STABILITY AND HOPF BIFURCATION IN A SYMMETRIC LOTKA-VOLTERRA PREDATOR-PREY SYSTEM WITH DELAYS

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Abstract. This article concerns a symmetrical Lotka-Volterra predator-prey system with delays. By analyzing the associated characteristic equation of the original system at the positive equilibrium and choosing the delay as the bifurcation parameter, the local stability and Hopf bifurcation of the system are investigated. Using the normal form theory, we also establish the direction and stability of the Hopf bifurcation. Numerical simulations suggest an existence of Hopf bifurcation near a critical value of time delay.

1. Introduction

Since the pioneering work of Volterra [18], the delayed predator-prey models of Lotka-Volterra type have played an important role in the development of population dynamics. For instance, the predator-prey system

$$\dot{x}(t) = x(t)[r_1 - a_{11}x(t - \tau) - a_{12}y(t)],$$
$$\dot{y}(t) = y(t)[-r_2 + a_{21}x(t) - a_{22}y(t)] \tag{1.1}$$

was first proposed and discussed briefly by May [11] in 1973. The dynamics of system (1.1) or systems similar to (1.1) has been widely investigated, concerning boundedness of solutions, stability, permanence, persistence, periodic phenomena, bifurcation and chaos (see, for example, [1, 9, 12, 13, 14] and their references). In particular, research on the stability of positive equilibrium and periodic solutions arising from the Hopf bifurcation is very critical, which helps us to realize the law and trend for species quantity. Also, extensive models of various fields (such as biology, neural network, engineering systems, epidemic disease) have been embodied these properties, (see [6, 10, 19, 17]).

In mathematical biology, Lotka-Volterra type predator-prey with a delay have been well studied by Song and Wei [15], Yan and Li [20], Yan and Zhang [21] and so on. In addition, models involving two discrete delays are increasingly applied to the study of a variety of situations. Faria [6] considered the predator-prey system of the form

$$\dot{x}(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y(t - \tau_1)],$$
$$\dot{y}(t) = y(t)[-r_2 + a_{21}x(t - \tau_2) - a_{22}y(t)]. \tag{1.2}$$

The author took τ_2 as a bifurcation parameter and obtained the existence of local Hopf bifurcation. Moreover, Song et. al took the sum of two delays $\tau_1 + \tau_2$ as a parameter and showed that when τ passed through some critical values, Hopf bifurcation occurs.

In the present article, we focus on the following symmetrical Lotka-Volterra predator-prey system:

$$
\begin{aligned}
\dot{x}(t) &= x(t)[r_1 - ax(t) - bx(t - \tau_1) - cy(t - \tau_2)], \\
\dot{y}(t) &= y(t)[r_2 - ay(t) + cx(t - \tau_1) - by(t - \tau_2)],
\end{aligned}
\tag{1.3}
$$

where r_1, r_2, a, b, c, τ_1, τ_2 are constants with $a > 0$, $b > 0$, $c > 0$ and $\tau_1 \geq 0$, $\tau_2 \geq 0$. The variables $x(t)$ and $y(t)$ denote the population densities of prey and predator at time t, respectively. The initial conditions for system (1.3) are

$$
\begin{aligned}
x(\theta) &= \phi_1(\theta), \quad -\tau_1 \leq \theta \leq 0, \; \phi_1(0) > 0, \\
y(\theta) &= \phi_2(\theta), \quad -\tau_2 \leq \theta \leq 0, \; \phi_2(0) > 0.
\end{aligned}
$$

Obviously, in the case $b = 0$ and $a \neq b$, systems (1.3) is reduced to system (1.2). Systems (1.3) shows that, in the absence of predator species, the prey species are governed by general case of Hutchinson's equation

$$
\dot{x}(t) = x(t)[r_1 - ax(t) - bx(t - \tau_1)].
\tag{1.4}
$$

(There is an extensive literature on various aspects of (1.4), see, e.g., [7, 16]). In the presence of predators, there is a hunting term, $cy(t - \tau_2)$, with a certain delay τ_2, called the hunting delay. In the absence of preys, the feedback $cx(t - \tau_1)$ has a nonnegative delay which is the time of the predator maturation. For system (1.3), Saito et al studied the permanence and the global asymptotic stability of positive equilibrium including the cases $b \leq 0$. Our main purpose is to study stability and Hopf bifurcation of (1.3).

This article is organized as follows. In Section 2, the local stability of the positive equilibrium of system (1.3) is addressed. In Section 3, we show that the system (1.3) with $\tau_1 = \tau_2$ undergoes Hopf bifurcation as the delays cross some critical values. In Section 4, following the procedure of deriving normal form due to Faria and Magalhães [4, 5], the direction and stability of the Hopf bifurcation are determined. Finally, we give some numerical simulations to justify the theoretical analysis.

2. LOCAL STABILITY

The following lemma will be needed in analyzing the characteristic equation of the linearize system.

Lemma 2.1 ([3]). *Let $f(\lambda, \tau) = \lambda^2 + a\lambda + b\lambda e^{-\tau\lambda} + c + de^{-\tau\lambda}$, where a, b, c, d, τ are real numbers and $\tau \geq 0$. Then, as τ varies, the sum of the multiplicities of zeros of f in the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

In the rest of this paper, we assume that system (1.3) has a unique positive equilibrium $E(x^*, y^*)$:

$$
x^* = \frac{r_1(a + b) - r_2 c}{(a + b)^2 + c^2} > 0, \quad y^* = \frac{r_2(a + b) + r_1 c}{(a + b)^2 + c^2} > 0,
$$

and $x^* \neq y^*$.

By using the transformation $\overline{x}(t) = x(t) - x^*$, $\overline{y}(t) = y(t) - y^*$, and dropping the bars for simplicity of notation, system (1.3) can be transformed into the system

$$
\begin{aligned}
\dot{x}(t) &= -ax^*x(t) - bx^*x(t - \tau_1) - cx^*y(t - \tau_2) - ax^2(t) \\
&\quad - bx(t)x(t - \tau_1) - cx(t)y(t - \tau_2), \\
\dot{y}(t) &= -ay^*y(t) + cy^*x(t - \tau_1) - by^*y(t - \tau_2) - ay^2(t) \\
&\quad + cx(t - \tau_1)y(t) - by(t)y(t - \tau_2).
\end{aligned}
\tag{2.1}
$$

Then we obtain the linearized system

$$
\begin{aligned}
\dot{x}(t) &= -ax^*x(t) - bx^*x(t - \tau_1) - cx^*y(t - \tau_2), \\
\dot{y}(t) &= -ay^*y(t) + cy^*x(t - \tau_1) - by^*y(t - \tau_2).
\end{aligned}
\tag{2.2}
$$

The associated characteristic equation of system (2.2) is

$$
\begin{aligned}
&\lambda^2 + \lambda(ax^* + ay^* + bx^*e^{-\lambda\tau_1} + by^*e^{-\lambda\tau_2}) + a^2x^*y^* + abx^*y^*e^{-\lambda\tau_2} \\
&+ abx^*y^*e^{-\lambda\tau_1} + (b^2x^*y^* + c^2x^*y^*)e^{-\lambda(\tau_1+\tau_2)} = 0.
\end{aligned}
\tag{2.3}
$$

Obviously, $\lambda = 0$ is not a root of (2.3). When $\tau_1 = \tau_2 = 0$, the characteristic equation (2.3) becomes

$$
\lambda^2 + \lambda(ax^* + ay^* + bx^* + by^*) + a^2x^*y^* + 2abx^*y^* + b^2x^*y^* + c^2x^*y^* = 0. \tag{2.4}
$$

Letting λ_1, λ_2 be the characteristic roots of (2.4), then $\lambda_1 + \lambda_2 = -(ax^* + ay^* + bx^* + by^*) < 0$, $\lambda_1\lambda_2 = a^2x^*y^* + 2abx^*y^* + b^2x^*y^* + c^2x^*y^* > 0$. Thus, all roots of (2.4) have negative real parts.

When $\tau_2 = 0$, Equation (2.3) becomes

$$
\lambda^2 + p\lambda + q + se^{-\lambda\tau_1} + l\lambda e^{-\lambda\tau_1} = 0, \tag{2.5}
$$

where $p = ax^* + ay^* + by^*$, $q = (a^2 + ab)x^*y^*$, $s = (ab + b^2 + c^2)x^*y^*$, $l = bx^*$. We first study (2.5) with the delay τ_1. We consider the stable intervals of τ_1 by taking τ_1 as a parameter and applying the lemma in [3]. Secondly, we investigate (2.3) in the stable interval of τ_1. Choosing τ_2 as a parameter and using the lemma in [3] once more, we will find a stable interval (depending on τ_1) for τ_2. This will be the stable interval for (2.3).

By putting $\tau_1 = 0$ in (2.5), we obtain

$$
\lambda^2 + \lambda(l + p) + s + q = 0.
$$

Based on the above analysis, we know that two roots of the above equation have negative real parts. In addition, $\lambda = i\sigma$ is a root of (2.5) if and only if σ satisfies:

$$
-\sigma^2 + i\sigma p + q + se^{-i\sigma\tau_1} + li\sigma e^{-i\sigma\tau_1} = 0.
$$

Separating the real and imaginary parts, we have

$$
\begin{aligned}
\sigma^2 - q &= s\cos\sigma\tau_1 + l\sigma\sin\sigma\tau_1, \\
p\sigma &= s\sin\sigma\tau_1 - l\sigma\cos\sigma\tau_1,
\end{aligned}
$$

which leads to

$$
\sigma^4 + \sigma^2(p^2 - 2q - l^2) + q^2 - s^2 = 0.
$$

In this article we will use the following hypotheses:

(H1) $q^2 < s^2$,

(H2) $q^2 > s^2$, $2q + l^2 - p^2 > 0$, $(2q + l^2 - p^2)^2 > 4(q^2 - s^2)$.

Then we easily obtain the following results.

Lemma 2.2. *For* (2.5), *we have*

1. *If* (H1) *holds, then when* $\tau_1 = \tau_{1,k}^{(1)}$, *Equation* (2.5) *has a pair of purely imaginary roots* $\pm i\sigma_+$;
2. *If* (H2) *holds, then when* $\tau_1 = \tau_{1,k}^{(1)}(\tau_1 = \tau_{1,k}^{(2)})$, *Equation* (2.5) *has a pair of purely imaginary roots* $\pm i\sigma_+(\pm i\sigma_-)$;
3. *If neither* (H1) *nor* (H2) *hold, then* (2.5) *has no purely imaginary root,*

where

$$\sigma_\pm^2 = \frac{2q + l^2 - p^2 \pm \sqrt{(2q + l^2 - p^2)^2 - 4(q^2 - s^2)}}{2},$$

$$\tau_{1,k}^{(1)} = \frac{1}{\sigma_+} \arccos \frac{\sigma_+^2 s - qs - lp\sigma_+^2}{s^2 + l^2\sigma_+^2} + \frac{2k\pi}{\sigma_+}, \tag{2.6}$$

$$\tau_{1,k}^{(2)} = \frac{1}{\sigma_-} \arccos \frac{\sigma_-^2 s - qs - lp\sigma_-^2}{s^2 + l^2\sigma_-^2} + \frac{2k\pi}{\sigma_-}$$

Let $\lambda_{n,k} = \mu_{n,k}(\tau_1) + i\sigma_{n,k}(\tau_1)(n = 1, 2)$ denote the roots of (2.5) satisfying

$$\mu_{1,k}(\tau_{1,k}^{(1)}) = 0, \quad \sigma_{1,k}(\tau_{1,k}^{(1)}) = \sigma_+,$$

$$\mu_{2,k}(\tau_{1,k}^{(2)}) = 0, \quad \sigma_{2,k}(\tau_{1,k}^{(2)}) = \sigma_-.$$

In what follows, we can verify the following transversality conditions.

Lemma 2.3. *Assume that either* (H1) *or* (H2) *holds, then*

$$\frac{d\operatorname{Re}\lambda_{1,k}(\tau_{1,k}^{(1)})}{d\tau_1} > 0, \quad \frac{d\operatorname{Re}\lambda_{2,k}(\tau_{1,k}^{(2)})}{d\tau_1} < 0. \tag{2.7}$$

Proof. Differentiating the characteristic equation (2.5) with respect to τ_1, we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{2\lambda + p}{\lambda(\lambda^2 + p\lambda + q)} + \frac{l}{\lambda^2 l + \lambda s} - \frac{\tau_1}{\lambda}.$$

Thus, if $\lambda = i\sigma_+$, then

$$\operatorname{sign}\left\{ \operatorname{Re}(\frac{d\lambda}{d\tau_1})\big|_{\tau_1 = \tau_{1,k}^{(1)}} \right\} = \operatorname{sign}\left\{ \operatorname{Re}(\frac{d\tau}{d\lambda})\big|_{\lambda = i\sigma_+} \right\}$$

$$= \operatorname{sign}\left\{ \frac{\sigma_+^4 l^2 + 2\sigma_+^2 s^2 - 2qs^2 + p^2 s^2 - l^2 q^2}{[(q - \sigma_+^2)^2 + p^2\sigma_+^2](\sigma_+^2 l^2 + s^2)} \right\}.$$

Since

$$\sigma_+^4 l^2 + 2\sigma_+^2 s^2 - 2qs^2 + p^2 s^2 - l^2 q^2$$

$$= \frac{1}{2}l^2 \sqrt{(2q + l^2 - p^2)^2 - 4(q^2 - s^2)}$$

$$\times [\sqrt{(2q + l^2 - p^2)^2 - 4(q^2 - s^2)} + (2q + l^2 - p^2)]$$

$$+ s^2 \sqrt{(2q + l^2 - p^2)^2 - 4(q^2 - s^2)},$$

thus, either (H1) or (H2) implies that $\operatorname{sign}\left\{ \operatorname{Re}(\frac{d\lambda}{d\tau_1})\big|_{\tau_1 = \tau_{1,k}^{(1)}} \right\} > 0$. If $\lambda = i\sigma_-$, we can similarly prove that

$$\operatorname{sign}\left\{ \operatorname{Re}(\frac{d\lambda}{d\tau_1})\big|_{\tau_1 = \tau_{1,k}^{(2)}} \right\} < 0.$$

The proof is complete.   □

From Lemmas 2.2 and 2.3, we have the following result.

Lemma 2.4. *For Equation* (2.5)*, the following statements are true:*

1. Assume that (H1) *holds, then for any* $\tau_1 \in [0, \tau_{1,0}^{(1)})$*, all roots of* (2.5) *have negative real parts, and for* $\tau_1 > \tau_{1,0}^{(1)}$*, Equation* (2.5) *has at least one root with positive real parts.*

2. Assume that (H2) *holds, then there exists an integer* $k_1 > 0$ *such that there are* k_1 *switches from stability to instability to stability; that is, when* $\tau_1 \in (\tau_{1,k}^{(2)}, \tau_{1,k+1}^{(1)})$*,* $k = -1, 0, 1, \ldots, k_1 - 1$*, all roots of* (2.5) *have negative real parts, where* $\tau_{1,-1}^{(2)} = 0$*; When* $\tau_1 \in (\tau_{1,k}^{(1)}, \tau_{1,k}^{(2)})$ *and* $\tau_1 > \tau_{1,k_1}^{(1)}$*,* $k = 0, 1, \ldots, k_1 - 1$*, Equation* (2.5) *has at least one root with positive real parts.*

In what follows, we consider (2.3) with τ_1 in its stable interval. We will take τ_2 as a bifurcation parameter and have the following result.

Lemma 2.5. *Assume that all roots of* (2.5) *have negative real parts, then there exists* $\tau_2(\tau_1) > 0$ *such that when* $\tau_2 \in [0, \tau_2(\tau_1))$*, all roots of* (2.3) *have negative real parts.*

Proof. Since all roots of (2.5) have negative real parts, all roots of (2.3) with $\tau_2 = 0$ have negative real parts. Similar to the proof in [3], let $f(\lambda, \tau_1, \tau_2)$ be the left-hand side of (2.3). Obviously, $f(\lambda, \tau_1, \tau_2)$ is analysis in λ and τ_2. We take τ_2 as a bifurcation parameter, when τ_2 varies, the sum of the multiplicities of zero of $f(\lambda, \tau_1, \tau_2)$ in the open right half-plane can change only if a zero appears on or crosses the imaginary axis. Thus, there exists $\tau_2(\tau_1) > 0$, such that when $\tau_2 \in \tau_2(\tau_1)$, all roots of (2.3) have negative real parts. The proof is complete.   □

Following Lemma 2.5, we can get sufficient conditions for local stability of system (2.1).

Theorem 2.6. *For system* (2.1)*, the following statements hold.*

1. Suppose that (H1) *holds, then for any* $\tau_1 \in [0, \tau_{1,0}^{(1)})$*, there is a* $\tau_2(\tau_1) > 0$ *such that when* $\tau_2 \in [0, \tau_2(\tau_1))$*, the zero solution of system* (2.1) *is locally asymptotically stable.*

2. Suppose that (H2) *holds, then for any* $\tau_1 \in (\tau_{1,k}^{(2)}, \tau_{1,k+1}^{(1)})$*,* $k = -1, 0, 1, \ldots, k_1 - 1$*, there is* $\tau_2(\tau_1) > 0$ *such that when* $\tau_2 \in [0, \tau_2(\tau_1))$*, the zero solution of system* (2.1) *is locally asymptotically stable.*

3. Suppose that neither (H1) *nor* (H2) *holds, then for any* $\tau_1 \geq 0$*, there is a* $\tau_2(\tau_1) > 0$*, such that when* $\tau_2 \in [0, \tau_2(\tau_1))$*, the zero solution of system* (2.1) *is locally asymptotically stable.*

3. Hopf bifurcation

Let $\tau_1 = \tau_2 = \tau$. System (1.3) becomes

$$\dot{x}(t) = x(t)[r_1 - ax(t) - bx(t - \tau) - cy(t - \tau)],$$
$$\dot{y}(t) = y(t)[r_2 - ay(t) + cx(t - \tau) - by(t - \tau)].$$
(3.1)

and the characteristic equation (2.3) has the form

$$\lambda^2 + p_1\lambda + q_1 + (s_1\lambda + l_1)e^{-\lambda\tau} + m_1 e^{-2\lambda\tau} = 0,$$
(3.2)

where
$$p_1 = ax^* + ay^*, \quad q_1 = a^2 x^* y^*, \quad l_1 = 2abx^* y^*,$$
$$s_1 = bx^* + by^*, \quad m_1 = b^2 x^* y^* + c^2 x^* y^*.$$
(3.3)

Lemma 3.1. *If $b^2 + c^2 > a^2$, then there exists $\sigma_+ (> 0)$ such that (3.2) with $\tau = \tau_k$ has a simple pair of conjugate purely imaginary roots $\pm i\sigma_+$, where*

$$\tau_k = \frac{1}{\sigma_+} \arccos \frac{(l_1 - s_1 p_1)\sigma_+^2 - l_1(q_1 - m_1)}{(-\sigma_+^2 + q_1)^2 - m_1^2 + p_1^2 \sigma_+^2} + \frac{2k\pi}{\sigma_+}, \quad k = 0, 1, \dots.$$

Proof. Obviously, $\lambda = 0$ is not a root of (3.2). If $\lambda = i\sigma$ is a root of (3.2), then

$$-\sigma^2 + ip_1\sigma + q_1 + l_1 e^{-i\sigma\tau} + i\sigma s_1 e^{-i\sigma\tau} + m_1 e^{-2i\sigma\tau} = 0. \tag{3.4}$$

Multiplying both sides of (3.4) by $e^{i\sigma\tau}$, we have

$$(-\sigma^2 + ip_1\sigma + q_1)e^{i\sigma\tau} + (l_1 + i\sigma s_1) + m_1 e^{-i\sigma\tau} = 0. \tag{3.5}$$

Separating the real parts and imaginary parts of (3.5) leads to

$$\cos\sigma\tau = (l_1 - s_1 p_1)\sigma^2 - l_1(q_1 - m_1),$$
$$[(-\sigma^2 + q_1)^2 - m_1^2 + p_1^2\sigma^2]\sin\sigma\tau = s_1\sigma^3 + [l_1 p_1 - s_1(q_1 + m_1)]\sigma,$$
(3.6)

and thus,

$$[(-\sigma^2 + q_1)^2 - m_1^2 + p_1^2\sigma^2]^2$$
$$= [(l_1 - s_1 p_1)\sigma^2 - l_1(q_1 - m_1)]^2 + [s_1\sigma^3 + [l_1 p_1 - s_1(q_1 + m_1)]\sigma]^2.$$

Let $y = \sigma^2$, define

$$f(y) = [(-y + q_1)^2 - m_1^2 + p_1^2 y]^2 - [(l_1 - s_1 p_1)y - l_1(q_1 - m_1)]^2$$
$$- y[s_1 y + l_1 p_1 - s_1(q_1 + m_1)]^2.$$
(3.7)

Since $f(y)$ is a quartic function, we have $f(y) \to +\infty$ as $|y| \to +\infty$. On the other hand,

$$f(0) = (q_1^2 - m_1^2)^2 - l_1^2(q_1 - m_1)^2 = (q_1 - m_1)^2(q_1 + m_1 + l_1)(q_1 + m_1 - l_1) > 0,$$

Define

$$F(y) = [(-y + q_1)^2 - m_1^2 + p_1^2 y]^2,$$
$$G(y) = -[(l_1 - s_1 p_1)y - l_1(q_1 - m_1)]^2,$$
$$H(y) = -y[s_1 y + l_1 p_1 - s_1(q_1 + m_1)]^2,$$

Then $f(y) = F(y) + G(y) + H(y)$. Assume that $g(y) = (-y + q_1)^2 - m_1^2 + p_1^2 y$, then

$$g(0) = q_1^2 - m_1^2 = (q_1 + m_1)(q_1 - m_1) = (q_1 + m_1)x^* y^*(a^2 - b^2 - c^2),$$

In view of $b^2 + c^2 > a^2$, we have $g(0) < 0$. On the other hand, $g(y) \to +\infty$ as $y \to +\infty$. Thus, there is a $z_0 > 0$ such that $g(z_0) = 0$; i.e., $F(z_0) = 0$. It is easily to see that positive roots of F, G and H are mutually different when $x^* \neq y^*$. Consequently, $G(z_0) < 0$, $H(z_0) < 0$. Thus, $f(z_0) < 0$, which together with $f(0) > 0$ imply that there exists a $y_0 > 0$ such that $f(y_0) = 0$. Obviously, there exists another positive root because $f(y) \to +\infty$ as $y \to +\infty$. Furthermore, one of the two roots is a simple root at least. Let y_0 be such a simple root and $y_0 = \sigma^2$. Substituting $y_0 = \sigma^2$ into (3.6), we have

$$\tau_k = \frac{1}{\sigma_+} \arccos \frac{(l_1 - s_1 p_1)\sigma_+^2 - l_1(q_1 - m_1)}{(-\sigma_+^2 + q_1)^2 - m_1^2 + p_1^2\sigma^2} + \frac{2k\pi}{\sigma_+}, \quad k = 0, 1, \dots. \tag{3.8}$$

The proof is complete.   □

Let $\lambda(\tau) = \mu(\tau) + i\sigma(\tau)$ be the root of (3.2) near $\tau = \tau_k$ satisfying $\mu(\tau_k) = 0$, $\sigma(\tau_k) = \sigma_+$, and $k = 0, 1, 2, \ldots$, where τ_k is defined by (3.8).

Lemma 3.2 ([14])**.** *Assume that $b^2 + c^2 > a^2$ and $a > b$, then*

$$\text{sign} \left\{ \text{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_k} \right\} > 0.$$

The proof of the above lemma is similar to that in [14, p. 549] and is omitted. By Lemma 3.1 and 3.2, we have the following result.

Theorem 3.3. *Assume that $b^2 + c^2 > a^2$ and $a > b$. Then*
1. *When $\tau \in [0, \tau_0)$, the equilibrium $E = (x^*, y^*)$ of (3.1) is locally asymptotically stable;*
2. *When $\tau > \tau_0$, the equilibrium $E = (x^*, y^*)$ of (3.1) is unstable;*
3. *System (3.1) undergoes a Hopf bifurcation at $E = (x^*, y^*)$ and $\tau = \tau_k$.*

4. Direction and stability of the Hopf bifurcation

In this section, we shall determine the direction and stability of the bifurcating periodic solutions by employing the normal form theory duo to Faria and Magalhães [4, 5].

Let $z_1(t) = x(\tau t) - x^*$, $z_2(t) = y(\tau t) - y^*$. Then system (3.1) can be rewritten as a functional differential equation in $C([-1, 0], \mathbb{R}^2)$,

$$\begin{aligned}
\dot{z}_1(t) &= x^*\tau[-az_1(t) - bz_1(t-1) - cz_2(t-1)] - \tau z_1(t)[az_1(t) \\
&\quad + bz_1(t-1) + cz_2(t-1)], \\
\dot{z}_2(t) &= y^*\tau[cz_1(t-1) - az_2(t) - bz_2(t-1)] + \tau z_2(t)[cz_1(t-1) \\
&\quad - az_1(t) - bz_2(t-1)].
\end{aligned} \tag{4.1}$$

and its linearized system is

$$\begin{aligned}
\dot{z}_1(t) &= \tau[-ax^* z_1(t) - bx^* z_1(t-1) - cx^* z_2(t-1)], \\
\dot{z}_2(t) &= \tau[cy^* z_1(t-1) - ay^* z_2(t) - by^* z_2(t-1)].
\end{aligned} \tag{4.2}$$

The characteristic equation associated with system (4.2) has the form

$$\lambda^2 + \lambda\tau(ay^* + ax^* + bx^* e^{-\lambda} + by^* e^{-\lambda}) + \tau^2 x^* y^* (a^2 + 2abe^{-\lambda} + b^2 e^{-2\lambda} + c^2 e^{-2\lambda}) = 0. \tag{4.3}$$

By the change of variable $\lambda = \xi\tau$, Equation (4.3) becomes

$$\xi^2 + p_1\xi + q_1 + (s_1\xi + l_1)e^{-\xi\tau} + m_1 e^{-2\xi\tau} = 0, \tag{4.4}$$

where p_1, q_1, s_1, l_1, m_1 are defined by (3.3). Equation (4.4) is the same as (3.2). Therefore, from Section 3, we know that (4.4) has a simple pair of conjugate complex roots $\xi(\tau) = \mu(\tau) \pm i\sigma(\tau)$ which satisfy

$$\mu(\tau_k) = 0, \quad \sigma(\tau_k) = \sigma_+, \quad \mu'(\tau_k) \neq 0,$$

where τ_k is given in (3.8). Thus, Eq. (4.3) has two complex roots $\lambda(\tau) = \tau\mu(\tau) \pm i\tau\sigma(\tau)$, which satisfy

$$\frac{d\,\text{Re}\,\lambda(\tau)}{d\tau}|_{\tau=\tau_k} = \tau\frac{d\,\text{Re}\,\xi(\tau)}{d\tau}|_{\tau=\tau_k} + \text{Re}\,\xi(\tau)|_{\tau=\tau_k} = \tau_k\mu'(\tau_k) \neq 0.$$

Letting $z = (z_1, z_2)^T$, then system (4.1) can further be written as

$$\dot{z}(t) = L(\tau)z_t + F(z_t, \tau), \tag{4.5}$$

where

$$L(\tau)\varphi = \tau \begin{pmatrix} -ax^*\varphi_1(0) - bx^*\varphi_1(-1) - cx^*\varphi_2(-1) \\ cy^*\varphi_1(-1) - ay^*\varphi_2(0) - by^*\varphi_2(-1) \end{pmatrix},$$

$$F(\varphi, \tau) = \tau \begin{pmatrix} -a\varphi_1^2(0) - b\varphi_1(0)\varphi_1(-1) - c\varphi_1(0)\varphi_2(-1) \\ -a\varphi_2^2(0) + c\varphi_1(-1)\varphi_2(0) - b\varphi_2(0)\varphi_2(-1) \end{pmatrix},$$

and $\varphi = (\varphi_1, \varphi_2)^T \in C([-1, 0], \mathbb{R}^2)$.

Introducing a new parameter $\alpha = \tau - \tau_k$, system (4.5) is equivalent to

$$\dot{z}(t) = L(\tau_k)(z_t) + F_0(z_t, \alpha), \tag{4.6}$$

where $F_0(z_t, \alpha) = L(\alpha)z_t + F(z_t, \tau_k + \alpha)$.

Let A_0 be the infinitesimal generator corresponding to $\dot{z}(t) = L(\tau_k)z_t$. Then A_0 has a pair of purely imaginary characteristic roots $\pm i\sigma_k (\sigma_k = \tau_k\sigma_+)$, which are simple, and no other characteristic roots with zero real part. Define $\Lambda = \{-i\sigma_k, i\sigma_k\}$ and denote by P the invariant space of A_0 corresponding to Λ, where the dimension of P equals to 2. The phase space is $C = C([-1, 0], \mathbb{R}^2)$, so we can decompose the space C as $C = P \oplus Q$ by applying the formal adjoint theory for functional differential equations in [8]. Consider complex coordinates and still write as $C([-1, 0], \mathbb{C}^2)$. Suppose that Φ_1, Φ_2 are the eigenvectors of the operator A_0 associated with eigenvalues $i\sigma_k, -i\sigma_k$, respectively. Thus, $\Phi = (\Phi_1, \Phi_2)$ is a basis of P, $\Phi_1(\theta) = e^{i\sigma_k\theta}v^T = e^{i\sigma_k\theta}(1, v_2)^T$, $\Phi_2(\theta) = \overline{\Phi_1(\theta)}$, $-1 \leq \theta \leq 0$, where $v = (v_1, v_2)^T$ is a vector in $\mathbb{C}^2$ and

$$v_2 = -\frac{i\sigma_k + \tau_k ax^* + \tau_k bx^* e^{-i\sigma_k}}{\tau_k cx^* e^{-i\sigma_k}},$$

which satisfies

$$L(\tau_k)(\Phi_1) = i\sigma_k v^T.$$

For $\Phi = (\Phi_1, \Phi_2)$, we note that $\dot{\Phi} = \Phi B$, where B is a diagonal matrix of the form $\text{diag}(i\sigma_k, -i\sigma_k)$. Also, the formal adjoint operator A_0^* of A_0 has eigenvectors Ψ_1, Ψ_2 corresponding to $i\sigma_k, -i\sigma_k$. $\Psi(s) = \text{col}(\Psi_1(s), \Psi_2(s))$ constructs a basis of the adjoint space P^* of P and $\Psi_1(s) = e^{-i\sigma_k s}(u_1, u_2)$, $\Psi_2(s) = \overline{\Psi_1(s)}$, $0 \leq s \leq 1$. We know $-\dot{\Psi}_1(0) = \int_{-1}^{0} \Psi_1(-\theta)d\eta(\theta)$, therefore, we can choose

$$u_2 = \frac{i\sigma_k + \tau_k ax^* + \tau_k bx^* e^{-i\sigma_k}}{\tau_k cy^* e^{-i\sigma_k}}u_1.$$

In order to have $(\Psi, \Phi) = ((\Psi_j, \Phi_i), i, j = 1, 2) = I_2$; i.e., second order identical matrix, where $(\cdot, \cdot)$ is a bilinear inner product form

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi,$$

we have

$$u_1 = \frac{y^*}{(1 + i\sigma_k)(y^* - x^*v_2^2) + \tau_k ax^*y^*(1 - v_2^2)}.$$

Using that the enlarged phase space $BC : \{\varphi : [-1, 0] \to \mathbb{C}^2 | \varphi$ is continuous on $[-1, 0)$, it exists $\lim_{\theta \to 0^-} \varphi(\theta)$. The elements of BC have the form $\psi = \varphi + X_0 \alpha$, where $\varphi \in C$, $\alpha \in \mathbb{C}^2$, $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & -1 \le \theta < 0. \end{cases}$$

The projection of C onto P, $\varphi \mapsto \Phi(\Psi, \varphi)$, associated with the decomposition $C = P \oplus Q$ is replaced by $\pi : BC \mapsto P$ such that

$$\pi(\varphi + X_0 \alpha) = \Phi[(\Psi, \varphi) + \Psi(0)\alpha].$$

Thus, we have the decomposition $BC = P \oplus \ker \pi$. Using the decomposition $z_t = \Phi x(t) + y_t$, $x(t) \in \mathbb{C}^2$, $y_t \in \ker \pi \cap C^1 =: Q^1$, the equation (4.6) is equivalent to the system

$$\dot{x} = Bx + \Psi(0)F_0(\Phi x + y, \alpha),$$
$$\frac{d}{dt}y = A_{Q^1}y + (I - \pi)X_0 F_0(\Phi x + y, \alpha). \tag{4.7}$$

Consider the Taylor expansion as follows:

$$\Psi(0)F_0(\Phi x + y, \alpha) = \frac{1}{2}f_2^1(x, y, \alpha) + \frac{1}{3!}f_3^1(x, y, \alpha) + h.o.t., \tag{4.8}$$

$$(I - \pi)X_0 F_0(\Phi x + y, \alpha) = \frac{1}{2}f_2^2(x, y, \alpha) + \frac{1}{3!}f_3^2(x, y, \alpha) + h.o.t.,$$

where $f_j^1(x, y, \alpha)$ and $f_j^2(x, y, \alpha)$ are homogeneous polynomials in (x, y, α) of degree j, $j = 2, 3$, with coefficients in $\mathbb{C}^2$ and $\ker \pi$, respectively, h.o.t. stands for higher order terms. Thus, the normal form method gives a normal form on the center manifold of the origin for (4.7) as

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0, \alpha) + \frac{1}{3!}g_3^1(x, 0, \alpha) + h.o.t, \tag{4.9}$$

where g_2^1, g_3^1 are the second and third order terms in (x, α), respectively. Always following [5], $V_j^{m+p}(X)$ denotes the linear space of the homogeneous polynomials of degree j in $m + p$ real variables, $x = (x_1, \ldots, x_m), \alpha = (\alpha_1, \ldots, \alpha_p)$ with coefficients in X; that is,

$$V_j^{m+p}(X) = \Big\{ \sum_{|(q,l)|=j} c_{(q,l)}x^q \alpha^l : (q, l) \in N_0^{m+p}, c_{(q,l)} \in X \Big\}.$$

The operators M_j^1 are defined by

$$(M_j^1 p)(x, \alpha) = D_x p(x, \alpha)Bx - Bp(x, \alpha), \quad j \ge 2,$$

and M_j^1 act on $V_j^3(\mathbb{C}^2) = \text{Im}(M_j^1) \oplus \ker(M_j^1)$, and

$$\ker(M_j^1) = \text{span}\{x^q \alpha^l e_k : (q, \bar{\lambda}) = \lambda_k, k = 1, 2, q \in \mathbb{N}_0^2, l \in \mathbb{N}_0, |(q, l)| = j\},$$

where e_1, e_2 is the canonical basis of $\mathbb{C}^2$. Hence,

$$\ker(M_2^1) = \text{span}\Big\{ \begin{pmatrix} x_1\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\alpha \end{pmatrix} \Big\},$$

$$\ker(M_3^1) = \text{span}\Big\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\alpha^2 \end{pmatrix} \Big\}.$$

From (4.8), we have

$$f_2^1(x, y, \alpha) = \Psi(0)[2L(\alpha)(\Phi x + y) + F_2(\Phi x + y, \tau_k)], \tag{4.10}$$

where

$$F_2(\varphi, \tau_k) = 2\tau_k \begin{pmatrix} -a\varphi_1^2(0) - b\varphi_1(0)\varphi_1(-1) - c\varphi_1(0)\varphi_2(-1) \\ -a\varphi_2^2(0) + c\varphi_1(-1)\varphi_2(0) - b\varphi_2(0)\varphi_2(-1) \end{pmatrix},$$

Therefore, we obtain

$$f_2^1(x, 0, \alpha) = \begin{pmatrix} 2A_1 x_1 \alpha + 2A_2 x_2 \alpha + a_{20} x_1^2 + a_{02} x_2^2 \\ 2\bar{A}_2 x_1 \alpha + 2\bar{A}_1 x_2 \alpha + \bar{a}_{02} x_1^2 + \bar{a}_{20} x_2^2 \end{pmatrix}, \tag{4.11}$$

here

$$A_1 = \frac{i\sigma_k}{\tau_k}(u_1 + u_2 v_2), \quad A_2 = \frac{-i\sigma_k}{\tau_k}(u_1 + u_2 \bar{v}_2),$$

$$a_{20} = 2i\sigma_k\left(\frac{u_1}{x^*} + \frac{u_2 v_2^2}{y^*}\right), \quad a_{02} = -2i\sigma_k\left(\frac{u_1}{x^*} + \frac{u_2 \bar{v}_2^2}{y^*}\right). \tag{4.12}$$

Since the second order terms in (α, x) of the normal form on the center manifold are given by

$$\frac{1}{2}g_2^1(x, 0, \alpha) = \frac{1}{2}\operatorname{Proj}_{\ker(M_2^1)} f_2^1(x, 0, \alpha),$$

it follows that

$$\frac{1}{2}g_2^1(x, 0, \alpha) = \begin{pmatrix} A_1 x_1 \alpha \\ \bar{A}_1 x_2 \alpha \end{pmatrix}, \tag{4.13}$$

where $A_1 = \frac{i\sigma_k}{\tau_k}(u_1 + u_2 v_2)$. We notice that

$$g_3^1(x, 0, \alpha) \in \ker(M_3^1) = \operatorname{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \alpha^2 \end{pmatrix} \right\}.$$

However, the terms $O(|x|\alpha^2)$ are irrelevant to determine the generic Hopf bifurcation. Thus, we only need to compute the coefficients of $\begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}$. Note that

$$\frac{1}{3!}g_3^1(x, 0, \alpha) = \frac{1}{3!}\operatorname{Proj}_{\ker(M_3^1)} \tilde{f}_3^1(x, 0, \alpha) = \frac{1}{3!}\operatorname{Proj}_s \tilde{f}_3^1(x, 0, 0) + O(|x|\alpha^2),$$

where

$$s := \operatorname{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \right\}$$

and

$$\tilde{f}_3^1(x, 0, 0) = f_3^1(x, 0, 0) + \frac{3}{2}[(D_x f_2^1)U_2^1 - (D_x U_2^1)g_2^1]_{(x,0,0)} + \frac{3}{2}[(D_y f_2^1)h]_{(x,0,0)},$$

here $\tilde{f}_3^1(x, 0, 0)$ is the third order terms of the equation which is obtained after computing the second terms of the normal form.

Now we compute $\frac{1}{3!}g_3^1(x, 0, \alpha)$ step by step.

We first compute $\operatorname{Proj}_s[(D_x f_2^1)U_2^1]_{(x,0,0)}$. Following [5], we know that

$$U_2^1(x, 0) = (M_2^1)^{-1} P_{I,2}^1 f_2^1(x, 0, 0).$$

From (4.11), we can easily see that

$$f_2^1(x, 0, 0) = \begin{pmatrix} a_{20} x_1^2 + a_{02} x_2^2 \\ \bar{a}_{02} x_1^2 + \bar{a}_{20} x_2^2 \end{pmatrix}.$$

According to the definition of M_2^1, the equation $M_2^1 U_2^1(x,0) = f_2^1(x,0,0)$ can be written as the following differential equations:

$$x_1 \frac{\partial u_1}{\partial x_1} - x_2 \frac{\partial u_1}{\partial x_2} - u_1 = \frac{1}{i\sigma_k}(a_{20}x_1^2 + a_{02}x_2^2),$$

$$x_1 \frac{\partial u_2}{\partial x_1} - x_2 \frac{\partial u_2}{\partial x_2} + u_2 = \frac{1}{i\sigma_k}(\bar{a}_{02}x_1^2 + \bar{a}_{20}x_2^2).$$

For the equation, we can easily obtain

$$U_2^1(x,0) = \begin{pmatrix} \frac{1}{i\sigma_k}(a_{20}x_1^2 - \frac{1}{3}a_{02}x_2^2) \\ \frac{1}{i\sigma_k}(\frac{1}{3}\bar{a}_{02}x_1^2 - \bar{a}_{20}x_2^2) \end{pmatrix}.$$

Thus,

$$\text{Proj}_s[(D_x f_2^1)U_2^1]_{(x,0,0)} = \begin{pmatrix} \frac{2}{3i\sigma_k}|a_{02}|^2 x_1^2 x_2 \\ \frac{-2}{3i\sigma_k}|a_{02}|^2 x_1 x_2^2 \end{pmatrix}.$$

From (4.13), we know $g_2^1(x,0,0) = 0$. Thus, $[(D_x U_2^1)g_2^1]_{(x,0,0)} = 0$.

Next, we compute $\text{Proj}_s[(D_y f_2^1)h]_{(x,0,0)}$, where $h = (h^1, h^2)^T$ is a second order homogeneous polynomial in (x_1, x_2, α) with coefficients in Q^1, which has the form

$$h = h(x_1, x_2, \alpha) = h_{110}x_1x_2 + h_{101}x_1\alpha + h_{011}x_2\alpha + h_{200}x_1^2 + h_{020}x_2^2 + h_{002}\alpha^2$$

and $h = h(x_1, x_2, \alpha)$ is the unique solution in $V_2^3(Q^1)$ of the equation

$$(M_2^2 h)(x, \alpha) = (I - \pi)X_0[2L(\alpha)(\Phi x) + F_2(\Phi x, \tau_k)].$$

As in [5], we know that

$$(M_2^2 h)(x, \alpha) = D_x h(x, \alpha)Bx - A_{Q^1}(h(x, \alpha))$$

$$= D_x h(x, \alpha)Bx - \dot{h}(x, \alpha) - X_0[L(\tau_k)(h(x, \alpha)) - \dot{h}(x, \alpha)(0)]$$

$$= (I - \pi)X_0[2L(\alpha)(\Phi x) + F_2(\Phi x, \tau_k)].$$

thus, $h(x, \theta)$ is also evaluated by the system

$$\dot{h}(x) - D_x h(x)Bx = \Phi\Psi(0)[F_2(\Phi x, \tau_k)], \tag{4.14}$$

$$\dot{h}(x)(0) - L(\tau_k)(h(x)) = F_2(\Phi x, \tau_k), \tag{4.15}$$

where $\dot{h}$ denotes the derivative of $h(x)(\theta)$ relative to θ. According to (4.10), it follows that

$$f_2^1(x, y, 0) = \Psi(0)F_2(\Phi x + y, \tau_k)$$

$$= \begin{pmatrix} u_1 & u_2 \\ \bar{u}_1 & \bar{u}_2 \end{pmatrix} \tau_k \begin{pmatrix} -ad_1^2 - bd_1d_2 - cd_1n_2 \\ -an_1^2 + cd_2n_1 - bn_1n_2 \end{pmatrix},$$

where

$$d_1 = x_1 + x_2 + y_1(0), \quad d_2 = e^{-i\sigma_k}x_1 + e^{i\sigma_k}x_2 + y_1(-1),$$

$$n_1 = v_2 x_1 + \bar{v}_2 x_2 + y_2(0), \quad l_2 = e^{-i\sigma_k}v_2 x_1 + e^{i\sigma_k}\bar{v}_2 x_2 + y_2(-1).$$

Then we have

$$[(D_y f_2^1)h]_{(x,0,0)}$$

$$= 2\tau_k \begin{pmatrix} u^T \begin{pmatrix} -2ad_1'h^1(0) - bd_1'h^1(-1) - bd_2'h^1(0) - cd_1'h^2(-1) - cn_2'h^1(0) \\ -2an_1'h^2(0) + cd_2'h^2(0) + cn_1'h^1(-1) - bn_1'h^2(-1) - bn_2'h^2(0) \end{pmatrix} \\ \bar{u}^T \begin{pmatrix} -2ad_1'h^1(0) - bd_1'h^1(-1) - bd_2'h^1(0) - cd_1'h^2(-1) - cn_2'h^1(0) \\ -2an_1'h^2(0) + cd_2'h^2(0) + cn_1'h^1(-1) - bn_1'h^2(-1) - bn_2'h^2(0) \end{pmatrix} \end{pmatrix},$$

where

$$d_1' = x_1 + x_2, \quad d_2' = e^{-i\sigma_k}x_1 + e^{i\sigma_k}x_2,$$
$$n_1' = v_2 x_1 + \bar{v}_2 x_2, \quad n_2' = e^{-i\sigma_k}v_2 x_1 + e^{i\sigma_k}\bar{v}_2 x_2. \tag{4.16}$$

Thus,

$$\text{Proj}_s[(D_y f_2^1)h)]_{(x,0,0)} = \begin{pmatrix} 2c_3 x_1^2 x_2 \\ 2\bar{c}_3 x_1 x_2^2 \end{pmatrix},$$

where

$$c_3 = u^T \tau_k$$
$$\times \begin{pmatrix} Ah_{110}^1(0) + \overline{A}h_{200}^1(0) + Bh_{110}^1(-1) + \overline{B}h_{200}^1(-1) + Ch_{110}^2(-1) + \overline{C}h_{200}^2(-1) \\ Dh_{110}^1(-1) + \overline{D}h_{200}^1(-1) + Eh_{110}^2(0) + \overline{E}h_{200}^2(0) + Fh_{110}^2(-1) + \overline{F}h_{200}^2(-1) \end{pmatrix}$$

and

$$A = -2a - be^{-i\sigma_k} - ce^{-i\sigma_k}v_2, \quad B = -b, \quad C = -c,$$
$$D = cv_2, \quad E = -2av_2 + ce^{-i\sigma_k} - be^{-i\sigma_k}v_2, \quad F = -bv_2.$$

Thus, we only need to compute $h_{110}(\theta)$ and $h_{200}(\theta)$. From (4.14) and (4.15), we know that $h_{110} = (h_{110}^1, h_{110}^2)^T$ and $h_{200} = (h_{200}^1, h_{200}^2)^T$ are respectively the solution of the following two systems

$$\dot{h}_{110} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\dot{h}_{110}(0) - L(\tau_k)(h_{110}) = 2\tau_k \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \tag{4.17}$$

and

$$\dot{h}_{200} - 2i\sigma_k h_{200} = (\Phi_1 \Phi_2)\begin{pmatrix} a_{20} \\ \bar{a}_{02} \end{pmatrix},$$
$$\dot{h}_{200}(0) - L(\tau_k)(h_{200}) = 2\tau_k \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \tag{4.18}$$

where

$$a_1 = -2a - be^{i\sigma_k} - be^{-i\sigma_k} - ce^{i\sigma_k}\bar{v}_2 - ce^{-i\sigma_k}v_2 = 0,$$
$$b_1 = -2av_2\bar{v}_2 + ce^{-i\sigma_k}\bar{v}_2 + ce^{i\sigma_k}v_2 - be^{i\sigma_k}v_2\bar{v}_2 - be^{-i\sigma_k}v_2\bar{v}_2 = 0,$$
$$a_2 = -a - be^{-i\sigma_k} - cv_2 e^{-i\sigma_k},$$
$$b_2 = -av_2^2 + ce^{-i\sigma_k}v_2 - bv_2^2 e^{-i\sigma_k}.$$

Solving (4.17) and (4.18), we obtain

$$h_{110}(\theta) = c_1, \tag{4.19}$$

and

$$h_{200}(\theta) = -\frac{1}{i\sigma_k}(a_{20}e^{i\sigma_k\theta}v + \frac{1}{3}\bar{a}_{02}e^{-i\sigma_k\theta}\bar{v}) + e^{2i\sigma_k\theta}c_2, \tag{4.20}$$

where $c_1 = (c_1^{(1)}, c_1^{(2)})^T$, $c_2 = (c_2^{(1)}, c_2^{(2)})^T$ and

$$c_1^{(1)} = \frac{2a_1(a+b)y^* - 2b_1 cx^*}{x^*y^*[(a+b)^2 + c^2]}, \quad c_1^{(2)} = \frac{2a_1 cy^* + 2b_1(a+b)x^*}{x^*y^*[(a+b)^2 + c^2]},$$

$$c_2^{(1)} = \Big(2\tau_k a_2(2i\sigma_k + \tau_k ay^* + \tau_k by^* e^{-2i\sigma_k}) - 2\tau_k^2 b_2 cx^* e^{-2i\sigma_k}\Big)$$

$$\div \Big(\tau_k^2 c^2 x^* y^* e^{-4i\sigma_k} + (2i\sigma_k + \tau_k ax^*$$

$$+ \tau_k bx^* e^{-2i\sigma_k})(2i\sigma_k + \tau_k ay^* + \tau_k by^* e^{-2i\sigma_k})\Big),$$

$$c_2^{(2)} = \Big(2\tau_k b_2(2i\sigma_k + \tau_k ax^* + \tau_k bx^* e^{-2i\sigma_k}) + 2\tau_k^2 a_2 cy^* e^{-2i\sigma_k}\Big)$$

$$\div \Big(\tau_k^2 c^2 x^* y^* e^{-4i\sigma_k} + (2i\sigma_k + \tau_k ax^*$$

$$+ \tau_k bx^* e^{-2i\sigma_k})(2i\sigma_k + \tau_k ay^* + \tau_k by^* e^{-2i\sigma_k})\Big).$$

Finally, we compute $\mathrm{Proj}_s f_3^1(x,0,0)$. From (4.8), we can see that $f_3^1(x,0,0) = 0$.
Summarizing the above steps, we obtain

$$\frac{1}{3!}g_3^1(x,0,0) = \begin{pmatrix} A_3 x_1^2 x_2 \\ \bar{A}_3 x_1 x_2^2 \end{pmatrix},$$

where

$$A_3 = \frac{1}{6i\sigma_k}|a_{02}|^2 + \frac{1}{2}c_3. \tag{4.21}$$

Accordingly, the normal form of (4.9) has the form

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x,0,\alpha) + \frac{1}{3!}g_3^1(x,0,\alpha) + h.o.t.$$

$$= Bx + \begin{pmatrix} A_1 x_1 \alpha \\ \bar{A}_1 x_2 \alpha \end{pmatrix} + \begin{pmatrix} A_3 x_1^2 x_2 \\ \bar{A}_3 x_1 x_2^2 \end{pmatrix} + O(|x|\alpha^2 + |x|^4).$$

The normal form (4.9) relative to P can be written in real coordinates (w_1, w_2) through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$. Followed by the use of polar coordinates (ρ, ξ), $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, this formal form becomes

$$\dot{\rho} = k_1\alpha\rho + k_2\rho^3 + O(\alpha^2\rho + |(\rho,\alpha)|^4),$$
$$\dot{\xi} = -\sigma_k + O(|(\rho,\alpha)|), \tag{4.22}$$

where $k_1 = \mathrm{Re}\,A_1$, $k_2 = \mathrm{Re}\,A_3$. Following [2], we know that the sign of $k_1 k_2$ determines the direction of the bifurcation and the sign of k_2 determines the stability of the nontrivial periodic solution bifurcating from Hopf bifurcation. Therefore, we have the following theorem

Theorem 4.1. *The flow of (4.6) on the center manifold of the origin at $\tau = \tau_k$ is given by (4.22). Hopf bifurcation is supercritical if $k_1 k_2 < 0$ and subcritical if $k_1 k_2 > 0$. Moreover, the nontrivial periodic solution is stable if $k_2 < 0$ and unstable if $k_2 > 0$.*

We remark that by the same method and calculation, for the nonsymmetric system, we can also obtain a similar conclusion. Thus, the symmetry does not affect the emergence of periodic solutions.

5. Numerical simulations

In the section, we will give numerical simulations to illustrate the conclusions obtained. Consider system (3.1) with $r_1 = 3/2$, $r_2 = 1$, $a = 1/2$, $b = 1/3$, $c = 1$;

that is,

$$\dot{x}(t) = x(t)[\frac{3}{2} - \frac{1}{2}x(t) - \frac{1}{3}x(t-\tau) - y(t-\tau)],$$
$$\dot{y}(t) = y(t)[1 - \frac{1}{2}y(t) + x(t-\tau) - \frac{1}{3}y(t-\tau)].$$
(5.1)

In this case, system (5.1) has unique positive equilibrium $(x^*, y^*) = (\frac{9}{61}, \frac{84}{61})$. By means of the software Maple, we know that $y_0 = 0.0599$ is a simple root of (3.7). By computation, we obtain $\sigma_+ = 0.2448, \tau_0 = 4.4898, \sigma_0 = \tau_0\sigma_+ = 1.0989$. Applying Theorems 3.3 and 4.1, we have the following theorem.

Theorem 5.1. *The characteristic equation of* (5.1) *at* $(x^*, y^*) = (\frac{9}{61}, \frac{84}{91})$ *has a simple pair of purely imaginary roots* $\pm i\sigma_0$ *for* $\tau = \tau_0$ *and the other roots has non-zero real parts. System* (5.1) *with* $\tau_0 = 4.4898$ *has a supercritical Hopf bifurcation and the nontrivial periodic solution bifurcating from Hopf bifurcation of* (5.1) *is stable on the center manifold.*

Proof. From Theorem 3.3, we know that the equilibrium (x^*, y^*) is locally asymptotically stable when $\tau \in [0, 4.4898)$ and is unstable when $\tau > 4.4898$. In the following, we will consider the direction and stability of Hopf bifurcation at $\tau = \tau_0 = 4.4898$. By means of software Maple, we obtain the following values:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.9197 - 1.1994i \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.2107 - 0.3369i \\ 0.0226 + 0.0602i \end{pmatrix},$$

and

$$a_{20} = 5.1560 + 3.3283i, \quad a_{02} = -4.9973 - 2.9057i,$$
$$c_3 = -24.0701 - 5.0921i.$$

According to (4.12) and (4.21), we obtain

$$A_1 = 0.0756 + 0.0744i, \quad A_3 = -12.0351 - 7.6141i.$$

Thus, $k_1 = 0.0756$, $k_2 = -12.0351$. Applying Theorem 4.1, we obtain that system (5.1) has a supercritical Hopf bifurcation at $\tau = \tau_0$ and the nontrivial periodic solution associated with Hopf bifurcation at $\tau = \tau_0$ is stable on the center manifold. The proof is complete. $\square$

The numerical simulations for $\tau = 4.4898$ are shown in Figure 1.

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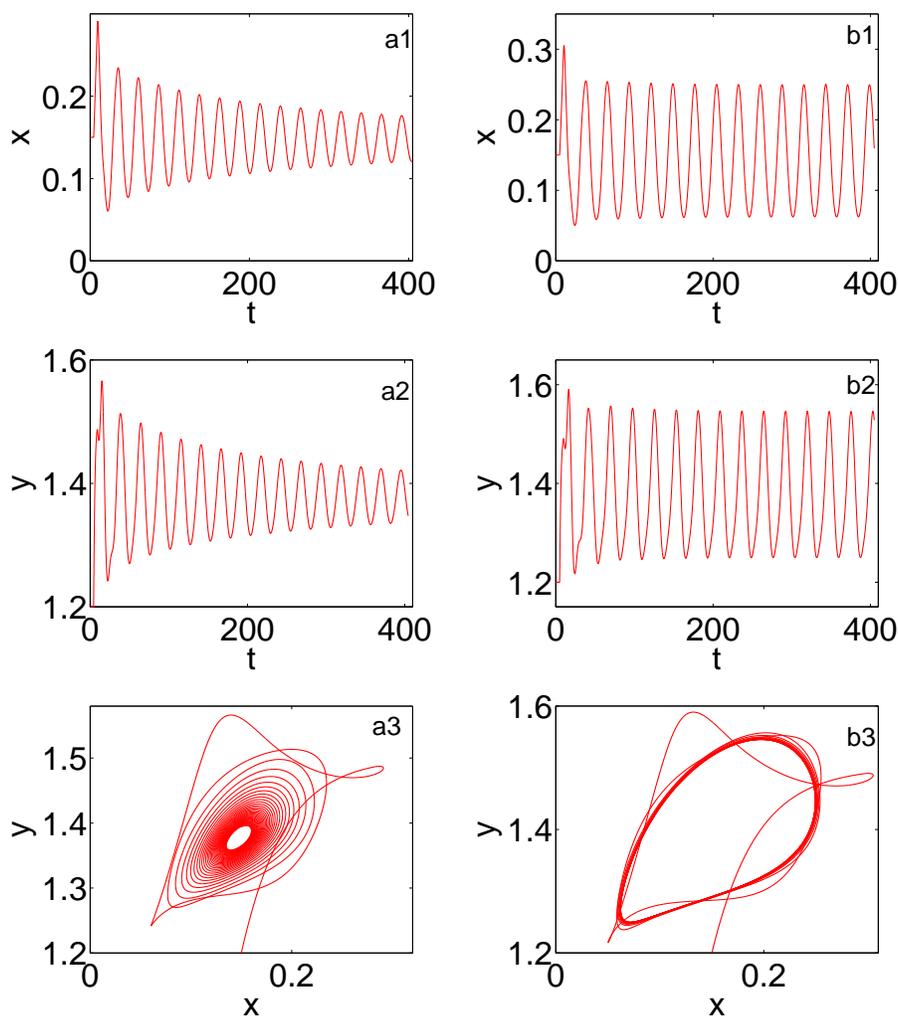
FIGURE 1. Trajectories for system (5.1) with $\tau = 4.41$ (left column) and $\tau = 4.49$ (right column), on the t-x plane (top row), t-y plane (middle row), and x-y plane (bottom row)

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