpha<\bar\alpha, \exists \tilde u_\varepsilon(x)\in H_0^1(\Omega')\setminus\{0\}$ such that

$$\frac{\|\tilde u_\varepsilon\|^2_{\mathbb{B}^N}}{[\int_{\Omega'}|x|^\alpha|\tilde u_\varepsilon|^{2^*}dV_{\mathbb{B}^N}]^{2/2^*}} < S.$$

So let $v(x)=\tilde u_\varepsilon(x)$, then

$$\begin{aligned}
0<\max_{t\geq0}I(tv) &= \max_{t\geq 0}[\frac{1}{2}\int_\Omega[|\nabla_{\mathbb{B}^N}(tv)|^2-\lambda(tv)^2]dV_{\mathbb{B}^N} - \frac{1}{2^*}\int_\Omega d(x)^\alpha|tv|^{2^*}dV_{\mathbb{B}^N}]\\
&= (\frac{1}{2}-\frac{1}{2^*})[\int_\Omega(|\nabla_{\mathbb{B}^N}v|^2-\lambda v^2)dV_{\mathbb{B}^N}/(\int_\Omega d(x)^\alpha|v|^{2^*}dV_{\mathbb{B}^N})]^{N/2}\\
&< \frac{1}{N}S^{N/2}.
\end{aligned}$$

Since

$$I(u) \geq \frac{1}{2}\int_{\Omega'}(|\nabla_{\mathbb{B}^N}u|^2-\lambda u^2)\ dV_{\mathbb{B}^N} - \frac{1}{2^*}\int_{\Omega'}|u|^{2^*}\ dV_{\mathbb{B}^N}$$

$$\geq \frac{1}{2}\|u\|^2_{\mathbb{B}^N} - \frac{1}{2^* S(\mathbb{B}^N)^{\frac{2^*}{2}}} \int_{\Omega'} |\nabla_{\mathbb{B}^N} u|^2 \, dV_{\mathbb{B}^N},$$

then there exists $r > 0$ such that

$$b = \inf_{\|u\|_{\mathbb{B}^N} = r} I(u) > 0 = I(u).$$

For

$$\begin{aligned} I(t_0 v) &= \frac{1}{2}\int_{\Omega'} (|\nabla_{\mathbb{B}^N}(t_0 v)|^2 - \lambda (t_0 v)^2) dV_{\mathbb{B}^N} - \frac{1}{2^*}\int_{\Omega'} d(x)^\alpha |t_0 v|^{2^*} dV_{\mathbb{B}^N} \\ &= \frac{t_0^2}{2}\int_{\Omega'} (|\nabla_{\mathbb{B}^N} v|^2 - \lambda v^2) dV_{\mathbb{B}^N} - \frac{t_0^{2^*}}{2^*}\int_{\Omega'} d(x)^\alpha |v|^{2^*} dV_{\mathbb{B}^N} \\ &= \frac{t_0^2}{2}\|v\|^2_{\mathbb{B}^N} - \frac{t_0^{2^*}}{2^*}\int_{\Omega'} d(x)^\alpha |v|^{2^*} dV_{\mathbb{B}^N} \to -\infty \quad \text{as } t_0 \to +\infty, \end{aligned}$$

then there exists $t_0 > 0$, such that when $\|t_0 v\|_{\mathbb{B}^N} > r$, we have $I(t_0 v) < 0$. Thus

$$\max_{t \in [0,1]} I(t \cdot (t_0 v)) < \frac{S^{N/2}}{N}.$$

From Lemma 2.1, Lemma 2.2 and the Mountain Pass theorem, we know that I has a critical value and problem (1.1) has a nontrivial solution u. Multiplying the equation by u^- and integrating, we find $u^- = 0$, and u is a solution of (1.1). □

Acknowledgments. This work was supported by grants 11201140 from the National Natural Sciences Foundations of China, and 212120 from the Key Project of Chinese Ministry of Education, China.

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