Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 10, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NONLINEAR CONVECTION IN REACTION-DIFFUSION EQUATIONS UNDER DYNAMICAL BOUNDARY CONDITIONS

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Abstract. We study the blow-up phenomena for positive solutions of nonlinear reaction-diffusion equations including a nonlinear convection term $\partial_t u = \Delta u - g(u) \cdot \nabla u + f(u)$ in a bounded domain of $\mathbb{R}^N$ under the dissipative dynamical boundary conditions $\sigma \partial_t u + \partial_\nu u = 0$. Some conditions on g and f are discussed to state if the positive solutions blow up in finite time or not. Moreover, for certain classes of nonlinearities, an upper-bound for the blow-up time can be derived and the blow-up rate can be determined.

1. Introduction

We consider the nonlinear parabolic problem

$$\begin{gathered}\partial_t u = \Delta u - g(u) \cdot \nabla u + f(u) \quad \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 \quad \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \overline{\Omega},\end{gathered} \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}^N$, $f : \mathbb{R} \to \mathbb{R}$, Ω is a bounded domain of $\mathbb{R}^{\mathbb{N}}$ with $\mathcal{C}^2$-boundary $\partial\Omega$. We denote by $\nu : \partial\Omega \to \mathbb{R}^N$ the outer unit normal vector field, and by ∂_ν the outer normal derivative.

These equations arise in different areas, especially in population growth, chemical reactions and heat conduction. For instance, in the case of a heat transfer in a medium Ω, the first equation $\partial_t u = \Delta u - g(u) \cdot \nabla u + f(u)$ is a heat equation including a nonlinear convection term $g(u) \cdot \nabla u$ and a nonlinear source f. On the boundary $\partial\Omega$, if σ is positive, the dynamical boundary conditions describe the fact that a heat wave with the propagation speed $\frac{1}{\sigma}$ is sent into the region into an infinitesimal layer near the boundary due to the heat flux across the boundary (see [6] and [11]).

There are various results in the literature about the theory of blow-up for semilinear parabolic equations, in particular for reaction-diffusion equations, see e.g. [8, 9, 10, 12]. In this work, we discuss a problem involving a nonlinear convection term. Whereas a Burgers' equation has been studied in [5] in the one-dimensional case, we now consider a more general convection term and we set in a regular domain of $\mathbb{R}^N$. After recalling some qualitative properties in Section 2, we construct a

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2000 *Mathematics Subject Classification.* 35K55, 35B44.
Key words and phrases. Nonlinear parabolic problem; dynamical boundary conditions; lower and upper-solution; blow-up; global solution.

Submitted July 10, 2012. Published January 9, 2013.

global upper-solution for Problem (1.1) in Section 3 and we deduce some conditions on f and g guaranteeing global existence of the solutions (Theorem 3.3). In Section 4, we investigate two methods to ensure the blow-up of solutions of Problem (1.1). The first one is an eigenfunction method valid for the model problem

$$\begin{gathered} \partial_t u = \Delta u - g(u)\cdot\nabla u + u^p \quad \text{in } \overline{\Omega} \text{ for } t>0,\\ \sigma\partial_t u + \partial_\nu u = 0 \quad \text{on } \partial\Omega \text{ for } t>0,\\ u(\cdot,0) = u_0 \quad \text{in } \overline{\Omega}, \end{gathered} \tag{1.2}$$

with $p>1$ (Theorem 4.2). We also derive some upper bounds for the blow-up time. The second method, devoted to the problem

$$\begin{gathered} \partial_t u = \Delta u - g(u)\cdot\nabla u + e^{pu} \quad \text{in } \overline{\Omega} \text{ for } t>0,\\ \sigma\partial_t u + \partial_\nu u = 0 \quad \text{on } \partial\Omega \text{ for } t>0,\\ u(\cdot,0) = u_0 \quad \text{in } \overline{\Omega}, \end{gathered} \tag{1.3}$$

with $p>0$, requires a self-similar lower-solution which blows up in finite time (Theorem 4.3). We prove the blow-up of solutions of Problem (1.3). Finally, in Section 5, we determine the blow-up rate of the solutions of Problem (1.2) in the L^∞-norm when approaching the blow-up time (Theorem 5.2).

Throughout, we shall assume the dissipativity condition

$$\sigma \geq 0 \quad \text{on } \partial\Omega\times(0,\infty). \tag{1.4}$$

To study classical solutions, we always assume that the parameters in the equations of Problem (1.1) are smooth

$$\sigma \in \mathcal{C}_b^1(\partial\Omega\times(0,\infty)), \tag{1.5}$$

$$f\in\mathcal{C}^1(\mathbb{R}), \quad f(s)>0 \quad \text{for } s>0, \tag{1.6}$$

$$g\in\mathcal{C}^1(\mathbb{R},\mathbb{R}^N). \tag{1.7}$$

The initial data is continuous, non-trivial and non-negative in $\overline{\Omega}$

$$u_0\in\mathcal{C}(\overline{\Omega}), \quad u_0\not\equiv 0, \quad u_0\geq 0. \tag{1.8}$$

Let $T=T(\sigma,u_0)$ denote the maximal existence time of the unique maximal classical solution of Problem (1.1),

$$u_\sigma\in\mathcal{C}(\overline{\Omega}\times[0,T))\cap\mathcal{C}^{2,1}(\overline{\Omega}\times(0,T))$$

with the coefficient σ in the boundary conditions and the initial data u_0. As for the well-posedness and the local existence of the solutions of Problem (1.1), we refer to [2], [6] and [7]. From [6], since the convection term depends linearly on the gradiant ∇u of the solution, the maximal existence time T is the blow-up time of the solution with respect to the L^∞-norm:

$$T=\inf\left\{\ s>0 : \lim_{t\nearrow s}\sup_{\overline{\Omega}}|u(x,t)|=\infty\right\}.$$

2. Qualitative properties

The aim of this section is to compare the solutions for different parameters σ and initial data u_0 and to summarize some positivity results on the classical solutions of Problem (1.1).

Using the maximum principle from [2], we extend some results obtained in [3] in the case of reaction-diffusion to our problem with convection.

Theorem 2.1. *Assume hypotheses* (1.4)–(1.8). *Suppose that* σ *does not depend on time*

$$\sigma \in \mathcal{C}^1(\partial\Omega). \tag{2.1}$$

Then the solution u *of Problem* (1.1) *satisfies*

$$\begin{gathered} u > 0 \quad \text{in } \overline{\Omega} \times (0, T(\sigma, u_0)), \\ \partial_t u \geq 0 \quad \text{in } \overline{\Omega} \times [0, T(\sigma, u_0)), \\ \partial_t u > 0 \quad \text{in } \overline{\Omega} \times (0, T(\sigma, u_0)). \end{gathered}$$

Moreover, for all $\xi \in (0, T(\sigma, u_0))$*, there exists* $d > 0$ *such that*

$$\partial_t u > d \quad \text{in } \overline{\Omega} \times [\xi, T(\sigma, u_0)).$$

Proof. Let $\tau \in (0, T(\sigma, u_0))$. Since u is $\mathcal{C}^{2,1}(\overline{\Omega} \times [0, \tau])$ and because f and g are smooth ((1.6) and (1.7)), we can define these constants

$$C = \sup_{\overline{\Omega} \times [0,\tau]} g(u), \quad M = \sup_{\overline{\Omega} \times [0,\tau]} g'(u) \cdot \nabla u - f'(u).$$

First, the positivity principle [2, Corollary 2.4] applied to Problem (1.1) implies $u \geq 0$ in $\overline{\Omega} \times [0, \tau]$ since $f \geq 0$ by condition (1.6). Thus we obtain

$$\begin{gathered} \partial_t u \geq \Delta u - g(u) \cdot \nabla u \geq \Delta u - C|\nabla u| \quad \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + \partial_\nu u = 0 \quad \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 \quad \text{in } \overline{\Omega}. \end{gathered}$$

The strong maximum principle from [2] implies

$$m := \min_{\overline{\Omega} \times [0,\tau]} u = \min_{\overline{\Omega}} u_0 \,,$$

and if this minimum m is attained in $\overline{\Omega} \times (0, \tau]$, $u \equiv m$ in $\overline{\Omega} \times [0, \tau]$. Since $f > 0$ in $(0, \infty)$, the first equation in Problem (1.1) leads to $m = 0$, and we obtain $u_0 \equiv 0$, a contradiction with equation (1.8). Hence $u > m \geq 0$ in $\overline{\Omega} \times (0, \tau]$. Then, since the coefficients in the equations of Problem (1.1) are sufficiently smooth, classical regularity results in [13] imply that $u \in \mathcal{C}^{2,2}(\overline{\Omega} \times [0, \tau])$. Thus $y = \partial_t u \in \mathcal{C}^{2,1}(\overline{\Omega} \times [0, \tau])$ and satisfies

$$\begin{gathered} \partial_t y = \Delta y - g(u) \cdot \nabla y - (g'(u) \cdot \nabla u) y + f'(u) y \quad \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t y + \partial_\nu y = 0 \quad \text{on } \partial\Omega \text{ for } t > 0. \end{gathered}$$

By continuity, condition (1.8) implies $y(\cdot, 0) \geq 0$ in $\overline{\Omega}$. Again, Corollary 2.4 from [2] implies $y \geq 0$ in $\overline{\Omega} \times [0, \tau]$. To apply properly the strong maximum principle, we have to introduce $w = y e^{Mt} \geq 0$. By definition of C and M, we obtain

$$\begin{gathered} \partial_t w \geq \Delta w - g(u) \cdot \nabla w \geq \Delta w - C|\nabla w| \quad \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t w + \partial_\nu w \geq 0 \quad \text{on } \partial\Omega \text{ for } t > 0. \end{gathered}$$

Again, the strong maximum principle from [2] implies

$$\tilde{m} := \min_{\overline{\Omega} \times [0,\tau]} w = \min_{\overline{\Omega}} w(\cdot, 0) \,,$$

and if this minimum $\tilde{m}$ is attained in $\overline{\Omega} \times (0, \tau]$, $w \equiv \tilde{m}$ in $\overline{\Omega} \times [0, \tau]$. In particular, if $\tilde{m} = 0$, we have $\partial_t u \equiv 0$ in $\overline{\Omega} \times [0, \tau]$, thus $u(\cdot, t) = u_0$ for all $t \in [0, \tau]$. Hence u attains its minimum in $\overline{\Omega} \times (0, \tau]$, which is impossible according to the first part of the proof. Thus w and $\partial_t u$ are positive in $\overline{\Omega} \times (0, \tau]$.

Finally, let $\xi \in (0,\tau)$. Because y is continuous and thanks to the previous point, there exists $d > 0$ such that $y(\cdot,\xi) > d$ in $\overline{\Omega}$. As y satisfies

$$\partial_t y = \Delta y - g(u)\cdot\nabla y - \Big(g'(u)\cdot\nabla u + f'(u)\Big)y \quad \text{in } \Omega\times[\xi,\tau],$$
$$\sigma\partial_t y + \partial_\nu y = 0 \quad \text{on } \partial\Omega\times[\xi,\tau],$$

the weak maximum principle from [2] implies

$$\min_{\overline{\Omega}\times[\xi,\tau]} y = \min_{\overline{\Omega}} y(\cdot,\xi)\,.$$

Hence $y > d$ in $\overline{\Omega}\times[\xi,\tau]$. Note that d depends only on ξ, not on τ. Without this step, we only have $y \geq \tilde{m}e^{-M\tau}$ which may vanish as $\tau \to T(\sigma,u_0)$. $\square$

Let $0 \leq \sigma_1 \leq \sigma_2$ be two coefficients satisfying condition (1.5), $v_0 \leq u_0$ be two initial data fulfilling hypothesis (1.8) and w_0 a function in $\mathcal{C}_0(\overline{\Omega})$ with $0 \leq w_0 \leq v_0$. Denote by u_{σ_1}, u_{σ_2}, v and w the maximal solutions of the following four problems

$$\partial_t u_{\sigma_1} = \Delta u_{\sigma_1} - g(u_{\sigma_1})\cdot\nabla u_{\sigma_1} + f(u_{\sigma_1}) \quad \text{in } \Omega \text{ for } t>0,$$
$$\sigma_1\partial_t u_{\sigma_1} + \partial_\nu u_{\sigma_1} = 0 \quad \text{on } \partial\Omega \text{ for } t>0,$$
$$u_{\sigma_1}(\cdot,0) = u_0 \quad \text{in } \overline{\Omega};$$

$$\partial_t u_{\sigma_2} = \Delta u_{\sigma_2} - g(u_{\sigma_2})\cdot\nabla u_{\sigma_2} + f(u_{\sigma_2}) \quad \text{in } \Omega \text{ for } t>0,$$
$$\sigma_2\partial_t u_{\sigma_2} + \partial_\nu u_{\sigma_2} = 0 \quad \text{on } \partial\Omega \text{ for } t>0,$$
$$u_{\sigma_2}(\cdot,0) = u_0 \quad \text{in } \overline{\Omega};$$

$$\partial_t v = \Delta v - g(v)\cdot\nabla v + f(v) \quad \text{in } \Omega \text{ for } t>0,$$
$$\sigma_2\partial_t v + \partial_\nu v = 0 \quad \text{on } \partial\Omega \text{ for } t>0,$$
$$v(\cdot,0) = v_0 \quad \text{in } \overline{\Omega};$$

and

$$\partial_t w = \Delta w - g(w)\cdot\nabla w + f(w) \quad \text{in } \Omega \text{ for } t>0,$$
$$w = 0 \quad \text{on } \partial\Omega \text{ for } t>0,$$
$$w(\cdot,0) = w_0 \quad \text{in } \overline{\Omega}.$$

Let $T(\sigma_1,u_0)$, $T(\sigma_2,u_0)$, $T(\sigma_2,v_0)$ and $T(w_0)$ be their respective maximal existence times. For the reader convenience, we recall some results stemming from the comparison principle [2].

Theorem 2.2 ([4]). *Under the aforementioned hypotheses, we have*

$$T(\sigma_2,u_0) \leq T(\sigma_2,v_0) \leq T(w_0),$$
$$0 \leq w \leq v \leq u_{\sigma_2} \quad \text{in } \overline{\Omega}\times[0,T(\sigma_2,u_0))\,.$$

In addition, if $u_0 \in \mathcal{C}^2(\overline{\Omega})$ with

$$\Delta u_0 - g(u_0)\cdot\nabla u_0 + f(u_0) \geq 0 \text{ in } \Omega, \tag{2.2}$$

we have

$$T(\sigma_1,u_0) \leq T(\sigma_2,u_0),$$

$$u_{\sigma_2} \le u_{\sigma_1} \quad \text{in } \overline{\Omega} \times [0, T(\sigma_1, u_0))\,.$$

An important fact comes from the last statement of Theorem 2.1. For any positive solution u of Problem (1.1), the maximum principle implies that for any $s \in (0, T(\sigma, u_0))$, there exists $c > 0$ such that $u(\cdot, s) \ge c$ in $\overline{\Omega}$. Then, consider the solution $\tilde{u}$ of (1.1) with the constant initial data c and $\tilde{\sigma} = \sup \sigma$ in the boundary conditions. Theorem 2.2 implies $\tilde{u} \le u$. Since c satisfies (2.2), Theorem 2.1 leads to $\partial_t \tilde{u} > d > 0$. Thus, $\tilde{u}$ can be big enough after a long time (maybe it blows up). So does u, even if u_0 does not satisfy condition (2.2).

3. Existence of global solutions

In this section, we give some conditions on the function g in the convection term, which ensure the existence of global solutions to Problem (1.1) for various reaction terms f. We use the comparison method from [2]. Thus, we just need to find an appropriate upper-solution of Problem (1.1) which does not blow up. This is our first lemma.

Lemma 3.1. *Let $\alpha > 0$ and $K > 0$ be two real numbers and let $\eta \in \mathcal{C}^1([0, \infty))$ with $\eta' \ge \alpha^2$. For any integer $1 \le j \le N$, the function U defined in $\Omega \times [0, \infty)$ by*

$$U(x, t) = K \exp\big(\alpha x_j + \eta(t)\big),$$

satisfies

$$\begin{gathered} \partial_t U \ge \Delta U - g(U) \cdot \nabla U + f(U) \quad \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t U + \partial_\nu U \ge 0 \quad \text{on } \partial\Omega \text{ for } t > 0, \\ U(\cdot, 0) > 0 \quad \text{in } \overline{\Omega}, \end{gathered}$$

if

$$\alpha g_j(\omega) \ge \frac{f(\omega)}{\omega} \quad \text{for all } \omega \ge 0 \tag{3.1}$$

and if

$$\sigma(x, t) \ge \frac{\alpha}{\eta'(t)} \quad \text{for all } t > 0. \tag{3.2}$$

Proof. A simple computation of the derivatives of U leads us to

$$\partial_t U - \Delta U + g(U) \cdot \nabla U = \big(\eta' - \alpha^2\big) U + \alpha g_j(U) U \quad \text{in } \Omega \text{ for } t > 0.$$

Since we assume $\eta' \ge \alpha^2$, hypothesis (3.1) implies

$$\partial_t U - \Delta U + g(U) \cdot \nabla U - f(U) \ge 0 \quad \text{in } \Omega \times (0, \infty).$$

Furthermore, on the boundary $\partial\Omega$, for $t > 0$, we have

$$\begin{aligned} \sigma \partial_t U + \partial_\nu U &= \big(\sigma \eta'(t) + \alpha \nu_j(x)\big) U \\ &\ge \big(\sigma \eta'(t) - \alpha\big) U \ge 0, \end{aligned} \tag{3.3}$$

by hypothesis (3.2) since ν is normalized, and clearly $U(x, 0) = K \exp\big(\alpha x_j + \eta(0)\big) > 0$ in $\overline{\Omega}$. □

Remark 3.2. In the case of the Dirichlet boundary conditions, we can use this upper-solution with the special choice $\eta \equiv 0$ (see [14]). However, for the dynamical boundary conditions, we must use a positive time-dependent η because our solutions are not bounded, see Theorem 2.1.

Now we can state the following theorems for a nonlinear reaction term f growing as a power of u (Problem (1.2)), or as an exponential function (Problem (1.3)).

Theorem 3.3. *Let σ be a coefficient fulfilling conditions* (1.4), (1.5) *and such that there exists $\delta > 0$ with*

$$\inf_{\partial\Omega} \sigma \geq \delta \sup_{\partial\Omega} \sigma \quad \textit{for } t > 0 \quad \textit{and} \quad \Big(\sup_{x\in\partial\Omega} \sigma(x,\cdot)\Big)^{-1} \in L^1_{\rm loc}(\mathbb{R}^+).$$

Assume u_0 satisfies condition (1.8). *If there exists an integer $1 \leq j \leq N$ such that*

$$\liminf_{\omega\to\infty} \frac{g_j(\omega)}{\omega^{p-1}} > 0, \tag{3.4}$$

then the solution of Problem (1.2) *is a global solution.*

Proof. In view of Theorem 2.1 and (3.4), we can suppose that u_0 is sufficiently big such that there exists $C > 0$ with

$$g_j(u) \geq Cu^{p-1} \text{ in } \Omega \text{ for } t > 0.$$

For

$$\eta(t) = C\delta^{-1} \int_0^t \Big(\sup_{x\in\partial\Omega} \sigma(x,s)\Big)^{-1} ds + C^2 t.$$

we have $\eta' \geq C^2$ and (3.2) is satisfied. Let K be a positive number such that

$$K \geq u_0(x) e^{-Cx_j - \eta(0)} \quad \text{for all } x \in \overline{\Omega}.$$

Then by hypotheses (1.5), (1.8) and (3.2), the function U defined in Lemma 3.1 is an upper-solution of (1.2) since $U(\cdot,0) \geq u_0$ in $\overline{\Omega}$. Using the comparison principle from [2], the unique solution u of Problem (1.1) satisfies

$$0 \leq u(x,t) \leq U(x,t) \quad \text{for all } x \in \overline{\Omega} \text{ and } t > 0\,,$$

thus u does not blow up. $\square$

This theorem holds in particular for a nonlinearity g in the form

$$g(u) = (\alpha_1 u^{q_1}, \dots, \alpha_i u^{q_i}, \dots, \alpha_N u^{q_N})$$

with at least one integer j such that $\alpha_j > 0$ and $q_j \geq p - 1$. A similar result can be derived for Problem (1.3).

Theorem 3.4. *Under the aforementioned assumptions, the solution of Problem* (1.3) *is a global solution if the convection term $g(u)\cdot\nabla u$ has (at least) one component g_j satisfying $g_j(u) = \alpha_j e^{q_j u}$ with $\alpha_j > 0$ and $q_j > p$.*

Proof. Thanks to $q_j > p$, condition (3.1) is fulfilled because $\alpha_j e^{q_j u} \geq \alpha_j e^{pu}/u$ for u sufficiently big. $\square$

Remark 3.5. Condition (3.4) is optimal for Problem (1.2), see Theorems 3.3 and 4.2. But it can be improved in some special cases, for example, if the reaction term is $f(u) = u \ln u$. Lemma 3.1 implies that all solutions of Problem (1.1) are global if one component g_j of g satisfies $g_j(u) \geq \alpha_j \ln u$. In fact, in that case, every positive solution of (1.1) is global, without any assumption on the convection term g, since $\int_c^\infty 1/f(y)\,dy = \infty$ for $c > 0$, see [6, Theorem 3.2].

Condition (3.2) on σ allows us to consider fast decaying functions σ, but, to ensure global existence, it is essential that σ does not vanish on the whole $\partial\Omega$. Indeed let us prove the following blow-up result related to the Neumann boundary conditions, for $\sigma \equiv 0$ on $\partial\Omega$.

Theorem 3.6. *Assume that $\sigma \equiv 0$, u_0 fulfills hypothesis* (1.8) *and f is positive in* $(0, \infty)$ *such that*

$$\int_c^\infty \frac{1}{f(y)}\,dy < \infty \quad \text{for some } c > 0\,. \tag{3.5}$$

Then every positive solution of (1.1) *blows up in finite time.*

Proof. Let u be a non-trivial positive solution of

$$\begin{gathered} \partial_t u = \Delta u - g(u)\cdot\nabla u + f(u) \quad \text{in } \Omega \text{ for } t > 0, \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 \quad \text{in } \overline{\Omega}. \end{gathered} \tag{3.6}$$

Using the maximum principle from [2], we have $u(\cdot, \xi) > 0$ in $\overline{\Omega}$ for $\xi > 0$. Hence, without loss of generality, we suppose $u_0 > c$ in $\overline{\Omega}$. Now, consider the maximal solution z of the ODE $\dot{z} = f(z)$ with the initial data $z(0) = \inf\{u_0(x) : x \in \overline{\Omega}\}$. Condition (3.5) implies that its maximal existence time T_z is finite:

$$T_z = \int_{z(0)}^\infty \frac{1}{f(y)}\,dy < \infty.$$

Since $\nabla z = 0$, z is a lower solution of Problem (3.6). Using the comparison principle from [2], we obtain $z(t) \le u(\cdot, t)$ in $\overline{\Omega}$ for $t > 0$. Thus, u must blow up in finite time with $0 < T < T_z$. $\square$

Remark 3.7. This section illustrates the damping effect of the dissipative dynamical boundary conditions: we have shown that for nontrivial $\sigma \ge 0$ the maximal existence time of the solutions of Problem (1.1) can be strictly greater than the ones under the Neumann boundary conditions.

4. Blow-up

In this section, we investigate the blow-up in finite time for the solutions of Problems (1.2) and (1.3). Let G be a primitive of g and suppose that there exist $\alpha > 0$ and $q < p$ such that

$$G(\omega) \le \alpha\omega^q \quad \text{for } \omega > 0. \tag{4.1}$$

By applying the eigenfunction method (see [4, 9, 12]), we obtain some conditions on the initial data u_0 which guarantee the finite time blow-up and we derive some upper bounds for the blow-up times. This is a general technique which can be applied to the following problem, where the boundary behaviour of the solutions is not involved:

$$\begin{gathered} \partial_t u = \Delta u - g(u)\cdot\nabla u + u^p \quad \text{in } \overline{\Omega} \text{ for } t > 0, \\ u \ge 0 \quad \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = u_0 \quad \text{in } \overline{\Omega}. \end{gathered} \tag{4.2}$$

Henceforth, we denote by λ the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and by φ an eigenfunction associated to λ satisfying

$$\varphi \in H_0^1(\Omega), \quad 0 < \varphi \le 1 \quad \text{in } \Omega. \tag{4.3}$$

Theorem 4.1. *Let $\alpha > 0$, $1 < q < p$, $m = p/(p-q)$ and suppose G satisfies condition* (4.1)*. Assume hypotheses* (1.4), (1.5), (1.7) *and* (1.8) *are fulfilled. If*

$$\int_\Omega u_0\varphi^m\,dx > (2|\Omega|^{p-1}C)^{1/p} \tag{4.4}$$

with

$$C = (p-1)|\Omega|\Big(\frac{4\lambda}{p-q}\Big)^{\frac{1}{p-1}} + \Big(\frac{4q}{p-q}\Big)^{\frac{q}{p-q}}\alpha^m\int_\Omega |\nabla\varphi|^m\,dx\,,$$

then the maximal classical solutions u of Problem (4.2) *blow up in finite time T satisfying*

$$T \le \frac{2\int_\Omega u_0\varphi^m\,dx}{(p-1)\Big(|\Omega|^{1-p}\Big(\int_\Omega u_0\varphi^m\,dx\Big)^p - 2C\Big)} =: \tilde{T}. \tag{4.5}$$

Proof. Define

$$M(t) = \int_\Omega u(x,t)\varphi(x)^m\,dx.$$

Thus,

$$\dot{M}(t) = \int_\Omega \Delta u\varphi^m\,dx - \int_\Omega g(u)\cdot\nabla u\,\varphi^m\,dx + \int_\Omega u^p\varphi^m\,dx.$$

First, we prove that

$$\int_\Omega \Delta u\varphi^m\,dx \ge -m\lambda|\Omega|^{\frac{p-1}{p}}\Big(\int_\Omega u^p\varphi^m\,dx\Big)^{1/p}. \tag{4.6}$$

Observe that the behaviour of φ and $\partial_\nu\varphi$ on $\partial\Omega$ imply

$$\int_{\partial\Omega}\partial_\nu u\varphi^m\,ds = 0 \quad\text{and}\quad \int_{\partial\Omega} u\partial_\nu(\varphi^m)\,ds \le 0, \tag{4.7}$$

since $u \ge 0$ on $\partial\Omega$ for $t > 0$. As in [14], Equation (4.7) and Green's formula yield

$$\int_\Omega \Delta u\varphi^m\,dx \ge -m\lambda\int_\Omega u\varphi^m\,dx. \tag{4.8}$$

Since $\varphi \le 1$, $\int_\Omega u\varphi^m\,dx \le \int_\Omega u\varphi^{\frac{m}{p}}\,dx$ and by Hölder's inequality, (4.6) holds.
Now, we show that

$$-\int_\Omega g(u)\cdot\nabla u\,\varphi^m\,dx \ge -m\alpha\Big(\int_\Omega|\nabla\varphi|^m\,dx\Big)^{1/m}\Big(\int_\Omega u^p\varphi^m\,dx\Big)^{q/p}. \tag{4.9}$$

By Green's formula and by definition of G and φ, we have

$$-\int_\Omega g(u)\cdot\nabla u\,\varphi^m\,dx = -\int_\Omega \operatorname{div}(G(u))\varphi^m\,dx = m\int_\Omega (G(u)\cdot\nabla\varphi)\varphi^{m-1}\,dx\,.$$

Equation (4.1) and Hölder's inequality lead to

$$\begin{aligned}\Big|\int_\Omega (G(u)\cdot\nabla\varphi)\varphi^{m-1}\,dx\Big| &\le \alpha\int_\Omega u^q\varphi^{m-1}|\nabla\varphi|\,dx\\ &\le \alpha\Big(\int_\Omega|\nabla\varphi|^m\,dx\Big)^{1/m}\Big(\int_\Omega u^p\varphi^{\frac{(m-1)p}{q}}\,dx\Big)^{q/p},\end{aligned}$$

and (4.9) is satisfied. Henceforth, introduce

$$C_1 = m\lambda|\Omega|^{\frac{p-1}{p}},\quad C_2 = m\alpha\Big(\int_\Omega|\nabla\varphi|^m\,dx\Big)^{1/m}.$$

Then we obtain

$$\dot{M}(t) \geq \int_\Omega u^p\, \varphi^m\, dx - C_1\Big(\int_\Omega u^p \varphi^m\, dx\Big)^{1/p} - C_2\Big(\int_\Omega u^p \varphi^m\, dx\Big)^{q/p}. \tag{4.10}$$

Set

$$\varepsilon_1 = \frac{p^{1/p}}{4^{1/p} C_1}, \quad \varepsilon_2 = \frac{p^{q/p}}{(4q)^{q/p} C_2}.$$

Recall Young's inequality: for $a > 0$ and $\varepsilon > 0$,

$$a = \frac{\varepsilon a}{\varepsilon} \leq \frac{\varepsilon^r a^r}{r} + \frac{1}{s\varepsilon^s}$$

for $r, s > 1$ with $r^{-1} + s^{-1} = 1$. It yields

$$C_1\Big(\int_\Omega u^p\, \varphi^m\, dx\Big)^{1/p} \leq \frac{1}{4}\int_\Omega u^p\, \varphi^m\, dx + \underbrace{\frac{p-1}{p\varepsilon_1^{\frac{p}{p-1}}}}_{:=C_3},$$

and in the same way we have

$$C_2\Big(\int_\Omega u^p\, \varphi^m\, dx\Big)^{q/p} \leq \frac{1}{4}\int_\Omega u^p\, \varphi^m\, dx + C_4,$$

with

$$C_4 = \frac{1}{m\varepsilon_2^m}\,.$$

Then

$$\dot{M}(t) \geq \frac{1}{2}\int_\Omega u^p\, \varphi^m\, dx - C$$

with $C = C_3 + C_4 > 0$. By (4.3) and Hölder's inequality, we obtain that

$$\dot{M}(t) \geq \frac{1}{2}|\Omega|^{1-p} M^p - C.$$

Since M is increasing with respect to t, by (4.4) we have

$$\dot{M}(t) \geq \Big(\frac{1}{2}|\Omega|^{1-p} - CM(0)^{-p}\Big) M^p,$$

and we can conclude that u can not exist globally. To derive an upper bound for the blow-up time, we integrate the previous differential inequality between 0 and $t > 0$. We obtain

$$M(t) \geq \Big(M(0)^{1-p} - (p-1)\Big(\frac{1}{2}|\Omega|^{1-p} - CM(0)^{-p}\Big)t\Big)^{\frac{-1}{p-1}}.$$

Hence M blows up before $\tilde{T} = M(0)^{1-p}(p-1)^{-1}\big(\frac{1}{2}|\Omega|^{1-p} - CM(0)^{-p}\big)^{-1}$, so does u. Thus, $T \leq \tilde{T}$. □

Note that Condition (4.4) on the initial data is only necessary to derive an upper bound for the maximal existence time. Thanks to Theorem 2.1, we obtain the following result.

Theorem 4.2. *Let $q < p$ and suppose G satisfies*

$$\limsup_{\omega\to\infty} \frac{G(\omega)}{\omega^q} < \infty.$$

Assume that σ and u_0 satisfy conditions (1.4), (1.5), (1.7) *and* (1.8). *All the positive solutions of Problem* (1.2) *blow up in finite time.*

Proof. Let u be a positive solution of Problem (1.2). Theorem 2.1 permits to ensure that there exist $t_0 > 0$ and $C > 0$ such that $u(\cdot, t_0)$ is big enough to satisfy Equation (4.4) and $G(u) \le Cu^q$ in Ω for $t > t_0$. Thus applying Theorem 4.1 to $v(x,t) = u(x, t+t_0)$, we prove that v blows up in a finite time T_v satisfying (4.5). Hence, u blows up in a finite time $T_u = t_0 + T_v$. □ □

Now, we prove the blow-up of positive solutions of Problem (1.3).

Theorem 4.3. *Assume σ and u_0 satisfy conditions (1.4), (1.5), (1.7) and (1.8). If*

$$\limsup_{\omega\to\infty} \frac{|g(\omega)|}{e^{q\omega}} < \infty,$$

then all the positive solutions of Problem (1.3) blow up in finite time.

Proof. Let u be a positive solution of Problem (1.3) and define $v = e^{\gamma u}$ with $\gamma \in (q,p)$ and $\gamma > 1/2$. As in the previous proof, we suppose that u is sufficiently big such that for some $C > 0$

$$|g(u)| \le Ce^{qu} \quad \text{in } \Omega \text{ for } t > 0. \tag{4.11}$$

Computing the derivatives of v, we obtain

$$\partial_t v = \Delta v - \frac{1}{v}|\nabla v|^2 - g(u)\cdot \nabla v + \gamma v^{\frac{p+\gamma}{\gamma}} \quad \text{in } \Omega \text{ for } t > 0.$$

Using condition (4.11), we obtain

$$\partial_t v \ge \Delta v - \frac{1}{v}|\nabla v|^2 - Cv^{q/\gamma}|\nabla v| + \gamma v^{\frac{p+\gamma}{\gamma}} \quad \text{in } \Omega \text{ for } t > 0.$$

Young's inequality

$$Cv^{q/\gamma}|\nabla v| \le \frac{C^2}{2}|\nabla v|^2 + \frac{1}{2}v^{2q/\gamma},$$

leads to

$$\partial_t v \ge \Delta v - \frac{2+C^2}{2}|\nabla v|^2 + \gamma v^{\frac{p+\gamma}{\gamma}} - \frac{1}{2}v^{\frac{2q}{\gamma}} \quad \text{in } \Omega \text{ for } t > 0,$$

since $v \ge 1$. Morevover, we have

$$\gamma v^{\frac{p+\gamma}{\gamma}} - \frac{1}{2}v^{\frac{2q}{\gamma}} \ge (\gamma - \frac{1}{2})v^{\frac{p+\gamma}{\gamma}}$$

by definition of γ. Thus, we obtain

$$\begin{gathered} \partial_t v \ge \Delta v - \mu|\nabla v|^2 + \kappa v^{\frac{p+\gamma}{\gamma}} \quad \text{in } \overline{\Omega} \text{ for } t > 0, \\ v \ge 0 \quad \text{on } \partial\Omega \text{ for } t > 0, \\ v(\cdot, 0) > 0 \quad \text{in } \overline{\Omega}, \end{gathered} \tag{4.12}$$

with $\mu = (2 + C^2)/2$ and $\kappa = \gamma - 1/2$. Without loss of generality (see Theorem 2.1), we can suppose that $v(\cdot, 0) \ge V(\cdot, 0)$ in $\overline{\Omega}$, where

$$V(x,t) = (1-\varepsilon t)^{\frac{-1}{p-1}} W\Big(\frac{|x|}{(1-\varepsilon t)^m}\Big),$$

with $0 < m < \min\{\frac{1}{2}, \frac{p-q}{q(p-1)}\}$, $W(y) = 1 + A/2 - y^2/(2A)$, $A > \frac{1}{m(p-1)}$ and $\varepsilon < \frac{2\kappa(p-1)}{2+A}$. According to Souplet & Weissler [15], V is a blowing-up sub-solution for Problem (4.12). By the comparison principle from [2], $v \ge V$ and u blows up in finite time. □

Remark 4.4. In this section, we point out the accelerating effect of the dynamical boundary conditions, in comparison with the Dirichlet boundary conditions. Indeed, we prove that, even if the initial data u_0 is small, the solutions of Problem (1.2) blow up in finite time. But, if we replace the dynamical boundary conditions by the Dirichlet boundary conditions in the second equation of Problem (1.2), it is well known that the solutions are global and decay to 0 if the initial data are small enough, see for instance references [16] and [17].

5. Growth Order

In this section, we are interested in the blow-up rate for Problem (1.2) when approaching the blow-up time T. For the convection term, we assume that

$$g(u) = (g_1(u), \dots, g_n(u)) \quad \text{with } g_i(u) = u^q \ \forall i = 1, \dots, n,\ 1 < q \in \mathbb{R}. \tag{5.1}$$

First, we derive a lower blow-up estimate for $p > q + 1$, valid for any non-negative initial data $u_0 \in \mathcal{C}(\overline{\Omega})$.

Lemma 5.1. *Let $p > q+1$, and assume hypotheses (1.4)–(1.8). Then the classical maximal solution u of (1.2) satisfies*

$$\|u(\cdot, t)\|_\infty \geq (p-1)^{\frac{-1}{p-1}} (T-t)^{\frac{-1}{p-1}}$$

for $0 < t < T$.

Proof. Let $t \in [0, T)$. Denote by $\zeta \in \mathcal{C}^1((0, t_1))$ the maximal solution of the IVP

$$\dot{\zeta} = \zeta^p \quad \text{in } (0, t_1)$$
$$\zeta(0) = \|u(\cdot, t)\|_\infty$$

with $t_1 = \frac{1}{p-1} \|u(\cdot, t)\|_\infty^{1-p}$. Introduce $v \in \mathcal{C}(\overline{\Omega} \times [0, T-t)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0, T-t))$ defined by $v(x, s) = u(x, s+t)$ for $x \in \overline{\Omega}$ and $s \in [0, T-t)$. Then v is the maximal solution of the problem

$$\partial_t v = \Delta v - g(v) \cdot \nabla v + v^p \quad \text{in } \Omega \text{ for } 0 < s < T-t,$$
$$\sigma \partial_t v + \partial_\nu v = 0 \quad \text{on } \partial\Omega \text{ for } 0 < s < T-t,$$
$$v(\cdot, 0) = u(\cdot, t) \quad \text{in } \overline{\Omega}.$$

The comparison principle from [2] implies that $t_1 \leq T - t$. $\square$

This result remains valid for Problem (1.1) as soon as blow-up occurs. We just need a positive function f such that an explicit primitive of $\frac{1}{f}$ is known. We improve the technique developed in [5, Theorem 2.3] for an one-dimensional Burgers' problem and inspired by Friedman and McLeod [10] to prove that the growth order of the solution of (1.2) amounts to $-1/(p-1)$ for $p > 2q + 1 > 3$, when the time t approaches the blow-up time T.

Theorem 5.2. *Suppose conditions (1.4), (1.8), (2.1) and (5.1) are fulfilled. For*

$$p > 2q + 1\,, \tag{5.2}$$

there exists a positive constant C such that the classical maximal solution u of (1.2) satisfies

$$\|u(\cdot, t)\|_\infty \leq \frac{C}{(T-t)^{1/p-1}} \quad \text{for } t \in [0, T).$$

Proof. Let $\beta > 1$ such that

$$p(p-1)(p-2q-1) = \frac{Nq^2}{\beta} > 0, \tag{5.3}$$

and choose $M > 1$ such that

$$M \geq \frac{Nq}{2(2q+1)} \beta^{\frac{2q}{p-2q-1}}.$$

First, for $\xi \in (0, T)$, we shall prove that there exists $\delta > 0$ such that

$$\partial_t u \geq \delta e^{-Mt}(u^p + \beta u^{2q+1})$$

in $\overline{\Omega} \times [\xi, T)$. Introduce

$$J = \partial_t u - \delta d(t) k(u)$$

with $d(t) = e^{-Mt}$ and $k(u) = u^p + \beta u^{2q+1}$. Note that classical regularity results from [13] yield $J \in \mathcal{C}^{2,1}\big(\overline{\Omega} \times [\xi, T)\big)$. We recall that Theorem 2.1 implies that there exists $c > 0$ such that $\partial_t u \geq c > 0$ in $\overline{\Omega} \times [\xi, T)$. Thus, we can choose $\delta > 0$ sufficiently small such that

$$J(\cdot, \xi) \geq 0 \quad \text{in } \overline{\Omega}.$$

The function J satisfies the boundary condition

$$\begin{aligned} \sigma\partial_t J + \partial_\nu J &= \partial_t(\sigma\partial_t u + \partial_\nu u) - \delta d k'(u)(\sigma\partial_t u + \partial_\nu u) - \sigma\delta d' k(u) \\ &= \sigma\delta M e^{-Mt} k(u) \geq 0. \end{aligned}$$

Furthermore, J satisfies

$$\partial_t J - \Delta J + g(u) \cdot \nabla J - (pu^{p-1} - g'(u) \cdot \nabla u) J = \delta d H(u) \text{ in } \overline{\Omega} \times [\xi, T),$$

where

$$H(u) := pu^{p-1}k(u) - k'(u)u^p + k''(u)|\nabla u|^2 - \frac{d'}{d} k(u) - k(u) g'(u) \cdot \nabla u.$$

To prove that $H(u) \geq 0$, we shall show that

$$\begin{aligned} q\sqrt{N} u^{q-1} |\nabla u| (u^p + \beta u^{2q+1}) &\leq M(u^p + \beta u^{2q+1}) + \beta(p-2q-1)u^{p+2q} \\ &\quad + (p(p-1)u^{p-2} + 2q(2q+1)\beta u^{2q-1})|\nabla u|^2. \end{aligned} \tag{5.4}$$

Inequality (5.4) is trivial in the case where $M \geq q\sqrt{N} u^{q-1}|\nabla u|$. Now, suppose that $M < q\sqrt{N}u^{q-1}|\nabla u|$. When $q\sqrt{N}u^{q+1} \leq 2q(2q+1)|\nabla u|$, we have $q\sqrt{N}u^{q-1}u^p|\nabla u| \leq p(p-1)u^{p-2}|\nabla u|^2$ and $q\sqrt{N}u^{3q}|\nabla u| \leq 2q(2q+1)u^{2q-1}|\nabla u|^2$ since $p > 3$ then (5.4) follows. In the case where $q\sqrt{N}u^{q+1} > 2q(2q+1)|\nabla u|$, since

$$u > \Big(\frac{2(2q+1)}{Nq} M\Big)^{1/2q} \geq \beta^{\frac{1}{p-2q-1}},$$

we obtain

$$u^p + \beta u^{2q+1} \leq 2u^p. \tag{5.5}$$

Moreover, (5.3) yields

$$\begin{aligned} 2\sqrt{N}\, q u^{q+1} |\nabla u| &= 2\sqrt{\beta p(p-1)(p-2q-1)}\, u^{q+1} |\nabla u| \\ &\leq \Big(\sqrt{\beta(p-2q-1)}\, u^{q+1} - \sqrt{p(p-1)}\, |\nabla u|\Big)^2 \\ &\quad + 2\sqrt{\beta p(p-1)(p-2q-1)}\, u^{q+1} |\nabla u| \\ &\leq \beta(p-2q-1) u^{2(q+1)} + p(p-1)|\nabla u|^2. \end{aligned}$$

Thus, multiplying by u^{p-2} leads to

$$2\sqrt{N}qu^{q-1}|\nabla u|u^p \le \beta(p-2q-1)u^{p+2q} + p(p-1)u^{p-2}|\nabla u|^2,$$

and by (5.5), the inequality (5.4) holds. Finally, we can conclude by the comparison principle from [2] that $J \ge 0$ in $\overline{\Omega} \times [\xi, T)$, in particular, $\partial_t u \ge \varepsilon u^p$ with $\varepsilon > 0$.

Now, we shall derive the upper blow-up rate estimate of $\|u(\cdot,t)\|_\infty$ for $t \in [\xi, T)$. For each $x \in \Omega$, the integral

$$\int_t^\tau \frac{\partial_t u(x,s)}{u^p(x,s)}\,ds = \int_{u(x,t)}^{u(x,\tau)} \frac{1}{\eta^p}\,d\eta$$

converges as $\tau \to T$. Integrating the inequality $\partial_t u \ge \varepsilon u^p$ leads to

$$\varepsilon(\tau - t) \le \frac{u(x,\tau)^{1-p} - u(x,t)^{1-p}}{1-p} \le \frac{u(x,t)^{1-p}}{p-1}\,.$$

Letting $\tau \to T$ implies $u(x,t) \le \Big(\varepsilon(p-1)(T-t)\Big)^{\frac{-1}{p-1}}$ and we can conclude as in the proof of Theorem 2.3 from [5]. $\square$

Acknowledgments. The authors would like to thank Dr. Mabel Cuesta for her helpful discussions and valuable suggestions.

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