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WEAK SOLUTIONS FOR NONLOCAL EVOLUTION VARIATIONAL INEQUALITIES INVOLVING GRADIENT CONSTRAINTS AND VARIABLE EXPONENT

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Abstract. In this article, we study a class of nonlocal quasilinear parabolic variational inequality involving $p(x)$-Laplacian operator and gradient constraint on a bounded domain. Choosing a special penalty functional according to the gradient constraint, we transform the variational inequality to a parabolic equation. By means of Galerkin's approximation method, we obtain the existence of weak solutions for this equation, and then through a priori estimates, we obtain the weak solutions of variational inequality.

1. Introduction

In this article, we are concerned with the existence of weak solutions for nonlocal (Kirchhoff type) parabolic variational inequality involving variable exponent. More precisely, we shall find a function $u \in \mathscr{K} = \{w(x,t) \in V(Q_T) \cap L^\infty(0,T;L^2(\Omega)) : w(x,0) = 0, |\nabla w(x,t)| \le 1 \text{ a.e. } (x,t) \in Q_T\}$ satisfying the follow inequality

$$\begin{aligned}&\int_{Q_T} \frac{\partial v}{\partial t}(v-u)\,dx\,dt + \int_0^T a\Big(t, \int_\Omega |\nabla u|^{p(x)} dx\Big) \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(v-u)\,dx\,dt \\ &\ge \int_{Q_T} f(v-u)\,dx\,dt,\end{aligned} \tag{1.1}$$

for all $v \in V(Q_T)$ with $\frac{\partial v}{\partial t} \in V'(Q_T)$, $v(x,0) = 0$, $|\nabla v(x,t)| \le 1$ a.e. $(x,t) \in Q_T$, where $V'(Q_T)$ is the dual space of variable exponent Sobolev space $V(Q_T)$ (see Definition 2.3 below).

In recent years, the research of nonlinear problems with variable exponent growth conditions has been an interesting topic. $p(\cdot)$-growth problems can be regarded as a kind of nonstandard growth problems and these problems possess very complicated nonlinearities, for instance, the $p(x)$-Laplacian operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is inhomogeneous. And these problems have many important applications in nonlinear elastic, electrorheological fluids and image restoration (see [9, 27, 30, 31, 32]). Many results have been obtained on this kind of problems, see [1, 2, 5, 6, 11, 12, 14, 15, 16, 25]. Especially, in [6, 25], the authors studied the existence and uniqueness

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of weak solutions for anisotropy parabolic variation inequalities in the framework of variable exponent Sobolev spaces. Motivating by their works, we study a class variational inequalities with gradient constrain and variable exponent. To the best of our knowledge, there are no papers dealing with parabolic equalities involving variable growth and gradient constraints. For the fundamental theory about variable exponent Lebesgue and Sobolev spaces, we refer to [13, 21]. The basic theory about Variational inequalities, we refer the reader to [7, 26] for the details.

The study of Kirchhoff-type problems has received considerable attention in recent years, see [3, 4, 10, 19, 18, 20, 28, 29]. This interest arises from their contributions to the modeling of many physical and biological phenomena. We refer the reader to [17, 24] for some interesting results and further references. In [3, 4], the authors discussed the asymptotic stability for Kirchhoff systems with variable exponent growth conditions

$$\begin{gathered} u_{tt} - M(\mathscr{F}u(t))\Delta_{p(x)}u + Q(t,x,u,u_t) + f(x,u) = 0 \quad \text{in } \mathbb{R}_0^+ \times \Omega \\ u(t,x) = 0 \quad \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{gathered}$$

where $M(\tau) = a + b\tau^{\gamma-1}$, $\tau \geq 0$ with $a, b \geq 0, a+b > 0$ and $\gamma > 1$, and $\mathscr{F}u(t) = \int_\Omega \{|Du(x,t)|^{p(x)}/p(x)\}dx$, $\Delta_{p(x)} = \operatorname{div}(|Du|^{p(x)-2}Du)$.

On the one hand, our motivation for investigating (1.1) arises from reaction-diffusion equations that model population density or heat propagation (see [8]). The following equation describes the density of a population (for instance of bacteria) subject to spreading

$$\begin{gathered} u_t = a(u)\Delta u + F(u) \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{gathered}$$

The diffusion coefficient a depends on a nonlocal quantity related to the total population in the domain Ω; that is, the diffusion of individuals is guided by the global state of the population in the medium. From an experimentalist point view, it certainly makes sense to introduce nonlocal quantities, since measurements are often averages. The function F describes the reaction or growth of the population.

On the other hand, we can use problem (1.1) to describe the motion of a nonstationary fluid or gas in a nonhomogeneous and anisotropic medium and the nonlocal term a appearing in (1.1) can describe a possible change in the global state of the fluid or gas caused by its motion in the considered medium.

This article is organized as follows. In section 2, we will give some necessary definitions and properties of variable exponent Lebesgue spaces and Sobolev spaces. Moreover, we introduce the space $V(Q_T)$ and give some necessary properties, which provides a basic framework to solve our problem. In section 3, using the penalty method, we consider class of parametrized parabolic equations, and obtain weak solutions by Galerkin's approximation. In section 4, we give the proof of main theorem to this paper.

2. Preliminaries

In this section, we first recall some important properties of variable exponent Lebesgue spaces and Sobolev spaces, see [12, 13, 21] for details.

2.1. **Variable exponent Lebesgue space and Sobolev space.** Let $\Omega \subset \mathbb{R}^N$ be a domain. A measurable function $p : \Omega \to [1, \infty)$ is called a variable exponent and we define $p^- = \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$. If p^+ is finite, then the exponent p is said to be bounded. The variable exponent Lebesgue space is

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is a measurable function}; \rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx < \infty\}$$

with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0 : \rho_{p(x)}(\lambda^{-1}u) \le 1\},$$

then $L^{p(x)}(\Omega)$ is a Banach space, and when p is bounded, we have the following relations

$$\min\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\} \le \rho_{p(x)}(u) \le \max\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\}.$$

That is, if p is bounded, then norm convergence is equivalent to convergence with respect to the modular $\rho_{p(x)}$. For bounded exponent the dual space $(L^{p(x)}(\Omega))'$ can be identified with $L^{p'(x)}(\Omega)$, where the conjugate exponent p' is defined by $p' = \frac{p}{p-1}$. If $1 < p^- \le p^+ < \infty$, then the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is separable and reflexive.

In the variable exponent Lebesgue space, Hölder's inequality is still valid. For all $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$ with $p(x) \in (1, \infty)$ the following inequality holds

$$\int_\Omega |uv| dx \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)} \le 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

Definition 2.1 ([11, 12])**.** We say a bounded exponent $p : \Omega \to \mathbb{R}$ is globally log-Hölder continuous if p satisfies the following two conditions:

(1) there is a constant $c_1 > 0$ such that
$$|p(y) - p(z)| \le \frac{c_1}{\log(e + |y - z|^{-1})}$$
for all points $y, z \in \Omega$;

(2) there exist constants $c_2 > 0$ and $p_\infty \in \mathbb{R}$ such that
$$|p(y) - p_\infty| \le \frac{c_2}{\log(e + |y|^{-1})}$$
for all $y \in \Omega$.

The Variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

and equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)},$$

then the space $W^{1,p(x)}(\Omega)$ is a Banach space. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with the norm of $\|\cdot\|_{W^{1,p(x)}(\Omega)}$. If $1 < p^- \le p^+ < \infty$, then the space $W^{1,p(x)}(\Omega)$ is separable and reflexive.

Theorem 2.2 ([11, 12])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and assume that $p : \mathbb{R}^N \to (1, \infty)$ is a bounded globally log-Hölder continuous exponent such that $p^- > 1$, then for every $u \in W_0^{1,p(x)}(\Omega)$ we have*

$$\|u\|_{L^{p(x)}(\Omega)} \le c \operatorname{diam}(\Omega)\|\nabla u\|_{L^{p(x)}(\Omega)},$$

where the constant c only depends on the dimension N and the log-Hölder constant of p.

2.2. **Variable exponent Sobolev space $V(Q_T)$.**

Definition 2.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Denote $Q_T = \Omega \times (0,T), 0 < T < \infty$. Suppose that $p(x)$ is a bounded globally log-Hölder continuous function on $\overline{\Omega}$ with $p^- > 1$, we set

$$V(Q_T) = \{u \in L^2(Q_T) : |\nabla u| \in L^{p(x)}(Q_T), u(\cdot,t) \in W_0^{1,p(x)}(\Omega) \quad \text{a.e. } t \in (0,T)\},$$

with the norm

$$\|u\| = \|u\|_{L^2(Q_T)} + \|\nabla u\|_{L^{p(x)}(Q_T)}.$$

Remark 2.4. Following the standard proof for Sobolev spaces, we can prove that $V(Q_T)$ is a Banach space, and it's easy to check that $V(Q_T)$ can be continuously embedded into the space $L^r(0,T;W_0^{1,p^-}(\Omega) \cap L^2(\Omega))$, where $r = \min\{p^-, 2\}$. It is worth to mention the paper [6] where the space $V(Q_T)$ is defined in a similar way.

By the same method in [11], we have the following theorem.

Theorem 2.5 ([11])**.** *The space $C_0^\infty(Q_T)$ is dense in $V(Q_T)$.*

Since $C_0^\infty(Q_T) \subset C^\infty(0,T;C_0^\infty(\Omega))$, we have the following result.

Lemma 2.6. *The space $C^\infty(0,T;C_0^\infty(\Omega))$ is dense in $V(Q_T)$.*

Let $V'(Q_T)$ denote the dual space of $V(Q_T)$.

Theorem 2.7 ([6, 11])**.** *A function $g \in V'(Q_T)$ if and only if there exist $\bar{g} \in L^2(Q_T)$ and $\bar{G} \in (L^{p'(x)}(Q_T))^N$ such that*

$$\int_{Q_T} g\varphi \, dx \, dt = \int_{Q_T} \bar{g}\varphi \, dx \, dt + \int_{Q_T} \bar{G}\nabla\varphi \, dx \, dt. \tag{2.1}$$

Remark 2.8. It follows from the proof of Theorem 2.7 that $V(Q_T)$ is reflexive and

$$V'(Q_T) \hookrightarrow L^{s'}(0,T;W^{-1,(p^+)'}(\Omega) + L^2(\Omega)), \quad \text{where } s = \max\{p^+, 2\}.$$

Similar to that in [11], we give the following definition.

Definition 2.9. We define the space $W(Q_T) = \{u \in V(Q_T) : \frac{\partial u}{\partial t} \in V'(Q_T)\}$ with the norm

$$\|u\|_{W(Q_T)} = \|u\|_{V(Q_T)} + \|\frac{\partial u}{\partial t}\|_{V'(Q_T)},$$

where $\frac{\partial u}{\partial t}$ is the weak derivative of u with respect to time variable t defined by

$$\int_{Q_T} \frac{\partial u}{\partial t}\varphi \, dx \, dt = -\int_{Q_T} u\frac{\partial \varphi}{\partial t} \, dx \, dt, \quad \text{for all } \varphi \in C_0^\infty(Q_T).$$

Lemma 2.10 ([11])**.** *The space $W(Q_T)$ is a Banach space.*

By the method in [11], we have the following result.

Theorem 2.11. *The space $C^\infty(0,T;C_0^\infty(\Omega))$ is dense in $W(Q_T)$.*

The following theorem can be proved similarly to that in [11], thus we omit its proof.

Theorem 2.12 ([1, 11]). *$W(Q_T)$ can be embedded continuously in $C(0,T;L^2(\Omega))$. Furthermore, for all $u,v \in W(Q_T)$ and $s,t \in [0,T]$ the following rule for integration by parts is valid*
$$\int_s^t \int_\Omega \frac{\partial u}{\partial t} v \,dx\,d\tau = \int_\Omega u(x,t)v(x,t)dx - \int_\Omega u(x,s)v(x,s)dx - \int_s^t \int_\Omega u\frac{\partial v}{\partial t}\,dx\,d\tau.$$

The following theorem gives a relation between almost everywhere convergence and weak convergence.

Theorem 2.13 ([9]). *Let $p(x) : Q_T \to \mathbb{R}$ be a bounded globally log-Hölder continuous function, with $p^- > 1$. If $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^{p(x)}(Q_T)$ and $u_n \to u$ a.e. $(x,t) \in Q_T$ as $n \to \infty$, then there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \to u$ weakly in $L^{p(x)}(Q_T)$ as $n \to \infty$.*

We will give a compact embedding for $V(Q_T)$ in the following.

Theorem 2.14 ([23]). *Let $B_0 \subset B \subset B_1$ be three Banach spaces, where B_0, B_1 are reflexive, and the embedding $B_0 \subset B$ is compact. Denote $W = \{v : v \in L^{p_0}(0,T;B_0), \frac{\partial v}{\partial t} \in L^{p_1}(0,T;B_1)\}$, where T is a fixed positive number, $1 < p_i < \infty$, $i = 0,1$, then W can be compactly embedded into $L^{p_0}(0,T;B)$.*

Theorem 2.15. *Let F be a bound subset in $V(Q_T)$ and $\{\frac{\partial u}{\partial t} : u \in F\}$ be bounded in $V'(Q_T)$, then F is relatively compact in $L^r(0,T;L^2(\Omega))$.*

Proof. Since $p^- > \frac{2N}{N+2}$ $(N \geq 2)$, the embedding $W_0^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. By Remarks 2.4 and 2.8, the embeddings $V(Q_T) \hookrightarrow L^r(0,T;W_0^{1,p^-}(\Omega) \cap L^2(\Omega))$ and
$$V'(Q_T) \hookrightarrow L^{s'}(0,T;W^{-1,(p^+)'}(\Omega) + L^2(\Omega)) \hookrightarrow L^{s'}(0,T;W^{-1,\lambda}(\Omega))$$
are continuous, where $\lambda = \min\{2,(p^+)'\}$. As the embedding $L^2(\Omega) \hookrightarrow W^{-1,\lambda}(\Omega)$ is continuous, by Theorem 2.14, F is relatively compact in $L^r(0,T;L^2(\Omega))$. $\square$

3. Existence of solutions for parabolic equations

In this section, for $\varepsilon \in (0,1)$, we consider the following nonlocal parabolic equation with Diriclet boundary-value conditions:
$$\begin{gathered}
\frac{\partial u}{\partial t} - a\Big(t, \int_\Omega |\nabla u|^{p(x)}dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \\
- \frac{1}{\varepsilon}\operatorname{div}\big((|\nabla u|^{p(x)-2} - 1)^+\nabla u\big) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\
u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\
u(x,0) = 0, \quad x \in \Omega,
\end{gathered} \tag{3.1}$$
where $(|\nabla u|^{p(x)-2} - 1)^+ = \max\{|\nabla u|^{p(x)-2} - 1, 0\}$. We assume that

- (H1) $a(t,s) : [0,\infty) \times [0,\infty) \to (0,\infty)$ is a continuous function and there exists two positive constants a_0 and a_1 such that
$$a_0 \leq a(t,s) \leq a_1 \quad \text{for each } (t,s) \in [0,\infty) \times [0,\infty).$$
- (H2) $p(x) : \Omega \to (1,\infty)$ is a global log-Hölder continuous function. Denote $p^- = \inf_{x\in\overline{\Omega}} p(x)$, $p^+ = \sup_{x\in\overline{\Omega}} p(x)$. And there holds
$$2 < p^- \leq p(x) \leq p^+ < \infty \quad \text{for each } x \in \Omega.$$

(H3) $f \in V'(Q_T)$.

Definition 3.1. A function $u_\varepsilon \in V(Q_T) \cap C(0,T;L^2(\Omega))$ with $\frac{\partial u_\varepsilon}{\partial t} \in V'(Q_T)$ is called a weak solution of (3.1), if

$$\begin{aligned}&\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt + \int_0^T a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)}dx\Big)\int_\Omega |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi\,dx\,dt\\ &+ \int_{Q_T} \frac{1}{\varepsilon}(|\nabla u_\varepsilon|^{p(x)-2}-1)^+\nabla u_\varepsilon\nabla\varphi\,dx\,dt = \int_{Q_T} f\varphi\,dx\,dt,\end{aligned}$$

for all $\varphi \in V(Q_T)$.

Since $f \in V'(Q_T)$, there exists a sequence $f_n \in C_0^\infty(Q_T)$ such that $\lim_{n\to\infty} f_n = f$ in $V'(Q_T)$. Similar to that in [14, 15], we choose a sequence $\{w_j\}_{j=1}^\infty \subset C_0^\infty(\Omega)$ such that $C_0^\infty(\Omega) \subset \overline{\cup_{n=1}^\infty V_n}^{C^1(\bar\Omega)}$ and $\{w_j\}_{j=1}^\infty$ is a standard orthogonal basis in $L^2(\Omega)$, where $V_n = \operatorname{span}\{w_1, w_2, \dots, w_n\}$.

Theorem 3.2. *Let assumptions* (H1)–(H3) *hold and let* $\varepsilon \in (0,1)$ *be fixed. Then there exists a weak solution for equation* (3.1).

Proof. (i) **Galerkin approximation.** For each $n \in \mathbb{N}$, we want to find the approximate solutions to problem (3.1) in the form

$$u_n(x,t) = \sum_{j=1}^n (\eta_n(t))_j w_j(x).$$

First we define a vector-valued function $P_n(t,\upsilon) : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ as

$$\begin{aligned}(P_n(t,\nu))_i &= a\Big(t, \int_\Omega |\sum_{j=1}^n \nu_j\nabla\omega_j|^{p(x)}dx\Big)\int_\Omega |\sum_{j=1}^n \nu_j\nabla w_j|^{p(x)-2}\big(\sum_{j=1}^n \nu_j\nabla w_j\big)\nabla w_i dx\\ &\quad+ \int_\Omega \frac{1}{\varepsilon}\big(|\sum_{j=1}^n \nu_j\nabla w_j\big|^{p(x)-2}-1\big)^+\big(\sum_{j=1}^n \nu_j\nabla w_j\big)\nabla w_i dx,\end{aligned}$$

where $\nu = (\nu_1, \cdots, \nu_n)$. Since a and p are continuous, from the definition of $P_n(t,\nu)$, $P_n(t,\nu)$ is continuous with respect to t and ν.

We consider the following ordinary differential systems

$$\begin{gathered}\eta'(t) + P_n(t,\eta(t)) = F_n,\\ \eta(0) = 0,\end{gathered} \tag{3.2}$$

where $(F_n)_i = \int_\Omega f_n w_i dx$.

Multiplying (3.2) by $\eta(t)$, we arrive at the equality

$$\eta'(t)\eta(t) + P_n\left(t,\eta(t)\right)\eta(t) = F_n\eta(t).$$

Since

$$\begin{aligned}P_n(t,\eta)\eta &= a\Big(t, \int_\Omega |\sum_{j=1}^n \eta_j\nabla\omega_j|^{p(x)}dx\Big)\int_\Omega |\sum_{j=1}^n \eta_j\nabla w_j|^{p(x)}dx\\ &\quad+ \int_\Omega \frac{1}{\varepsilon}(|\sum_{j=1}^n \nu_j\nabla w_j|^{p(x)-2}-1)^+|\sum_{j=1}^n \nu_j\nabla w_j|^2dx \geq 0,\end{aligned}$$

by Young's inequality, there holds
$$\frac{1}{2}\frac{\partial|\eta(t)|^2}{\partial t} \le |F_n||\eta(t)| \le \frac{1}{2}|F_n|^2 + \frac{1}{2}|\eta(t)|^2.$$
Then integrating with respect to t from 0 to t, we obtain
$$|\eta(t)|^2 \le C_n + \int_0^t |\eta(s)|^2 ds.$$
By Gronwall's inequality, we obtain that $|\eta(t)| \le C_n(T)$. We denote
$$L_n = \max_{(t,\eta)\in[0,T]\times B(\eta(0),C_n(T))} |F_n - P_n(t,\eta)|, \quad \tau_n = \min\{T, \frac{C_n(T)}{L_n}\},$$
where $B(\eta(0), C_n(T))$ is the ball of radius $C_n(T)$ with the center at the point $\eta(0)$ in $\mathbb{R}^n$. By Peano's Theorem we know that (3.2) admits a C^1 solution in $[0,\tau_n]$. Let $\eta(\tau_n)$ be a initial value, then we can repeat the above process and get a C^1 solution in $[t_n, 2\tau_n]$. Without lost of generality, we assume that $T = [\frac{T}{\tau_n}]\tau_n + (\frac{T}{\tau_n})\tau_n, 0 < (\frac{T}{\tau_n}) < 1$, where $[\frac{T}{\tau_n}]$ is the integer part of $\frac{T}{\tau_n}$, $(\frac{T}{\tau_n})$ is the decimal part of $\frac{T}{\tau_n}$. We can divide $[0,T]$ into $[(i-1)\tau_n, i\tau_n]$, $i = 1,\dots,L$ and $[L\tau_n, T]$ where $L = [\frac{T}{\tau_n}]$, then there exist C^1 solution $\eta_n^i(t)$ in $[(i-1)\tau_n, i\tau_n]$, $i = 1,\dots,L$ and $\eta_n^{L+1}(t)$ in $[L\tau_n, T]$. Therefore, we obtain a solution $\eta_n(t) \in C^1[0,T]$ defined by
$$\eta_n(t) = \begin{cases} \eta_n^1(t), & \text{if } t \in [0,\tau_n], \\ \eta_n^2(t), & \text{if } t \in (\tau_n, 2\tau_n], \\ \dots \\ \eta_n^L(t), & \text{if } t \in ((L-1)\tau_n, L\tau_n], \\ \eta_n^{L+1}(t), & \text{if } t \in (L\tau_n, T]. \end{cases}$$
Thus, we obtain the approximate solutions sequence $u_n = \sum_{j=1}^n (\eta_n(t))_j w_j(x)$. From (3.2), for $1 \le i \le n$, we have
$$\begin{aligned} &\int_\Omega \frac{\partial u_n}{\partial t} w_i dx + a\Big(t, \int_\Omega |\nabla u_n|^{p(x)} dx\Big) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla w_i dx \\ &+ \int_\Omega \frac{1}{\varepsilon}(|\nabla u_n|^{p(x)-2} - 1)^+ \nabla u_n \nabla w_i dx \\ &= \int_\Omega f_n w_i dx. \end{aligned} \tag{3.3}$$
Multiplying by $(\eta_n(t))_i$, summing up i from 1 to n, and integrating with respect to t from 0 to τ, where $\tau \in (0,T]$, we obtain
$$\begin{aligned} &\int_0^\tau \int_\Omega \frac{\partial u_n}{\partial t} u_n \, dx\, dt + \int_0^\tau a\Big(t, \int_\Omega |\nabla u_n|^{p(x)} dx\Big) \int_\Omega |\nabla u_n|^{p(x)} \, dx\, dt \\ &+ \int_0^\tau \int_\Omega \frac{1}{\varepsilon}(|\nabla u_n|^{p(x)-2} - 1)^+ |\nabla u_n|^2 \, dx\, dt \\ &= \int_0^\tau \int_\Omega f_n u_n \, dx\, dt. \end{aligned} \tag{3.4}$$

Remark 3.3. The approximate solutions u_n depends on ε; For convenience, we omit the ε. For all $\varphi \in C^1(0,T;V_k)$, $k \le n$, there holds
$$\int_0^\tau \int_\Omega \frac{\partial u_n}{\partial t} \varphi \, dx\, dt + \int_0^\tau a\Big(t, \int_\Omega |\nabla u_n|^{p(x)} \nabla u_n dx\Big) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi \, dx\, dt$$

$$
\begin{aligned}
&+\int_0^\tau\int_\Omega \frac{1}{\varepsilon}(|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n\nabla\varphi\,dx\,dt\\
&=\int_0^\tau\int_\Omega f_n\varphi\,dx\,dt.
\end{aligned}
$$

(ii) **A priori estimates.** By (3.4), assumption (H1) and integration by parts, we arrive at the inequality

$$
\frac{1}{2}\int_\Omega |u_n(x,\tau)|^2-|u_n(x,0)|^2dx+a_0\int_0^\tau\int_\Omega|\nabla u_n|^{p(x)}\,dx\,dt\le\|f_n\|_{V'(Q_\tau)}\|u_n\|_{V(Q_\tau)},
$$

where $Q_\tau=\Omega\times(0,\tau)$, $\tau\in(0,T]$. Since $u_n(x,0)=0$ and $f_n\to f$ in $V'(Q_T)$, $\|f_n\|_{V'(Q_T)}\le C$, where C independent of τ and n. Thus, we obtain

$$
\int_\Omega u_n^2(x,\tau)dx+a_0\int_0^\tau\int_\Omega|\nabla u_n|^{p(x)}\,dx\,dt\le C(\|u_n\|_{L^2(Q_\tau)}+\|\nabla u_n\|_{L^{p(x)}(Q_\tau)}). \tag{3.5}
$$

Without lost generality, we assume that $\|\nabla u_n\|_{L^{p(x)}(Q_\tau)}\ge 1$. Then

$$
\|\nabla u_n\|^{p^-}_{L^{p(x)}(Q_\tau)}\le\int_{Q_\tau}|\nabla u_n|^{p(x)}\,dx\,dt.
$$

By (3.5) and Young's inequality, there holds

$$
\int_\Omega u_n^2(x,\tau)dx+\frac{a_0}{2}\int_0^\tau\int_\Omega|\nabla u_n|^{p(x)}\,dx\,dt\le C(\|u_n\|_{L^2(Q_\tau)}+1).
$$

By Gronwall's inequality, we obtain $\|u_n\|_{L^\infty(0,T;L^2(\Omega))}\le C$. Therefore,

$$
\|u_n\|_{L^\infty(0,T;L^2(\Omega))}+\|u_n\|_{V(Q_T)}\le C. \tag{3.6}
$$

Combining assumption (H1), with (3.6), we have

$$
\begin{gathered}
\int_{Q_T}\Big|a\Big(t,\int_\Omega|\nabla u_n|^{p(x)}dx\Big)|\nabla u_n|^{p(x)-2}\nabla u_n\Big|^{p'(x)}\,dx\,dt\le C.\\
\int_{Q_T}|(|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n|^{p'(x)}\,dx\,dt\le C(\varepsilon),
\end{gathered}
$$

where $C(\varepsilon)$ is a constant independent of n on ε and $C(\varepsilon)\to\infty$ as $\varepsilon\to\infty$. Thus we obtain

$$
\begin{gathered}
\big\|a(t,\int_\Omega|\nabla u_n|^{p(x)}dx)|\nabla u_n|^{p(x)-2}\nabla u_n\big\|_{L^{p'(x)}(Q_T)}\le C,\\
\big\|(|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n\big\|_{L^{p'(x)}(Q_T)}\le C(\varepsilon).
\end{gathered} \tag{3.7}
$$

By Lemma 2.6, for all $\varphi\in V(Q_T)$, there exists a sequence $\varphi_n\in C^1(0,T;V_n)$ such that $\varphi_n\to\varphi$ strongly in $V(Q_T)$. By Remark 3.3, we have

$$
\begin{aligned}
&\Big|\int_{Q_T}\frac{\partial u_n}{\partial t}\varphi_n\,dx\,dt\Big|\\
&=\Big|-\int_{Q_T}a\Big(t,\int_\Omega|\nabla u_n|^{p(x)}dx\Big)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\varphi_n\,dx\,dt\\
&\quad-\int_{Q_T}\frac{1}{\varepsilon}(|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n\varphi_n\,dx\,dt+\int_{Q_T}f_n\varphi_n\,dx\,dt\Big|\\
&\le C\Big(\Big\|a\Big(t,\int_\Omega|\nabla u_n|^{p(x)}dx\Big)|\nabla u_n|^{p(x)-2}\nabla u_n\Big\|_{L^{p'(x,t)}(Q_T)}\|\nabla\varphi_n\|_{L^{p(x)}(Q_T)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon}\|(|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n\|_{L^{p'(x,t)}Q_T}\|\nabla\varphi_n\|+\|f_n\|_{V'(Q_T)}\|\varphi_n\|_{V(Q_T)}\Big)\\
&\le C(\varepsilon)\|\varphi_n\|_{V(Q_T)},
\end{aligned}
$$

where $C(\varepsilon)$ is a constant independent of n on ε. We immediately get that

$$
\|\frac{\partial u_n}{\partial t}\|_{V'(Q_T)}\le C(\varepsilon). \tag{3.8}
$$

(iii) **Passage to the limit.** From (3.6)-(3.8), we obtain a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$
\begin{gathered}
u_n \rightharpoonup u_\varepsilon \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)),\\
u_n \rightharpoonup u_\varepsilon \quad \text{weakly in } V(Q_T),\\
a\big(t,\int_\Omega |\nabla u_n|^{p(x)}dx\big)|\nabla u_n|^{p(x)-2}\nabla u_n \rightharpoonup \xi \quad \text{weakly in } \big(L^{p'(x)}(Q_T)\big)^N,\\
(|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n \rightharpoonup \eta \quad \text{weakly in } (L^{p'(x)}(Q_T))^N\\
\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u_\varepsilon}{\partial t} \quad \text{weakly in } V'(Q_T).
\end{gathered}
$$

Since $\int_\Omega u_n^2(x,T)dx\le C$, there exist a subsequence of $\{u_n(x,T)\}$ (still denoted by $\{u_n(x,T)\}$) and a function $\tilde u$ in $L^2(\Omega)$ such that $u_n(x,T)\to\tilde u$ weakly in $L^2(\Omega)$. Then for any $\varphi(x)\in C_0^\infty(\Omega)$ and $\eta(t)\in C^1[0,T]$, there holds

$$
\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u_n}{\partial t}\varphi\eta\,dx\,dt\\
&=\int_\omega u_n(x,T)\varphi\eta(T)dx-\int_\Omega u_n(x,0)\varphi\eta(0)dx-\int_0^T\int_\Omega u_n\varphi\eta'(t)\,dx\,dt.
\end{aligned}
$$

Letting $n\to\infty$, by integration by parts, we obtain

$$
\int_\Omega(\tilde u-u_\varepsilon(x,T))\eta(T)\varphi dx+\int_\Omega u_\varepsilon(x,0)\eta(0)\varphi dx=0.
$$

Choosing $\eta(T)=1,\eta(0)=0$ or $\eta(T)=0,\eta(0)=1$, by the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, we have $\tilde u=u_\varepsilon(x,T)$ and $u_\varepsilon(x,0)=0$ for almost every $x\in\Omega$. That is $u_n(x,T)\to u_\varepsilon(x,T)$ weakly in $L^2(\Omega)$, as $n\to\infty$, thus

$$
\int_\Omega u_\varepsilon^2(x,T)dx\le\liminf_{n\to\infty}\int_\Omega u_n^2(x,T)dx. \tag{3.9}
$$

In view of Remark 3.3, for all $\varphi\in C^1(0,T;V_k)$ where $k\le n$, letting $n\to\infty$ there holds

$$
\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi+\xi\nabla\varphi+\frac{1}{\varepsilon}\eta\nabla\varphi\,dx\,dt=\int_{Q_T}f\varphi\,dx\,dt, \tag{3.10}
$$

since $C^1\big(0,T;\cup_{k=1}^\infty V_k\big)$ is dense in $C^1(0,T;C^1(\overline{\Omega}))$, the above equality is valid for all $\varphi\in C^1(0,T;C_0^\infty(\Omega))$. Moreover, for all $\varphi\in V(Q_T)$, the above equality is valid. Thus, we can take $\varphi=u_\varepsilon$. By integration by parts, we have

$$
\frac12\int_\Omega|u_\varepsilon(x,T)|^2dx+\int_{Q_T}\xi\nabla u_\varepsilon+\frac{1}{\varepsilon}\eta\nabla u_\varepsilon\,dx\,dt=\int_{Q_T}fu_\varepsilon\,dx\,dt. \tag{3.11}
$$

We denote

$$
Y_n=\int_{Q_T}a\Big(t,\int_\Omega|\nabla u_n|^{p(x)}dx\Big)\big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\big)
$$

$$\times \big(\nabla u_n - \nabla u_\varepsilon\big)\,dx\,dt + \frac{1}{\varepsilon}\int_{Q_T}\big((|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n - (|\nabla u_\varepsilon|^{p(x)-2}-)^+\nabla u_\varepsilon\big)\big(\nabla u_n - \nabla u_\varepsilon\big)\,dx\,dt.$$

By (3.4), we obtain

$$\begin{aligned}0 \le Y_n = &\int_{Q_T} f_n u_n - \frac{1}{2}\int_\Omega |u_n(x,T)|^2 - |u_n(x,0)|^2 dx\\ &- \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_n|^{p(x)}dx\Big)|\nabla u_n|^{p(x)-2}\nabla u_n \nabla u_\varepsilon\,dx\,dt\\ &- \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_n|^{p(x)}dx\Big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt\\ &- \frac{1}{\varepsilon}\int_{Q_T}(|\nabla u_n|^{p(x)-2}-1)^+\nabla u_n\nabla u_\varepsilon\,dx\,dt\\ &- \frac{1}{\varepsilon}\int_{Q_T}(|\nabla u_\varepsilon|^{p(x)-2}-1)^+\nabla u_\varepsilon(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt\end{aligned} \tag{3.12}$$

By assumption (H1), the sequence $\{a\big(t, \int_\Omega |\nabla u_n|^{p(x)}dx\big)\}_{n=1}^\infty$ is equi-integrable and uniformly bounded in $L^1(0,T)$. Therefore, there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $\bar a(t)$ such that

$$a\Big(t, \int_\Omega |\nabla u_n|^{p(x)}dx\Big) \to \bar a(t) \quad \text{a.e. } t \in [0,T].$$

As

$$\Big|a\Big(t, \int_\Omega |\nabla u_n|^{p(x)}dx\Big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\Big|^{p'(x)} \le C|\nabla u_\varepsilon|^{p(x)} \in L^1(Q_T),$$

by the Lebesgue dominated convergence theorem, we obtain

$$\int_{Q_T}\Big|\Big[a\Big(t, \int_\Omega |\nabla u_n|^{p(x)}dx\Big) - \bar a(t)\Big]|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\Big|^{p'(x)}\,dx\,dt \to 0.$$

That is,

$$a\Big(t, \int_\Omega |\nabla u_n|^{p(x)}dx\Big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \to \bar a(t)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \quad \text{in } \big(L^{p'(x)}(Q_T)\big)^N. \tag{3.13}$$

Thus, from (3.9), (3.11)-(3.13), we obtain

$$\begin{aligned}0 \le \limsup_{n\to\infty} Y_n \le &\int_{Q_T} fu\,dx\,dt - \frac{1}{2}\int_\Omega |u(x,T)|^2 dx\\ &- \int_{Q_T}\xi\nabla u_\varepsilon\,dx\,dt - \frac{1}{\varepsilon}\int_{Q_T}\eta\nabla u_\varepsilon\,dx\,dt = 0;\end{aligned}$$

therefore $\lim_{n\to\infty} Y_n = 0$. Furthermore, by assumption (H1), there holds

$$\int_{Q_T}\big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\big)\big(\nabla u_n - \nabla u_\varepsilon\big)\,dx\,dt \to 0.$$

Since $p(x,t) \ge p^- > 2$, as $n \to \infty$, there holds

$$\begin{aligned}&\int_{Q_T}|\nabla u_n - \nabla u_\varepsilon|^{p(x)}\,dx\,dt\\ &\le C\int_{Q_T}(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon)(\nabla u_n - \nabla u_\varepsilon)\,dx\,dt \to 0.\end{aligned} \tag{3.14}$$

Therefore, from (3.14), we obtain $\nabla u_n \to \nabla u_\varepsilon$ in $(L^{p(x)}(Q_T))^N$. Thus, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that

$$\int_\Omega |\nabla u_n - \nabla u_\varepsilon|^{p(x)} dx \to 0 \quad \text{a.e. } t \in [0,T]. \tag{3.15}$$

Since

$$\begin{aligned}
&\Big|\int_\Omega |\nabla u_n|^{p(x)} - |\nabla u_\varepsilon|^{p(x)} dx\Big| \\
&\le \int_\Omega p(x)\big||\nabla u_n| + \theta(|\nabla u_n| - |\nabla u_\varepsilon|)\big|^{p(x)-1}\big||\nabla u_n| - |\nabla u_\varepsilon|\big| dx \\
&\le C\big\||\nabla u_n|^{p(x)-1} + |\nabla u_\varepsilon|^{p(x)-1}\big\|_{L^{p'(x)}(\Omega)}\|\nabla u_n - \nabla u_\varepsilon\|_{L^{p(x)}(\Omega)},
\end{aligned}$$

where $0 \le \theta \le 1$, by (3.15), we have

$$\int_\Omega |\nabla u_n|^{p(x)} \to \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx \quad \text{a.e. } t \in [0,T].$$

Thus, by the continuity of a, we obtain that

$$\bar a(t) = a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) \quad \text{a.e. } t \in [0,T]$$

Since $\nabla u_n \to \nabla u_\varepsilon$ in $(L^{p(x)}(Q_T))^N$, there exists a subsequence of $\{u_n\}$ (still labeled by $\{u_n\}$) such that $\nabla u_n \to \nabla u_\varepsilon$ for a.e. $(x,t) \in Q_T$, then

$$\begin{aligned}
&a\Big(t, \int_\Omega |\nabla u_n|^{p(x)} dx\Big)|\nabla u_n|^{p(x,t-2)}\nabla u_n \\
&\to a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \quad \text{a.e. } (x,t) \in Q_T.
\end{aligned}$$

By Theorem 2.13, we obtain $\xi = a\big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon$. Similarly, $\eta = (|\nabla u_\varepsilon|^{p(x,t)-2} - 1)^+\nabla u_\varepsilon$.

It follows from (3.10) that

$$\begin{aligned}
&\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi\,dx\,dt + \int_0^T a\Big(t, \int_\Omega |\nabla u_n|^{p(x)} dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u_\varepsilon \nabla\varphi \\
&+ \frac{1}{\varepsilon}(|\nabla u_\varepsilon|^{p(x,t)-2} - 1)^+\nabla u_\varepsilon\nabla\varphi\,dx\,dt = \int_{Q_T} f\varphi\,dx\,dt,
\end{aligned}$$

for all $\varphi \in V(Q_T)$. Since $u \in V(Q_T)$ and $\frac{\partial u}{\partial t} \in V'(Q_T)$, by Theorem 2.12, up to a set of measure zero, we have $u \in C(0,T;L^2(\Omega))$. □

4. Existence of solutions for the variational inequality

In this section, we prove our main theorem.

Theorem 4.1. *Under assumptions* (H1)–(H3) *there exists a function* $u(x,t) \in \mathscr{K}$ *such that*

$$\begin{aligned}
&\int_{Q_T} \frac{\partial v}{\partial t}(v-u)\,dx\,dt + \int_0^T a\Big(t, \int_\Omega |\nabla u|^{p(x)} dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(v-u)\,dx\,dt \\
&\ge \int_{Q_T} f(v-u)\,dx\,dt
\end{aligned}$$

for all $v \in V(Q_T)$ *with* $\frac{\partial v}{\partial t} \in V'(Q_T)$, $v(x,0) = 0$, $|\nabla v| \le 1$ *a.e.* $(x,t) \in Q_T$.

Proof. We will prove this theorem in three steps.
(Step 1) A priori estimates. In Definition 3.1, we take $\varphi = u_\varepsilon \chi_{(0,\tau)}$ as a test function, where $\chi_{(0,\tau)}$ is defined as the characteristic function of $(0,\tau)$, $\tau \in (0,T]$, then

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon \,dx\,dt + \int_{Q_\tau} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big)|\nabla u_\varepsilon|^{p(x)} \\
&+ \frac{1}{\varepsilon}(|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ |\nabla u_\varepsilon|^2 \,dx\,dt \\
&= \int_{Q_\tau} f(x,t) u_\varepsilon \,dx\,dt,
\end{aligned}$$

where $Q_\tau = \Omega \times (0,\tau)$. Similar to Section 3, we have

$$\int_\Omega |u_\varepsilon(x,\tau)|^2 dx + \int_{Q_\tau} |\nabla u_\varepsilon|^{p(x)} \,dx\,dt \le C, \quad \text{for all } \tau \in [0,T].$$

Therefore,

$$\frac{1}{\varepsilon}\int_{Q_T} (|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ |\nabla u_\varepsilon|^2 \,dx\,dt + \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|_{V(Q_T)} \le C. \tag{4.1}$$

Since

$$\begin{aligned}
&\int_{Q_T} \Big|a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x,t)} dx\Big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\Big|^{p'(x)} \,dx\,dt \\
&\le C \int_{Q_T} |\nabla u_\varepsilon|^{p(x)} \,dx\,dt \\
&\le C \max\{\|\nabla u_\varepsilon\|^{p-}_{L^{p(x)}(Q_T)}, \|\nabla u_\varepsilon\|^{p+}_{L^{p(x)}(Q_T)}\} \le C,
\end{aligned}$$

there holds

$$\Big\|\Big|a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\Big|\Big\|_{L^{p'(x)}(Q_T)} \le C.$$

(Step 2) Passage to the limit. By (4.1)-(**??**), there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$, still denoted by $\{u_\varepsilon\}_{\varepsilon>0}$, such that

$$\begin{gathered}
u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\
u_\varepsilon \rightharpoonup u \quad \text{weakly in } V(Q_T), \\
a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big)|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \rightharpoonup A \quad \text{weakly in } (L^{p'(x)}(Q_T))^N.
\end{gathered} \tag{4.2}$$

For all $\varphi \in V(Q_T)$, there holds

$$\int_{Q_T} [(|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ \nabla u_\varepsilon - (|\nabla \varphi|^{p(x)-2} - 1)^+ \nabla\varphi](\nabla u_\varepsilon - \nabla\varphi) \,dx\,dt \ge 0.$$

Since

$$\int_{Q_T} |(|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ \nabla u_\varepsilon|^{p'(x)} \,dx\,dt \le \int_{Q_T} (|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ |\nabla u_\varepsilon|^2 \,dx\,dt,$$

by (4.1), we obtain that $\int_{Q_T} |(|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ \nabla u_\varepsilon|^{p'(x)} \,dx\,dt \to 0$ as $\varepsilon \to 0$; that is, $\|(|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ \nabla u_\varepsilon\|_{L^{p'(x)}(Q_T)} \to 0$. From $\nabla u_\varepsilon \rightharpoonup u$ weakly in $(L^{p(x)}(Q_T))^N$, we have

$$\int_{Q_T} (|\nabla\varphi|^{p(x)-2} - 1)^+ \nabla\varphi(\nabla u - \nabla\varphi) \,dx\,dt \le 0$$

We take $\varphi = u + \lambda w$, where $0 < \lambda < 1$ and $w \in V(Q_T)$, then

$$\int_{Q_T} (|\nabla(u+\lambda w)|^{p(x)-2} - 1)^+ \nabla(u+\lambda w)\nabla w \, dx\, dt \le 0.$$

Since $|(|\nabla(u+\lambda w)|^{p(x)-2} - 1)^+ \nabla(u+\lambda w)\nabla w| \le C(|\nabla u|^{p(x)} + |\nabla w|^{p(x)}) \in L^1(Q_T)$ and $(|\nabla(u+\lambda w)|^{p(x)-2} - 1)^+ \nabla(u+\lambda w)\nabla w \to (|\nabla u|^{p(x)-2} - 1)^+ \nabla u \nabla w$ as $\lambda \to 0$, by the Lebesgue Dominated Convergence Theorem and the arbitrariness of w, we obtain

$$\int_{Q_T} (|\nabla u|^{p(x)-2} - 1)^+ |\nabla u|^2 \, dx\, dt = 0$$

Thus, $|\nabla u| \le 1$ a.e. $(x,t) \in Q_T$.

Taking $\varphi = v - u_\varepsilon$ as a test function in (3.1), where $v \in V(Q_T)$, $\frac{\partial v}{\partial t} \in V'(Q_T)$, $v(x,0) = 0$ and $|\nabla v| \le 1$ a.e. $(x,t) \in Q_T$, then

$$\begin{aligned}
&\int_{Q_T} \frac{\partial v}{\partial t}(v - u_\varepsilon) + a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla (v - u_\varepsilon) \\
&- f(x,t)(v - u_\varepsilon)\, dx\, dt \\
&= \int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}(v - u_\varepsilon) + a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla (v - u_\varepsilon) \\
&\quad - f(x,t)(v - u_\varepsilon)\, dx\, dt + \int_{Q_T} \frac{\partial (v - u_\varepsilon)}{\partial t}(v - u_\varepsilon)\, dx\, dt \\
&= \frac{1}{\varepsilon} \int_{Q_T} \big((|\nabla v|^{p(x)-2} - 1)^+ \nabla v - (|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ \nabla u_\varepsilon\big)(\nabla v - \nabla u_\varepsilon)\, dx\, dt \\
&\quad + \int_{Q_T} \frac{\partial (v - u_\varepsilon)}{\partial t}(v - u_\varepsilon)\, dx\, dt \ge 0,
\end{aligned}$$

and further

$$\begin{aligned}
&\int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)}\, dx\, dt \\
&\le \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla u\, dx\, dt \\
&\quad + \int_{Q_T} \frac{\partial v}{\partial t}(v - u_\varepsilon)\, dx\, dt - \int_{Q_T} f(x,t)(v - u_\varepsilon)\, dx\, dt.
\end{aligned} \tag{4.3}$$

For $k > 0$, we denote

$$u^{(k)} = \begin{cases} k, & u < -k, \\ u, & |u| \le k, \\ k, & u > k, \end{cases}$$

and $u_\mu^{(k)}(x,t) = \mu \int_0^t e^{\mu(s-t)} u^{(k)}(x,s) ds$. It's easy to check that $\frac{\partial u_\mu^{(k)}}{\partial t} = \mu(u^{(k)} - u_\mu^{(k)})$. From that in [6], we obtain $u_\mu^{(k)} \to u^{(k)}$ strongly in $L^2(Q_T)$ and weakly in $V(Q_T)$ as $\mu \to \infty$. Denote $A_k = \{(x,t) \in Q_T : |u| \le k\}$, then $u^{(k)} = u$ in A_k and $\operatorname{sgn}(u^{(k)} - u_\mu^{(k)}) = \operatorname{sgn}(u - u_\mu^{(k)})$ in $Q_T \setminus A_k$ (because $|u_\mu^{(k)}| \le k$). Thus,

$$\int_{Q_T} \frac{\partial u_\mu^{(k)}}{\partial t}(u_\mu^{(k)} - u)\, dx\, dt$$

$$
\begin{aligned}
&= \mu \int_{Q_T} (u^{(k)} - u_\mu^{(k)})(u_\mu^{(k)} - u)\,dx\,dt \\
&= -\mu \int_{A_k} (u - u_\mu^{(k)})^2\,dx\,dt - \mu \int_{Q_T\setminus A_k} (u^{(k)} - u_\mu^{(k)})(u - u_\mu^{(k)})\,dx\,dt \le 0.
\end{aligned}
$$

By a diagonal rule, we obtain a sequence denoted by v_k such that $v_k \to u$ strongly in $L^2(Q_T)$ and weakly in $V(Q_T)$ as $k \to \infty$, and $\limsup_{k\to\infty} \int_{Q_T} \frac{\partial v_k}{\partial t}(v_k - u)\,dx\,dt \le 0$. Taking $v = v_k$ in (4.3), we obtain

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)}\,dx\,dt \\
&\le \int_{Q_T} A\nabla u\,dx\,dt + \int_{Q_T} \frac{\partial v_k}{\partial t}(v_k - u)\,dx\,dt - \int_{Q_T} f(x,t)(v_k - u)\,dx\,dt.
\end{aligned}
$$

Letting $k \to \infty$, we have

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)}\,dx\,dt \\
&\le \int_{Q_T} A\nabla u\,dx\,dt = \lim_{\varepsilon\to 0} \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla u\,dx\,dt;
\end{aligned}
$$

that is,

$$
\limsup_{\varepsilon\to 0} \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla(u_\varepsilon - u)\,dx\,dt \le 0. \tag{4.4}
$$

As the sequence $\Big\{a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big)\Big\}_\varepsilon$ is uniformly bounded and equi-integrable in $L^1(Q_T)$, there exist a subsequence of $\{u_\varepsilon\}$ (for convenience still relabeled by $\{u_\varepsilon\}$) and a^* such that $a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) \to a^*$ for almost every $t \in [0,T]$. Since

$$
\Big| \Big(a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) - a^* \Big) |\nabla u|^{p(x)-2} \nabla u \Big|^{p'(x)} \le C |\nabla u|^{p(x)} \in L^1(Q_T),
$$

by the Lebesgue dominated convergence theorem, we obtain

$$
a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u|^{p(x)-2} \nabla u \to a^* |\nabla u|^{p(x)-2} \nabla u \quad \text{strongly in } L^{p'(x)}(Q_T).
$$

Since

$$
\begin{aligned}
0 &\le \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) (|\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon - |\nabla u|^{p(x)-2} \nabla u)(\nabla u_\varepsilon - \nabla u) \\
&= \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon (\nabla u_\varepsilon - \nabla u) \\
&\quad - a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u|^{p(x)-2} \nabla u (\nabla u_\varepsilon - \nabla u)\,dx\,dt,
\end{aligned}
$$

we have

$$
\liminf_{\varepsilon\to 0} \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla(u_\varepsilon - u)\,dx\,dt \ge 0. \tag{4.5}
$$

From (4.4)–(4.5) and $\nabla u_\varepsilon \rightharpoonup \nabla u$ weakly in $(L^{p(x)}(Q_T))^N$, there holds

$$
\lim_{\varepsilon\to 0} \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) (|\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon - |\nabla u|^{p(x)-2} \nabla u) \nabla(u_\varepsilon - u)\,dx\,dt = 0.
$$

Similar to Section 3, we have $\nabla u_\varepsilon \to \nabla u$ strongly in $(L^{p(x)}(Q_T))^N$ as $\varepsilon \to 0$. Thus there exists a subsequence of $\{u_\varepsilon\}$, still labeled by $\{u_\varepsilon\}$ such that $\nabla u_\varepsilon \to \nabla u$ a.e. $(x,t) \in Q_T$ and $\int_\Omega |\nabla u_\varepsilon|^{p(x)} dx \to \int_\Omega |\nabla u|^{p(x)} dx$ a.e. $t \in [0,T]$. Thus, we obtain that

$$A = a\Big(t, \int_\Omega |\nabla u|^{p(x)} dx\Big) |\nabla u|^{p(x)-2} \nabla u.$$

(Sep 3) Existence of weak solutions. By Fatou's Lemma,

$$\begin{aligned}
&\liminf_{\varepsilon \to 0} \int_{Q_T} a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)} \,dx\,dt \\
&\geq \int_{Q_T} a\Big(t, \int_\Omega |\nabla u|^{p(x)} dx\Big) |\nabla u|^{p(x)} \,dx\,dt.
\end{aligned}$$

For all $v \in V(Q_T)$ with $\frac{\partial v}{\partial t} \in V'(Q_T)$, $v(x,0) = 0$, $|\nabla v| \leq 1$ a.e. $(x,t) \in Q_T$, we take $\varphi = v - u_\varepsilon$ as a test function in (3.1), then

$$\begin{aligned}
&\int_{Q_T} \frac{\partial v}{\partial t}(v - u_\varepsilon) + a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla (v - u_\varepsilon) \\
&- f(x,t)(v - u_\varepsilon) \,dx\,dt \\
&= \frac{1}{\varepsilon} \int_{Q_T} \big((|\nabla v|^{p(x)-2} - 1)^+ \nabla v - (|\nabla u_\varepsilon|^{p(x)-2} - 1)^+ \nabla u_\varepsilon\big)(\nabla v - \nabla u_\varepsilon) \,dx\,dt \\
&\quad + \int_{Q_T} \frac{\partial (v - u_\varepsilon)}{\partial t}(v - u_\varepsilon) \,dx\,dt \geq 0,
\end{aligned}$$

and furthermore,

$$\begin{aligned}
&\liminf_{\varepsilon \to 0} \int_{Q_T} \frac{\partial v}{\partial t}(v - u_\varepsilon) + a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla v \\
&- f(x,t)(v - u_\varepsilon) \,dx\,dt \\
&\geq \int_{Q_T} a\Big(t, \int_\Omega |\nabla u|^{p(x)} dx\Big) |\nabla u|^{p(x)} \,dx\,dt.
\end{aligned}$$

Since

$$a\Big(t, \int_\Omega |\nabla u_\varepsilon|^{p(x)} dx\Big) |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \rightharpoonup a\Big(t, \int_\Omega |\nabla u|^{p(x)} dx\Big) |\nabla u|^{p(x)-2} \nabla u$$

weakly in $(L^{p'(x)}(Q_T))^N$, and $u_\varepsilon \rightharpoonup u$ weakly in $V(Q_T)$, there holds

$$\begin{aligned}
&\int_{Q_T} \frac{\partial v}{\partial t}(v - u) \,dx\,dt + \int_0^T a\Big(t, \int_\Omega |\nabla u|^{p(x,t)} dx\Big) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla (v - u) \,dx\,dt \\
&\geq \int_{Q_T} f(x,t)(v - u) \,dx\,dt.
\end{aligned}$$

Thus we have proved our main theorem. $\square$

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