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ASYMPTOTICALLY PERIODIC SOLUTIONS FOR DIFFERENTIAL AND DIFFERENCE INCLUSIONS IN HILBERT SPACES

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Abstract. Let H be a real Hilbert space and let $A : D(A) \subset H \to H$ be a (possibly set-valued) maximal monotone operator. We investigate the existence of asymptotically periodic solutions to the differential equation (inclusion) $u'(t) + Au(t) \ni f(t) + g(t)$, $t > 0$, where $f \in L^2_{\rm loc}(\mathbb{R}_+, H)$ is a T-periodic function ($T > 0$) and $g \in L^1(\mathbb{R}_+, H)$. Consider also the following difference inclusion (which is a discrete analogue of the above inclusion): $\Delta u_n + c_n A u_{n+1} \ni f_n + g_n$, $n = 0, 1, \dots$, where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are p-periodic sequences for a positive integer p and $(g_n) \in \ell^1(H)$. We investigate the weak or strong convergence of its solutions to p-periodic sequences. We show that the previous results due to Baillon, Haraux (1977) and Djafari Rouhani, Khatibzadeh (2012) corresponding to $g \equiv 0$, respectively $g_n = 0$, $n = 0, 1, \dots$, remain valid for $g \in L^1(\mathbb{R}_+, H)$, respectively $(g_n) \in l^1(H)$.

1. Introduction

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and the induced Hilbertian norm $\|\cdot\|$. Let $A : D(A) \subset H \to H$ be a (possibly multivalued) maximal monotone operator. Consider the following differential equation (inclusion)

$$\frac{du}{dt}(t) + Au(t) \ni f(t) + g(t), \quad t > 0, \tag{1.1}$$

where $f \in L^2_{\rm loc}(\mathbb{R}_+, H)$ is a T-periodic function for a given $T > 0$ and $g \in L^1(\mathbb{R}_+, H)$. In this paper we investigate the behavior at infinity of solutions to (1.1).

Consider also the following difference equation (inclusion) (which is the discrete analogue of (1.1))

$$\Delta u_n + c_n A u_{n+1} \ni f_n + g_n, \quad n = 0, 1, \dots, \tag{1.2}$$

where $(c_n) \subset (0, +\infty)$, $(f_n) \subset H$ are p-periodic sequences for a positive integer p, $(g_n) \in \ell^1(H) := \{u = (u_1, u_2, \dots) : \sum_{n=1}^{\infty} \|u_n\| < \infty\}$ and Δ is the difference operator defined as usual, i.e., $\Delta u_n = u_{n+1} - u_n$. We investigate the weak or strong convergence of solutions to p-periodic sequences.

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More precisely, in this article we show that the previous results due to Baillon, Haraux [1] and Djafari Rouhani, Khatibzadeh [2] related to the equations (inclusions),

$$\frac{du}{dt}(t) + Au(t) \ni f(t), \quad t > 0, \tag{1.3}$$

and

$$\Delta u_n + c_n A u_{n+1} \ni f_n, \quad n = 0, 1, \dots, \tag{1.4}$$

respectively, remain valid for (1.1) and (1.2), where $g \in L^1(\mathbb{R}_+, H)$ and $(g_n) \in l^1(H)$.

2. Preliminaries

To obtain our main results we recall the following results on the existence of asymptotically periodic solutions of the equations (1.3) and (1.4).

Lemma 2.1 ([1], [3, p. 169]). *Assume that A is the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, $A = \partial\varphi$. Let $f \in L^2_{\rm loc}(\mathbb{R}_+, H)$ be a T-periodic function (for a given $T > 0$). Then, equation* (1.3) *has a solution bounded on $\mathbb{R}_+$ if and only if it has at least a T-periodic solution. In this case all solutions of* (1.3) *are bounded on $\mathbb{R}_+$ and for every solution $u(t)$, $t \geq 0$, there exists a T-periodic solution q of* (1.3) *such that*

$$u(t) - q(t) \to 0, \quad \text{as } t \to \infty,$$

weakly in H. Moreover, every two periodic solutions of (1.3) *differ by an additive constant, and*

$$\frac{du_n}{dt} \to \frac{dq}{dt}, \quad \text{as } n \to \infty,$$

strongly in $L^2(0, T; H)$, where $u_n(t) = u(t + nT)$, $n = 1, 2, \dots$

Lemma 2.2 ([2], [4]). *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator. Let $c_n > 0$ and $f_n \in H$ be p-periodic sequences; i.e., $c_{n+p} = c_n$, $f_{n+p} = f_n$ $(n = 0, 1, \dots)$, for a given positive integer p. Then* (1.4) *has a bounded solution if and only if it has at least one p-periodic solution. In this case all solutions of* (1.4) *are bounded and for every solution (u_n) of* (1.4) *there exists a p-periodic solution (ω_n) of* (1.4) *such that*

$$u_n - \omega_n \to 0, \quad \text{weakly in } H, \text{ as } n \to \infty.$$

Moreover, every two periodic solutions differ by an additive constant vector.

3. Results on asymptotically periodic solutions

We begin this section with the following result regarding the continuous case, which is an extension of Lemma 2.1.

Theorem 3.1. *Assume that $A : D(A) \subset H \to H$ is the subdifferential of a proper, convex, lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, $A = \partial\varphi$. Let $f \in L^2_{\rm loc}(\mathbb{R}_+, H)$ be a T-periodic function ($T > 0$) and let $g \in L^1(\mathbb{R}_+, H)$. Then equation* (1.1) *has a bounded solution if and only if equation* (1.3) *has at least a T-periodic solution. In this case all solutions of* (1.1) *are bounded on $\mathbb{R}_+$ and for every solution $u(t)$ of* (1.1) *there exists a T-periodic solution $\omega(t)$ of* (1.3) *such that*

$$u(t) - \omega(t) \to 0, \quad \text{weakly in } H, \text{ as } t \to \infty.$$

Proof. If a solution $u(t)$, $t \geq 0$, of equation (1.1) is bounded on $\mathbb{R}_+$, then any other solution $\tilde{u}(t)$, $t \geq 0$, of equation (1.1) is also bounded, because

$$\|u(t) - \tilde{u}(t)\| \leq \|u(0) - \tilde{u}(0)\|. \tag{3.1}$$

If a solution $u(t)$ of (1.1) is bounded, then any solution $v(t)$ of (1.3) is bounded and conversely, because

$$\|u(t) - v(t)\| \leq \|u(0) - v(0)\| + \int_0^t \|g(s)\| ds \leq \|u(0) - v(0)\| + \int_0^\infty \|g(s)\| ds < \infty,$$

for $t \geq 0$. Thus, the first part of the theorem follows by Lemma 2.1. To prove the second part, we define $g_m : \mathbb{R}_+ \to H$ as follows:

$$g_m(t) = \begin{cases} g(t) & \text{for a.e. } t \in (0, m) \\ 0 & \text{if } t \geq m, \end{cases}$$

where $m = 1, 2, \dots$.

Let $u(t)$, $t \geq 0$, be an arbitrary bounded solution of (1.1). For each $m = 1, 2, \dots$ denote by $u_m(t)$, $t \geq 0$, the solution of the Cauchy problem

$$\frac{du_m(t)}{dt} + A(u_m(t)) \ni f(t) + g_m(t), \quad t > 0, \tag{3.2}$$

$$u_m(0) = u(0). \tag{3.3}$$

Since $u_m(t)$, $t \geq m$, is a solution of equation (1.3), it follows by Lemma 2.1 that there is a T-periodic solution $q_m(t)$ of (1.3), such that

$$u_m(t) - q_m(t) \to 0, \quad \text{weakly in } H, \text{ as } t \to \infty. \tag{3.4}$$

In fact, since any two periodic solutions of (1.3) differ by an additive constant (cf. Lemma 2.1), it follows that

$$q_m(t) = q(t) + c_m, \quad m = 1, 2, \dots,$$

for a fixed periodic solution $q(t)$ of (1.3), where (c_m) is a sequence in H. Thus, (3.4) becomes

$$u_m(t) - q(t) \to c_m \quad \text{as } t \to \infty, \tag{3.5}$$

weakly in H. Moreover,

$$\frac{dq(t)}{dt} + A(q(t) + c_m) \ni f(t). \tag{3.6}$$

On the other hand, it is easy to see that, for all $m < r$, we have

$$\|[u_m(t) - q(t)] - [u_r(t) - q(t)]\| = \|u_m(t) - u_r(t)\| \leq \|u(0) - u(0)\| + \int_m^r \|g(t)\| \, dt.$$

Therefore, taking the limit as $t \to \infty$, it follows (see (3.5)),

$$\|c_m - c_r\| \leq \int_m^r \|g(t)\| \, dt, \tag{3.7}$$

which shows that (c_m) is a convergent sequence; i.e., there exists a point $a \in H$, such that

$$\|c_m - a\| \to 0, \quad \text{as } m \to \infty. \tag{3.8}$$

Since A is maximal monotone (hence demiclosed), we can pass to the limit in (3.6), as $m \to \infty$, to deduce that $\omega(t) := q(t) + a$ is a solution of (1.3) (which is T-periodic). Note also that

$$\|u(t) - u_m(t)\| \le \int_m^t \|g(s)\|\, ds \le \int_m^\infty \|g(s)\|\, ds, \ t \ge m. \tag{3.9}$$

To conclude, we use the decomposition

$$\begin{aligned} u(t) - \omega(t) &= [u(t) - u_m(t)] + [u_m(t) - q_m(t)] + [q_m(t) - \omega(t)] \\ &= [u(t) - u_m(t)] + [u_m(t) - q(t) - c_m] + [(q(t) + c_m) - (q(t) + a)], \end{aligned}$$

which shows that $u(t) - \omega(t)$ converges weakly to zero, as $t \to \infty$ (cf. (3.5), (3.8), (3.9)). In other words, $u(t)$ is asymptotically periodic with respect to the weak topology of H. $\square$

It is well known that, even in the case $g \equiv 0$, the above result (Theorem 3.1) is not valid for a general maximal monotone operator A, so we cannot expect more in our case.

Theorem 3.2. Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator. Let $(g_n) \in \ell^1(H)$ and let $c_n > 0$, $f_n \in H$ be p-periodic sequences, i.e., $c_{n+p} = c_n$, $f_{n+p} = f_n$ $(n = 0, 1, \dots)$, for a given positive integer p. Then equation (1.2) has a bounded solution if and only if equation (1.4) has at least one p-periodic solution. In this case all solutions of (1.2) are bounded and for every solution (u_n) of (1.2) there exists a p-periodic solution (ω_n) of (1.4) such that

$$u_n - \omega_n \to 0, \quad \text{weakly in } H, \text{ as } n \to \infty.$$

Proof. Consider the initial condition

$$u_0 = x, \tag{3.10}$$

for a given $x \in H$. We can rewrite equation (1.2) in the form:

$$u_{n+1} - u_n + c_n A u_{n+1} \ni f_n + g_n.$$

The solution of the problem (1.2)-(3.10) is calculated successively from

$$u_{n+1} = \big(I + c_n A\big)^{-1} \big(u_n + f_n + g_n\big), \quad n = 0, 1, \dots,$$

in a unique manner, which will give a unique solution $(u_n)_{n \ge 0}$.

If a solution (u_n) of (1.2) is bounded, then any other solution $(\tilde{u}_n)$ of (1.2) is bounded, because

$$\|u_n - \tilde{u}_n\| \le \|u_0 - \tilde{u}_0\| \quad \text{for } n = 0, 1, \dots \tag{3.11}$$

If a solution (u_n) of (1.2) is bounded, then any solution (v_n) of (1.4) is bounded and conversely, because

$$\|u_n - v_n\| \le \|u_0 - v_0\| + \sum_{k=0}^{n-1} \|g_k\| \le \|u_0 - v_0\| + \sum_{k=0}^{\infty} \|g_k\| < \infty.$$

According to Lemma 2.2 the first part of the theorem is proved. For the second part we define $(g_{n,m})_{n,m \ge 0}$ as follows:

$$g_{n,m} = \begin{cases} g_n & \text{if } n < m, \\ 0 & \text{if } n \ge m. \end{cases}$$

Let (z_n) be an arbitrary solution of (1.2) (which is bounded). For each $m = 0, 1, \dots$ denote by $(z_{n,m})_{n\geq 0}$ the (unique) solution of the problem

$$z_{n+1,m} - z_{n,m} + c_n A z_{n+1,m} \ni f_n + g_{n,m} \tag{3.12}$$

$$z_{0,m} = z_0. \tag{3.13}$$

Note that $(z_{n,m})_{n\geq m}$ is a solution of equation (1.4). By Lemma 2.2 there is a p-periodic (with respect to n) solution $(\omega_{n,m})$ of (1.4) such that

$$z_{n,m} - \omega_{n,m} \to 0, \quad \text{weakly in } H, \text{ as } n \to \infty. \tag{3.14}$$

For each $m \geq 0$ we have

$$\begin{gathered}
\omega_{1,m} - \omega_{0,m} + c_0 A\omega_{1,m} \ni f_0, \\
\omega_{2,m} - \omega_{1,m} + c_1 A\omega_{2,m} \ni f_1, \\
\dots \\
\omega_{p,m} - \omega_{p-1,m} + c_{p-1} A\omega_{p,m} \ni f_{p-1},
\end{gathered}$$

where $\omega_{p,m} = \omega_{0,m}$. Since any two periodic solutions of (1.4) differ by an additive constant, we can write

$$\omega_{t,m} = \zeta_t + a_m \quad t \in \{0, 1, \dots, p-1\}, \tag{3.15}$$

where (ζ_t) is a an arbitrary but fixed periodic solution of (1.4), and $(a_m)_{m\geq 0}$ is a sequence in H. Thus

$$\begin{gathered}
\zeta_1 - \zeta_0 + c_0 A(\zeta_1 + a_m) \ni f_0, \\
\zeta_2 - \zeta_1 + c_1 A(\zeta_2 + a_m) \ni f_1, \\
\dots \\
\zeta_p - \zeta_{p-1} + c_{p-1} A(\zeta_p + a_m) \ni f_{p-1},
\end{gathered} \tag{3.16}$$

for all $m \geq 0$, where $\zeta_p = \zeta_0$. Also we can rewrite (3.14) as

$$z_{kp+t,m} \to \zeta_t + a_m, \quad \text{weakly in } H, \text{ as } k \to \infty, \tag{3.17}$$

for $m \geq 0$ and $t \in \{0, 1, \dots, p-1\}$. On the other hand, for $0 \leq m < r$, we have (cf. (3.12), (3.13))

$$\|z_{kp+t,m} - z_{kp+t,r}\| \leq \sum_{j=m}^{r-1} \|g_j\|.$$

According to (3.17) this implies

$$\|a_m - a_r\| \leq \sum_{j=m}^{r-1} \|g_j\| \leq \sum_{j=m}^{\infty} \|g_j\|, \tag{3.18}$$

for all $0 \leq m < r$, so there exists an $a \in H$ such that

$$\|a_m - a\| \to 0, \quad \text{as } m \to \infty. \tag{3.19}$$

Since A is maximal monotone (hence demiclosed), we can pass to the limit in (3.16) as $m \to \infty$ to obtain

$$\begin{gathered}
\zeta_1 - \zeta_0 + c_0 A(\zeta_1 + a) \ni f_0, \\
\zeta_2 - \zeta_1 + c_1 A(\zeta_2 + a) \ni f_1, \\
\dots \\
\zeta_p - \zeta_{p-1} + c_{p-1} A(\zeta_p + a) \ni f_{p-1},
\end{gathered}$$

where $\zeta_p = \zeta_0$. So $\omega_n := \zeta_n + a$ is a p-periodic solution of equation (1.4). We can also see that

$$\|z_n - z_{n,m}\| \le \|z_0 - z_{0,m}\| + \sum_{j=m}^{n-1} \|g_j\| \le \sum_{j=m}^{\infty} \|g_j\|. \tag{3.20}$$

Finally, for all natural n, we have $n = kp + t$, $t \in \{0, 1, \dots, p-1\}$, and

$$\begin{aligned} z_n - \omega_n &= [z_n - z_{n,m}] + [z_{n,m} - \omega_{t,m}] + [\omega_{t,m} - \omega_n] \\ &= [z_n - z_{n,m}] + [z_{kp+t,m} - \zeta_t - a_m] + [\zeta_t + a_m - \zeta_t - a], \end{aligned}$$

thus the conclusion of the theorem follows by (3.17), (3.19) and (3.20). $\square$

If in addition A is strongly monotone, then we can easily extend Theorem 2 in [4], as follows.

Theorem 3.3. *Assume that $A : D(A) \subset H \to H$ is a maximal monotone operator, that is also strongly monotone; i.e., there is a constant $b > 0$, such that*

$$(x_1 - x_2, y_1 - y_2) \ge b\|x_1 - x_2\|^2, \quad \forall x_i \in D(A),\ y_i \in Ax_i,\ i = 1, 2.$$

Let $c_n > 0$ and $f_n \in H$ be p-periodic sequences for a given positive integer p and $(g_n) \in \ell^1(H)$. Then equation (1.4) has a unique p-periodic solution (ω_n) and for every solution (u_n) of (1.2) we have

$$u_n - \omega_n \to 0, \quad \text{strongly in } H, \text{ as } n \to \infty.$$

The proof relies on arguments similar to the one above.

References

[1] J. B. Baillon, A. Haraux; *Comportement á l'infini pour les équations d'évolution avec forcing périodique*, Archive Rat. Mech. Anal., 67(1977), 101-109.

[2] B. Djafari Rouhani, H. Khatibzadeh; *Existence and asymptotic behaviour of solutions to first- and second-order difference equations with periodic forcing*, J. Difference Eqns Appl., DOI:10.1080/10236198.2012.658049.

[3] G. Moroşanu; *Nonlinear evolution equations and applications.* D.Reidel, Dordrecht-Boston-Lancaster-Tokyo, 1988.

[4] G. Moroşanu and F. Özpınar; *Periodic forcing for some difference equations in Hilbert spaces*, Bull. Belgian Math. Soc. (Simon Stevin), to appear.

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