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EXISTENCE OF BOUNDED SOLUTIONS FOR NONLINEAR FOURTH-ORDER ELLIPTIC EQUATIONS WITH STRENGTHENED COERCIVITY AND LOWER-ORDER TERMS WITH NATURAL GROWTH

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Abstract. In this article, we consider nonlinear elliptic fourth-order equations with the principal part satisfying a strengthened coercivity condition, and a lower-order term having a "natural" growth with respect to the derivatives of the unknown function. We assume that there is an absorption term in the equation, but we do not assume that the lower-order term satisfies the sign condition with respect to unknown function. We prove the existence of bounded generalized solutions for the Dirichlet problem, and present some a priori estimates.

1. Introduction

Skrypnyk [11] introduced a class of nonlinear elliptic equations of the form

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha \mathcal{A}_\alpha(x,u,\dots,D^m u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $m > 1$ and Ω is a bounded domain of $\mathbb{R}^n$. All generalized solutions to this equation are bounded and Hölder continuous. This class of equations is characterized by a strengthened coercivity condition on coefficients $\mathcal{A}_\alpha$, $1 \le |\alpha| \le m$. In a typical case this condition means that for every $x \in \Omega$ and every $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| \le m\}$, the following inequality holds:

$$\sum_{1\le|\alpha|\le m} \mathcal{A}_\alpha(x,\xi)\xi_\alpha \ge C\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=m} |\xi_\alpha|^p \Big\} \tag{1.2}$$

where $p \ge 2$, $mp < q < n$ and $C > 0$. At the same time, in [11] it was assumed that the lower-order term $\mathcal{A}_0$ may have the growth of a rate less than $nq/(n-q) - 1$ with respect to the function u and the growth rates are definitely less than q and p with respect to the derivatives $D^\alpha u$, $|\alpha| = 1$, and the derivatives $D^\alpha u$, $|\alpha| = m$, accordingly.

We observe that the proof of the boundedness of generalized solutions in [11] uses a modification of Moser's method [10]. Using an analogue of Stampacchia's method

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(see [12], [13] and [5]), a weaker (exact) condition on integrability of data was established in [8] to guarantee the boundedness of generalized solutions of nonlinear fourth-order equations with a strengthened coercivity. Moreover, a dependence of summability of generalized solutions of these equations on integrability of data was described in [8]. Analogous results for nonlinear high-order equations with a strengthened coercivity were obtained in [14].

In the present article, we consider a class of nonlinear fourth-order equations of type (1.1) with the principal part satisfying a strengthened coercivity condition like (1.2), where $m = 2$, and with the lower-order term $\mathcal{A}_0$ admitting, unlike [8, 11, 14], the growth of the rate q with respect to the derivatives $D^\alpha u$, $|\alpha| = 1$, and the growth of the rate p with respect to the derivatives $D^\alpha u$, $|\alpha| = 2$. The main result of the article is a theorem on the existence and L^∞-estimate of bounded generalized solutions of the Dirichlet problem for the equations under investigation. We note that in the case under consideration, q and p are exponents of an energy space corresponding to the given problem.

Similar results for nonlinear fourth-order equations with strengthened coercivity and a lower-order term of natural growth were established in [15] in the case where the lower-order term satisfies the sign condition $\mathcal{A}_0(x, u, Du, D^2u)u \geq 0$ and admits an arbitrary growth with respect to u. In the given article, we do not assume that the sign condition is satisfied. At the same time the presence of an absorption term in the left-hand side of the equation is required.

Existence and L^∞-estimate of bounded solutions of nonlinear elliptic second-order equations with natural growth lower-order terms were established for instance in [1]–[3]. At the same time, in [1, 2] it is not assumed that the lower-order terms satisfy the sign condition. Observe that in order to obtain an L^∞-estimate of a solution by Stampacchia's method, in [1, 2] superpositions of the solution and the functions

$$(\exp(\lambda|s - T_k(s)|) - 1)\,\mathrm{sign}(s - T_k(s)), \quad k > 0,\ s \in \mathbb{R}, \tag{1.3}$$

were used as test functions. Here $T_k(s) = \max\{\min\{s, k\}, -k\}$ is the standard cut-off function. The use of the function $(\exp(\lambda|s|) - 1)\,\mathrm{sign}\, s$ with a suitable $\lambda > 0$ in the test functions (superpositions) leads to the absorption of the lower-order term of natural growth by the coercive principal part of the equation (see [1, 2]).

In this article, for obtaining L^∞-estimates, we modify the method of [8] and use the functions

$$|s - h_k(s)|^{\lambda k} \exp(\lambda|s - h_k(s)|)\,\mathrm{sign}(s - h_k(s)), \quad k > 0,\ s \in \mathbb{R},$$

which play a role similar to that of functions (1.3) in the case of elliptic second-order equations with lower-order terms of natural growth. Here h_k is an odd "cut-off" function of the class $C^2(\mathbb{R})$ such that $h_k(s) = s$ if $|s| \leq k$, and $h_k'(s) = 0$ if $|s| \geq 2k$.

We remark that a theory of existence and properties of solutions of nonlinear elliptic fourth-order equations with coefficients satisfying a strengthened coercivity condition and L^1-right-hand sides was developed in [6, 7].

2. Preliminaries and the statement of the main result

Let $n \in \mathbb{N}$, $n > 2$, and let Ω be a bounded open set of $\mathbb{R}^n$.

We shall use the following notation: Λ is the set of all n-dimensional multi-indices α such that $|\alpha| = 1$ or $|\alpha| = 2$; $\mathbb{R}^{n,2}$ is the space of all mappings $\xi : \Lambda \to \mathbb{R}$; if $u \in W^{2,1}(\Omega)$, then $\nabla_2 u : \Omega \to \mathbb{R}^{n,2}$, and for every $x \in \Omega$ and for every $\alpha \in \Lambda$,

$(\nabla_2 u(x))_\alpha = D^\alpha u(x)$. If $r \in [1,+\infty]$, then $\|\cdot\|_r$ is the norm in $L^r(\Omega)$ and $r' = r/(r-1)$. For every measurable set $E \subset \Omega$ we denote by meas$E$ n-dimensional Lebesgue measure of the set E.

Let $p \in (1, n/2)$ and $q \in (2p, n)$. We denote by $W^{1,q}_{2,p}(\Omega)$ the set of all functions in $W^{1,q}(\Omega)$ that have the second-order generalized derivatives in $L^p(\Omega)$. The set $W^{1,q}_{2,p}(\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{W^{1,q}(\Omega)} + \Big(\sum_{|\alpha|=2} \int_\Omega |D^\alpha u|^p dx\Big)^{1/p}.$$

We denote by $\mathring{W}^{1,q}_{2,p}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,q}_{2,p}(\Omega)$.

We set $q^* = nq/(n-q)$. As is known (see for instance [4, Chapter 7]),

$$\mathring{W}^{1,q}(\Omega) \subset L^{q*}(\Omega), \tag{2.1}$$

and there exists a positive constant c depending only on n and q such that for every function $u \in \mathring{W}^{1,q}(\Omega)$,

$$\Big(\int_\Omega |u|^{q^*} dx\Big)^{1/q^*} \le c\Big(\sum_{|\alpha|=1} \int_\Omega |D^\alpha u|^q dx\Big)^{1/q}. \tag{2.2}$$

Next, let $c_0, c_1, c_2, c_3, c_4, c_5 > 0$, let g_1, g_2, g_3, g_4, g_5 be nonnegative summable functions on Ω, $g_5 \in L^{q'}(\Omega)$, and let $A_0 : \Omega \times \mathbb{R} \to \mathbb{R}$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ and $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$, $\alpha \in \Lambda$, be Carathéodory functions. We assume that for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and for every $\xi \in \mathbb{R}^{n,2}$ the following inequalities hold:

$$\sum_{|\alpha|=1} |A_\alpha(x,\xi)|^{q/(q-1)} \le c_1\big\{\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p\big\} + g_1(x), \tag{2.3}$$

$$\sum_{|\alpha|=2} |A_\alpha(x,\xi)|^{p/(p-1)} \le c_2\big\{\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p\big\} + g_2(x), \tag{2.4}$$

$$\sum_{\alpha\in\Lambda} A_\alpha(x,\xi)\xi_\alpha \ge c_3\big\{\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p\big\} - g_3(x), \tag{2.5}$$

$$|B(x,s,\xi)| \le c_4\big\{\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p\big\} + g_4(x), \tag{2.6}$$

$$A_0(x,s)s \ge c_0|s|^q, \tag{2.7}$$

$$|A_0(x,s)| \le c_5|s|^{q-1} + g_5(x). \tag{2.8}$$

Further, let

$$f \in L^{q^*/(q^*-1)}(\Omega). \tag{2.9}$$

We consider the Dirichlet problem

$$\sum_{\alpha\in\Lambda} (-1)^{|\alpha|} D^\alpha A_\alpha(x,\nabla_2 u) + A_0(x,u) + B(x,u,\nabla_2 u) = f \quad \text{in } \Omega, \tag{2.10}$$

$$D^\alpha u = 0, \quad |\alpha| = 0,1, \quad \text{on } \partial\Omega. \tag{2.11}$$

Observe that, by (2.3) and (2.4), for every $u, v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ and for every $\alpha \in \Lambda$ the function $A_\alpha(x,\nabla_2 u)D^\alpha v$ is summable on Ω. By (2.8), for every $u, v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ the function $A_0(x,u)v$ belongs to $L^1(\Omega)$, and by (2.6), for every $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ and for every $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ the function $B(x,u,\nabla_2 u)v$ is summable on Ω.

Moreover, it follows from (2.1) and (2.9) that for every $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$ the function fv is summable on Ω.

Definition 2.1. A generalized solution of problem (2.10), (2.11) is a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ such that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u) D^\alpha v + A_0(x, u)v + B(x, u, \nabla_2 u)v\Big\} dx = \int_\Omega f v dx. \tag{2.12}$$

The following theorem is the main result of the present article.

Theorem 2.2. *Let $r > n/q$, let the functions g_2, g_3, g_4 and f belong to $L^r(\Omega)$, and let for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^{n,2}$, $\xi \neq \xi'$, the following inequality holds:*

$$\sum_{\alpha\in\Lambda} [A_\alpha(x, \xi) - A_\alpha(x, \xi')](\xi_\alpha - \xi'_\alpha) > 0. \tag{2.13}$$

Then there exists a generalized solution u_0 of problem (2.10), (2.11) such that $u_0 \in L^\infty(\Omega)$ and

$$\|u_0\|_\infty \le C_1 \tag{2.14}$$

where C_1 is a positive constant depending only on n, p, q, r, $\operatorname{meas}\Omega$, c, c_0, c_2, c_3, c_4 and the functions g_2, g_3, g_4 and f.

Let us give an example of functions satisfying conditions (2.3)–(2.8) and (2.13).

Example 2.3. Let for every n-dimensional multiindex α, $|\alpha| = 1$, $A_\alpha : \Omega\times\mathbb{R}^{n,2} \to \mathbb{R}$ be the function defined by

$$A_\alpha(x, \xi) = \Big(\sum_{|\beta|=1} \xi_\beta^2 \Big)^{(q-2)/2} \xi_\alpha, \quad (x, \xi) \in \Omega \times \mathbb{R}^{n,2},$$

and let for every n-dimensional multiindex α, $|\alpha| = 2$, $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ be the function defined by

$$A_\alpha(x, \xi) = \Big(\sum_{|\beta|=2} \xi_\beta^2 \Big)^{(p-2)/2} \xi_\alpha, \quad (x, \xi) \in \Omega \times \mathbb{R}^{n,2}.$$

Then the functions A_α, $\alpha \in \Lambda$, satisfy inequalities (2.3)–(2.5) and (2.13). Next, let

$$B(x, s, \xi) = b(x)\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\}, \quad (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2},$$
$$A_0(x, s) = c_0|s|^{q-2}s, \quad (x, s) \in \Omega \times \mathbb{R},$$

where $c_0 > 0$ and $b \in L^\infty(\Omega)$. Then the function B satisfies inequality (2.6), and the function A_0 satisfies inequalities (2.7) and (2.8).

Observe that the coefficients of the biharmonic operator $\Delta^2 u$ do not satisfy condition (2.5).

We will prove Theorem 2.2 in Section 6. The key point of its proof is obtaining a priori energy- and L^∞-estimates for bounded generalized solutions of problem (2.10), (2.11). These estimates are contained in the following two theorems which will be established in Sections 4 and 5 respectively.

Theorem 2.4. *Let the functions g_2, g_3, g_4 and f belong to $L^{n/q}(\Omega)$, and let u be a generalized solution of problem* (2.10), (2.11) *such that $u \in L^\infty(\Omega)$. Then for every $\lambda > c_4/c_3$ we have*

$$\int_\Omega \Big(\sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p \Big) \exp(\lambda|u|)dx \le C_2 \tag{2.15}$$

where C_2 is a positive constant depending only on n, p, q, $\operatorname{meas}\Omega$, c, c_0, c_2, c_3, c_4, λ and the functions g_2, g_3, g_4 and f.

Theorem 2.5. *Let $r > n/q$, let the functions g_2, g_3, g_4 and f belong to $L^r(\Omega)$, and let u be a generalized solution of problem* (2.10), (2.11) *such that $u \in L^\infty(\Omega)$. Then*

$$\|u\|_\infty \le C_1 \tag{2.16}$$

where C_1 is the positive constant from Theorem 2.2.

Remark 2.6. The condition $r > n/q$ in the statements of Theorems 2.2 and 2.5 coincides with the condition of boundedness of generalized solutions of the Dirichlet problem considered in [8] for equation (2.10) with $A_0 \equiv 0$ and $B \equiv 0$.

Before proving Theorems 2.2, 2.4 and 2.5, we give several auxiliary results.

3. Auxiliary results

By analogy with [6, Lemma 2.2], we establish the following result.

Lemma 3.1. *Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$, $h \in C^2(\mathbb{R})$ and $h(0) = 0$. Then $h(u) \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:*

(a) *for every n-dimensional multi-index α, $|\alpha| = 1$,*

$$D^\alpha h(u) = h'(u) D^\alpha u \quad \text{a. e. in } \Omega,$$

(b) *for every n-dimensional multi-index α, $|\alpha| = 2$,*

$$D^\alpha h(u) = h'(u) D^\alpha u + h''(u) D^\beta u D^\gamma u \quad \text{a. e. in } \Omega,$$

where $\alpha = \beta + \gamma$, $|\beta| = |\gamma| = 1$.

Lemma 3.2. *Let h be an odd function on $\mathbb{R}$ such that $h \in C^1(\mathbb{R})$, $h \in C^2(\mathbb{R} \setminus \{0\})$ and h'' has a discontinuity of the first kind at the origin. Let*

$$u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega). \tag{3.1}$$

Then $h(u) \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:

(i) *for every n-dimensional multi-index α, $|\alpha| = 1$,*

$$D^\alpha h(u) = h'(u) D^\alpha u \quad \text{a. e. in } \Omega;$$

(ii) *for every n-dimensional multi-index α, $|\alpha| = 2$,*

$$D^\alpha h(u) = \begin{cases} h'(u) D^\alpha u + h''(u) D^\beta u D^\gamma u & \text{a. e. in } \{u \ne 0\}, \\ h'(0) D^\alpha u & \text{a. e. in } \{u = 0\} \end{cases}$$

where $\alpha = \beta + \gamma$, $|\beta| = |\gamma| = 1$.

Proof. Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. We define the function $H : \mathbb{R} \to \mathbb{R}$ by

$$H(s) = h(s) - h'(0)s, \quad s \in \mathbb{R}. \tag{3.2}$$

Let

$$w_\alpha = H'(u) D^\alpha u \quad \text{if } |\alpha| = 1, \tag{3.3}$$

and let

$$w_\alpha = \begin{cases} H'(u)D^\alpha u + H''(u) D^\beta u D^\gamma u & \text{in } \{u \neq 0\}, \\ 0 & \text{in } \{u = 0\} \end{cases} \tag{3.4}$$

if $|\alpha| = 2$ and $\alpha = \beta + \gamma$, $|\beta| = |\gamma| = 1$. Clearly,

$$w_\alpha \in L^q(\Omega) \text{ if } |\alpha| = 1 \quad \text{and} \quad w_\alpha \in L^p(\Omega) \text{ if } |\alpha| = 2. \tag{3.5}$$

We fix $\varepsilon > 0$. Let $H_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the function such that

$$H_\varepsilon(s) = \begin{cases} H(s) + (\frac{1}{2}\varepsilon H''(\varepsilon) - H'(\varepsilon))(s - \varepsilon) + \frac{1}{6}\varepsilon^2 H''(\varepsilon) - H(\varepsilon) & \text{if } s > \varepsilon, \\ H''(\varepsilon)s^3/(6\varepsilon) & \text{if } |s| \le \varepsilon, \\ H(s) + (\frac{1}{2}\varepsilon H''(\varepsilon) - H'(\varepsilon))(s + \varepsilon) - \frac{1}{6}\varepsilon^2 H''(\varepsilon) + H(\varepsilon) & \text{if } s < -\varepsilon. \end{cases}$$

We have

$$H_\varepsilon \in C^2(\mathbb{R}), \tag{3.6}$$

$$H'_\varepsilon(s) = \begin{cases} H'(s) + \varepsilon H''(\varepsilon)/2 - H'(\varepsilon) & \text{if } |s| > \varepsilon, \\ H''(\varepsilon)s^2/(2\varepsilon) & \text{if } |s| \le \varepsilon, \end{cases}$$

$$H''_\varepsilon(s) = \begin{cases} H''(s) & \text{if } |s| > \varepsilon, \\ H''(\varepsilon)s/\varepsilon & \text{if } |s| \le \varepsilon. \end{cases}$$

The following limit relations hold:

$$\lim_{\varepsilon \to 0} H_\varepsilon(s) = H(s) \quad \forall s \in \mathbb{R}, \tag{3.7}$$

$$\lim_{\varepsilon \to 0} H'_\varepsilon(s) = H'(s) \quad \forall s \in \mathbb{R}, \tag{3.8}$$

$$\lim_{\varepsilon \to 0} H''_\varepsilon(s) = \begin{cases} H''(s) & \text{if } s \in \mathbb{R} \setminus \{0\}, \\ 0 & \text{if } s = 0. \end{cases} \tag{3.9}$$

Using inclusions (3.1) and (3.6), the equality $H_\varepsilon(0) = 0$ and Lemma 3.1, we establish that $H_\varepsilon(u) \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$, $D^\alpha H_\varepsilon(u) = H'_\varepsilon(u) D^\alpha u$ if $|\alpha| = 1$, and $D^\alpha H_\varepsilon(u) = H'_\varepsilon(u) D^\alpha u + H''_\varepsilon(u) D^\beta u D^\gamma u$ if $|\alpha| = 2$ and $\alpha = \beta + \gamma$, $|\beta| = |\gamma| = 1$. Hence, using (3.1), (3.7)–(3.9) along with Dominated Convergence Theorem, we deduce that

$$\lim_{\varepsilon \to 0} \|H_\varepsilon(u) - H(u)\|_{L^q(\Omega)} = 0, \tag{3.10}$$

$$\lim_{\varepsilon \to 0} \sum_{|\alpha|=1} \|D^\alpha H_\varepsilon(u) - w_\alpha\|_{L^q(\Omega)} = 0, \quad \lim_{\varepsilon \to 0} \sum_{|\alpha|=2} \|D^\alpha H_\varepsilon(u) - w_\alpha\|_{L^p(\Omega)} = 0. \tag{3.11}$$

Using these limit relations, in the usual way we establish that for every $\alpha \in \Lambda$ there exists the generalized derivative $D^\alpha H(u)$, and $D^\alpha H(u) = w_\alpha$ a. e. on Ω. Then, by (3.5), (3.10) and (3.11), the function $H(u)$ belong to $\mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$, and (3.2)–(3.4) imply that assertions (i) and (ii) hold. The proof is complete. □

The next result is similar to the corresponding part of Stampacchia's lemma [13].

Lemma 3.3. *Let φ be a nonincreasing nonnegative function on $[0,+\infty)$. Let $C > 0$, $b_1 \geq 0$, $b_2 \geq 0$, $0 \leq \tau_1 < \tau_2$, $\gamma > 1$ and $k_0 \geq 0$. Let for every k and l such that $k_0 < k < l < 2k$ the following inequality holds:*

$$\varphi(l) \leq \frac{Ck^{\tau_1 k+b_1}}{(l-k)^{\tau_2 k+b_2}}[\varphi(k)]^{\gamma}. \tag{3.12}$$

Let $d > \max\{k_0, 1\}$ and

$$d^{(\tau_2-\tau_1)(k_0+d/2)+b_2-b_1} \geq 2^{2\tau_1 d+b_1+(2\gamma-1)(2\tau_2 d+b_2)/(\gamma-1)}C[\varphi(k_0)]^{\gamma-1}. \tag{3.13}$$

Then $\varphi(k_0+d)=0$.

Proof. We set $a=(2\tau_2 d+b_2)/(\gamma-1)$, and let for every $j\in\mathbb{N}$,

$$k_j = k_0 + d - \frac{d}{2^j}. \tag{3.14}$$

Then for every $j\in\mathbb{N}$ we have

$$k_0 < k_j < k_{j+1} < 2k_j, \quad k_{j+1}-k_j = \frac{d}{2^{j+1}}, \quad k_j < 2d, \quad k_j \geq k_0 + d/2.$$

Using these relations, (3.12) and the inequality $d>1$, we establish that for every $j\in\mathbb{N}$,

$$\varphi(k_{j+1}) \leq \frac{C2^{2\tau_1 d+b_1}\cdot 2^{(j+1)(2\tau_2 d+b_2)}}{d^{(\tau_2-\tau_1)(k_0+d/2)+b_2-b_1}}[\varphi(k_j)]^{\gamma}.$$

By means of the latter inequality and (3.13), we establish by induction on j, that for every $j\in\mathbb{N}$,

$$\varphi(k_j) \leq 2^{-a(j-1)}\varphi(k_0).$$

Using this result and relation (3.14) and taking into account that the function φ is nonincreasing and nonnegative, we deduce that $\varphi(k_0+d)=0$. The proof is complete. □

4. Proof of Theorem 2.4

Let the functions g_2, g_3, g_4 and f belong to $L^{n/q}(\Omega)$, and let u be a bounded generalized solution of problem (2.10), (2.11). We fix an arbitrary positive number λ such that

$$\lambda > c_4/c_3. \tag{4.1}$$

By c_i, $i=6,7,\dots$, we shall denote positive constants depending only on n, p, q, $\operatorname{meas}\Omega$, c, c_0, c_2, c_3, c_4, λ and the functions g_2, g_3, g_4 and f. We define the function $h:\mathbb{R}\to\mathbb{R}$ by

$$h(s) = (e^{\lambda|s|}-1)\operatorname{sign} s, \quad s\in\mathbb{R}.$$

We set $c_6 = c_3\lambda - c_4$. By (4.1), we have $c_6>0$. Elementary calculations show that

$$c_3 h' - c_4|h| > c_6 h' \quad \text{in } \mathbb{R}. \tag{4.2}$$

We set

$$I' = \int_{\{u\neq 0\}} \Big\{\sum_{|\alpha|=2}|A_\alpha(x,\nabla_2 u)|\Big\}\Big\{\sum_{|\beta|=1}|D^\beta u|^2\Big\}|h''(u)|dx,$$

$$\Phi = \sum_{|\alpha|=1}|D^\alpha u|^q + \sum_{|\alpha|=2}|D^\alpha u|^p.$$

By Lemma 3.2, $h(u) \in \overset{\circ}{W}{}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. Then, by (2.12), we have

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u) D^\alpha h(u) + A_0(x,u)h(u) + B(x,u,\nabla_2 u)h(u)\Big\} dx = \int_\Omega f h(u) dx.$$

From this equality and assertions (i) and (ii) of Lemma 3.2 we deduce that

$$\begin{aligned} &\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u) D^\alpha u\Big\} h'(u) dx + \int_\Omega A_0(x,u)h(u)dx \\ &\le \int_\Omega |B(x,u,\nabla_2 u)||h(u)|dx + \int_\Omega |f||h(u)|dx + I'. \end{aligned}$$

Hence, using (2.5)–(2.7) and the facts that $0 < h' = \lambda|h| + \lambda$ and sign $h(s) =$ sign s in $\mathbb{R}$, we obtain

$$\begin{aligned} &\int_\Omega \Phi(c_3 h'(u) - c_4|h(u)|)dx + c_0 \int_\Omega |u|^{q-1}|h(u)|dx \\ &\le \int_\Omega (\lambda g_3 + g_4 + |f|)|h(u)|dx + \lambda \int_\Omega g_3 dx + I'. \end{aligned}$$

From this and (4.2) it follows that

$$\begin{aligned} &c_6 \int_\Omega h'(u)\Phi dx + c_0 \int_\Omega |u|^{q-1}|h(u)|dx \\ &\le \int_\Omega (\lambda g_3 + g_4 + |f|)|h(u)|dx + \lambda \int_\Omega g_3 dx + I'. \end{aligned} \tag{4.3}$$

Let us estimate the integral I'. We fix an arbitrary $\varepsilon > 0$. It is obvious that

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1.$$

Using this equality and Young's inequality, we establish that if $\alpha \in \Lambda$, $|\alpha| = 2$, and $\beta \in \Lambda$, $|\beta| = 1$, then

$$|A_\alpha(x,\nabla_2 u)||D^\beta u|^2 \le \varepsilon^2 |A_\alpha(x,\nabla_2 u)|^{p/(p-1)} + \varepsilon^2 |D^\beta u|^q + \varepsilon^{2-2qp/(q-2p)} \quad \text{on} \quad \Omega.$$

From this and (2.4) we deduce that

$$\begin{aligned} I' &\le n(c_2+n)\varepsilon^2 \int_{\{u\neq 0\}} \Phi|h''(u)|dx + n\varepsilon^2 \int_{\{u\neq 0\}} g_2|h''(u)|dx \\ &\quad + n^3 \varepsilon^{2-2qp/(q-2p)} \int_{\{u\neq 0\}} |h''(u)|dx. \end{aligned}$$

Putting in this inequality $\varepsilon = (\frac{c_6}{2\lambda n(c_2+n)})^{1/2}$, and noting that $|h''| = \lambda h'$ and $|h''| = \lambda^2|h| + \lambda^2$ on $\mathbb{R} \setminus \{0\}$, we obtain

$$I' \le \frac{c_6}{2} \int_\Omega h'(u)\Phi dx + c_7 \int_\Omega (g_2+1)|h(u)|dx + c_8.$$

From this and (4.3) it follows that

$$\frac{c_6}{2} \int_\Omega h'(u)\Phi dx + c_0 \int_\Omega |u|^{q-1}|h(u)|dx \le c_9 \int_\Omega F|h(u)|dx + c_{10} \tag{4.4}$$

where $F = g_2 + \lambda g_3 + g_4 + |f| + 1$.

Now, we estimate the integral $\int_\Omega F|h(u)|dx$. We fix an arbitrary $K > 0$. It is clear that

$$\begin{aligned}&\int_\Omega F|h(u)|dx\\&= \int_{\{F>K,|u|\geq 1\}} F|h(u)|dx + \int_{\{F<K\}} F|h(u)|dx + \int_{\{F>K,|u|<1\}} F|h(u)|dx,\end{aligned} \tag{4.5}$$

$$\int_{\{F<K\}} F|h(u)|dx < K\int_\Omega |h(u)|dx, \tag{4.6}$$

$$\int_{\{F>K,|u|<1\}} F|h(u)|dx < (e^\lambda - 1)\int_\Omega F dx. \tag{4.7}$$

Before estimating the first integral in the right-hand side of equality (4.5), we remark that there exists a positive constant $c_{q,\lambda}$ depending only on q and λ such that

$$|h(s)| \leq c_{q,\lambda}|h(s/q)|^q \quad \text{for every } s \geq 1. \tag{4.8}$$

Note also that, by (2.2), assertion (i) of Lemma 3.2 and equality $(h'(s/q))^q = \lambda^{q-1}h'(s)$, $s \in \mathbb{R}$, we have

$$\Big(\int_\Omega |h(u/q)|^{q^*}dx\Big)^{q/q^*} \leq (c\lambda^{q-1}/q^q)\int_\Omega h'(u)\Phi dx. \tag{4.9}$$

Now, using Holder's inequality, (4.8) and (4.9), we obtain

$$\begin{aligned}\int_{\{F>K,|u|\geq 1\}} F|h(u)|dx &\leq \Big(\int_{\{F>K\}} F^{n/q}dx\Big)^{q/n}\Big(\int_{\{|u|\geq 1\}} |h(u)|^{n/(n-q)}dx\Big)^{(n-q)/n}\\&\leq c_{q,\lambda}\|F\|_{L^{n/q}(\{F>K\})}\Big(\int_\Omega |h(u/q)|^{q^*}dx\Big)^{q/q^*}\\&\leq c_{q,\lambda}c\lambda^{q-1}q^{-q}\|F\|_{L^{n/q}(\{F>K\})}\int_\Omega h'(u)\Phi dx.\end{aligned}$$

From this along with (4.5)–(4.7) it follows that

$$\int_\Omega F|h(u)|dx \leq c_{11}\|F\|_{L^{n/q}(\{F>K\})}\int_\Omega h'(u)\Phi dx + \int_\Omega K|h(u)|dx + c_{12}. \tag{4.10}$$

Now, choosing $K > 0$ such that $c_9c_{11}\|F\|_{L^{n/q}(\{F>K\})} < c_6/4$, from (4.4) and (4.10) we derive that

$$\frac{c_6}{4}\int_\Omega h'(u)\Phi dx + \int_\Omega c_0|u|^{q-1}|h(u)|dx \leq \int_\Omega c_9K|h(u)|dx + c_{13}. \tag{4.11}$$

It is clear that

$$\begin{aligned}c_0\int_\Omega |u|^{q-1}|h(u)|dx &\geq c_0\int_{\{c_0|u|^{q-1}>c_9K\}} |u|^{q-1}|h(u)|dx\\&> c_9\int_{\{c_0|u|^{q-1}>c_9K\}} K|h(u)|dx,\end{aligned} \tag{4.12}$$

$$\begin{aligned} & c_9 \int_\Omega K|h(u)|dx \\ & = c_9 \int_{\{c_0|u|^{q-1} > c_9 K\}} K|h(u)|dx + c_9 \int_{\{c_0|u|^{q-1} \le c_9 K\}} K|h(u)|dx \\ & \le c_9 \int_{\{c_0|u|^{q-1} > c_9 K\}} K|h(u)|dx + c_9 K(e^{\lambda(c_9 K/c_0)^{1/(q-1)}} - 1) \operatorname{meas}\Omega. \end{aligned} \tag{4.13}$$

From (4.11)–(4.13) it follows that

$$\frac{c_6}{4} \int_\Omega h'(u)\Phi dx \le c_{13}.$$

Hence, taking into account that for every $s \in \mathbb{R}$, $h'(s) = \lambda \exp(\lambda|s|)$, we deduce (2.15). The proof is comlete.

5. Proof of Theorem 2.5

Let $r > n/q$, let the functions g_2, g_3, g_4 and f belong to $L^r(\Omega)$, and let M be a majorant for $\|g_2\|_r$, $\|g_3\|_r$, $\|g_4\|_r$ and $\|f\|_r$. Let u be a generalized solution of problem (2.10), (2.11) such that

$$u \in L^\infty(\Omega). \tag{5.1}$$

In view of the assumption $r > n/q$, we have

$$qr/(r-1) < q^*. \tag{5.2}$$

We set

$$\tilde F = 1 + g_2 + g_3 + g_4 + |f|, \tag{5.3}$$

$$\Phi = \sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p.$$

Observe that, by Theorem 2.4, we have

$$\int_\Omega \Phi dx \le C_2. \tag{5.4}$$

By c_i, $i = 14, 15, \dots$, we shall denote positive constants depending only on n, p, q, $\operatorname{meas}\Omega$, c, c_0, c_2, c_3, c_4, C_2, r and M.

Step 1. Let φ be the function on $[0, +\infty)$ such that for every $s \in [0, +\infty)$,

$$\varphi(s) = \operatorname{meas}\{|u| \ge s\}.$$

Our main goal is to establish relation (3.12) for this function. Let us introduce some auxiliary numbers and functions. Let δ, ϑ, θ and t be positive numbers such that

$$1 + \delta(2 - 2qp/(q-2p)) > (r-1)/r, \tag{5.5}$$

$$\vartheta = q/n - 1/r - 2qp\delta/(q-2p), \tag{5.6}$$

$$\theta(q-1) - \vartheta q^* < 0, \tag{5.7}$$

$$t = 2 + 1/\delta. \tag{5.8}$$

We set

$$\lambda = 2c_4/c_3. \tag{5.9}$$

Without loss of generality, we may assume that

$$\lambda > 1. \tag{5.10}$$

By (2.2) and (5.4), for every $k > 0$ we have

$$\varphi(k) \le C_2^{q^*/q} c^{q^*} k^{-q^*}. \tag{5.11}$$

Therefore, there exists a positive number k_* depending only on n, p, q, t, θ, ϑ, c, c_2, c_3, c_4, C_2, and $\|\tilde{F}\|_r$ such that

$$\forall k \ge k_*, \quad 2(c_2+n)(\lambda t(t+1)nk)^2[\varphi(k)]^{1/(t-2)} < \min\{1/2, c_3/12\}, \tag{5.12}$$

$$\forall k \ge k_*, \quad (c/q)^q(\lambda(t+1))^{q-1}\|\tilde{F}\|_r k^{\theta(q-1)}[\varphi(k)]^{\vartheta} < c_3/6. \tag{5.13}$$

Observe that for establishing the last assertion we used not only (5.11) but also (5.7).

Let ψ be the function on $(0,+\infty)$ such that for every $s \in (0,+\infty)$,

$$\psi(s) = s - s^t + \frac{t-1}{t+1}s^{t+1}.$$

We set

$$k_0 = \max\{k_*, q, 1 + c_3/c_4, (12nt(c_2+n)/c_3)^{t/(t-1)}\} \tag{5.14}$$

and fix an arbitrary number $k \ge k_0$.

Let h_k and G_k be the functions on $\mathbb{R}$ such that

$$h_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ [\psi(\frac{|s|-k}{k}) + 1]k \operatorname{sign} s & \text{if } k < |s| < 2k, \\ \frac{2kt}{t+1} \operatorname{sign} s & \text{if } |s| \ge 2k, \end{cases}$$

and for every $s \in \mathbb{R}$, $G_k(s) = s - h_k(s)$.

Note that the function h_k was the main instrument in the realization of the Stampacchia's method for nonlinear elliptic fourth-order equations with strengthened coercivity in [8] and [15]. The functions of this type were introduced and used for other purposes in [6]. We consider the properties of the functions h_k and G_k, which are needed in this paper. We have

$$h_k \in C^2(\mathbb{R}), \tag{5.15}$$

$$|G_k(s)| = k\Big(\frac{|s|-k}{k}\Big)^t\Big(1 - \frac{t-1}{t+1}\cdot\frac{|s|-k}{k}\Big) \quad \text{if } k < |s| < 2k, \tag{5.16}$$

$$G_k'(s) = t\Big(\frac{|s|-k}{k}\Big)^{t-1} - (t-1)\Big(\frac{|s|-k}{k}\Big)^t \quad \text{if } k < |s| < 2k, \tag{5.17}$$

$$0 \le G_k' \le 1 \quad \text{in } \mathbb{R}, \tag{5.18}$$

$$|h_k''| \le \frac{t^2}{k} \quad \text{in } \mathbb{R}. \tag{5.19}$$

Moreover, the following assertions hold:

(A1) if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $k \le |s| \le k(1+\varepsilon)$, then

$$|h_k''(s)| \le \frac{t^2}{k}\varepsilon^{t-2};$$

(A2) if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $k(1+\varepsilon) \le |s| \le 2k$, then

$$|h_k''(s)| \le \frac{t}{k\varepsilon}G_k'(s);$$

(A3) if $k < l \leq 2k$, $s \in \mathbb{R}$ and $|s| \geq l$, then
$$|G_k(s)| \geq \frac{2}{t+1}(l-k)\Big(\frac{l-k}{k}\Big)^{t-1};$$
(A4) if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $k \leq |s| \leq k(1+\varepsilon)$, then
$$|G_k(s)| \leq k\varepsilon^t.$$

Proofs of assertions (A1)–(A3) are given in [8]. It remains to prove assertion (A4).

Let $\varepsilon \in (0,1)$, $s \in \mathbb{R}$, $k \leq |s| \leq k(1+\varepsilon)$ and $y = (|s|-k)/k$. Using (5.16) and the inequality $0 \leq y \leq \varepsilon < 1$, we obtain
$$|G_k(s)| = ky^t(1 - \frac{t-1}{t+1}y) \leq k\varepsilon^t.$$
Thus, assertion (A4) is valid.

We set
$$\mu = \lambda k. \tag{5.20}$$
Let $\Psi : \mathbb{R} \to \mathbb{R}$ be the function such that
$$\Psi(s) = |s|^\mu \exp(\lambda|s|) \operatorname{sign} s. \tag{5.21}$$
By (5.14), we have $\mu > 2$. Hence,
$$\Psi \in C^2(\mathbb{R}), \tag{5.22}$$
and for every $s \in \mathbb{R}$ the following equalities hold:
$$\Psi'(s) = |s|^{\mu-1} \exp(\lambda|s|)(\mu + \lambda|s|) = \lambda|\Psi(s)| + \mu|s|^{\mu-1}\exp(\lambda|s|), \tag{5.23}$$
$$\Psi''(s) = |s|^{\mu-2}\exp(\lambda|s|)(\mu(\mu-1) + 2\lambda\mu|s| + \lambda^2 s^2)\operatorname{sign} s. \tag{5.24}$$
Let us prove the following assertion:

(A5) if $s \in \mathbb{R}$, then
$$c_3\Psi'(G_k(s))G_k'(s) - c_4|\Psi(G_k(s))| \geq \frac{c_3}{2}\Psi'(G_k(s))G_k'(s). \tag{5.25}$$
Indeed, if $s \in \mathbb{R}$ and $|s| \leq k$, then both sides of inequality (5.25) are equal zero and therefore, this inequality is true.

Now, let $k < |s| < 2k$ and $y = (|s|-k)/k$. By (5.16), (5.17) and the inequality $y < 1$, we have
$$\begin{aligned}\frac{|G_k(s)|}{G_k'(s)} &= \frac{k}{t+1}\Big(y + \frac{t}{(t-1)(t-(t-1)y)} - \frac{1}{t-1}\Big)\\ &< \frac{k}{t+1}\Big(1 + \frac{t}{t-1} - \frac{1}{t-1}\Big) = \frac{2k}{t+1}.\end{aligned} \tag{5.26}$$
Using (5.20), (5.21), (5.23), (5.26) and the inequality $t > 1$, we obtain
$$\begin{aligned}&c_3\Psi'(G_k(s))G_k'(s) - c_4|\Psi(G_k(s))|\\ &= |G_k(s)|^{\mu-1}\exp(\lambda|G_k(s)|)G_k'(s)(c_3\mu + c_3\lambda|G_k(s)| - c_4|G_k(s)|/G_k'(s))\\ &\geq |G_k(s)|^{\mu-1}\exp(\lambda|G_k(s)|)G_k'(s)(c_3\mu + c_3\lambda|G_k(s)| - c_4 k)\\ &\geq \frac{c_3}{2}|G_k(s)|^{\mu-1}\exp(\lambda|G_k(s)|)G_k'(s)(2\mu - \frac{2c_4k}{c_3} + \lambda|G_k(s)|) = \frac{c_3}{2}\Psi'(G_k(s))G_k'(s).\end{aligned}$$
Thus, inequality (5.25) holds.

Finally, let $|s| \geq 2k$. Using (5.9), (5.21), (5.23) and the equality $G'_k(s) = 1$, we obtain

$$\begin{aligned} &c_3\Psi'(G_k(s))G'_k(s) - c_4|\Psi(G_k(s))| \\ &= |G_k(s)|^{\mu-1}\exp(\lambda|G_k(s)|)(c_3\mu + (\lambda c_3 - c_4)|G_k(s)|) \\ &\geq \frac{c_3}{2}|G_k(s)|^{\mu-1}\exp(\lambda|G_k(s)|)(\mu + \lambda|G_k(s)|) \\ &= \frac{c_3}{2}\Psi'(G_k(s))G'_k(s). \end{aligned}$$

Therefore, inequality (5.25) holds. Thus, inequality (5.25) holds for every $s \in \mathbb{R}$, and assertion (A5) is proved.

Step 2. Using inclusions (5.1), (5.15), (5.22), the equalities $G_k(0) = \Psi(0) = 0$ and Lemma 3.1, we establish that $\Psi(G_k(u)) \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:

(A6) for every n-dimensional multi-index $\alpha, |\alpha| = 1$,

$$D^\alpha\Psi(G_k(u)) = \Psi'(G_k(u))G'_k(u)D^\alpha u \quad \text{a. e. in} \quad \Omega;$$

(A7) for every n-dimensional multi-index $\alpha, |\alpha| = 2$,

$$\begin{aligned} D^\alpha\Psi(G_k(u)) = \Psi'(G_k(u))G'_k(u)D^\alpha u + \big[&\Psi''(G_k(u))(G'_k(u))^2 \\ &- \Psi'(G_k(u))h''_k(u)\big]D^\beta u D^\gamma u \quad \text{a. e. in } \Omega \end{aligned}$$

where $\alpha = \beta + \gamma$, $|\beta| = |\gamma| = 1$.

We set

$$I'_k = \int_\Omega \Big\{\sum_{|\alpha|=2}|A_\alpha(x,\nabla_2 u)|\Big\}\Big\{\sum_{|\beta|=1}|D^\beta u|^2\Big\}\Psi'(G_k(u))|h''_k(u)|dx,$$

$$I''_k = \int_\Omega \Big\{\sum_{|\alpha|=2}|A_\alpha(x,\nabla_2 u)|\Big\}\Big\{\sum_{|\beta|=1}|D^\beta u|^2\Big\}\Psi''(G_k(u))(G'_k(u))^2 dx.$$

Putting the function $\Psi(G_k(u))$ into (2.12) instead of v, we obtain

$$\begin{aligned} &\int_\Omega \Big\{\sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u)D^\alpha\Psi(G_k(u))\Big\}dx \\ &+ \int_\Omega A_0(x,u)\Psi(G_k(u))dx + \int_\Omega B(x,u,\nabla_2 u)\Psi(G_k(u))dx \\ &= \int_\Omega f\Psi(G_k(u))dx. \end{aligned}$$

From this equality and assertions (A6) and (A7) we deduce that

$$\begin{aligned} &\int_\Omega \Big\{\sum_{\alpha\in\Lambda} A_\alpha(x,\nabla_2 u)D^\alpha u\Big\}\Psi'(G_k(u))G'_k(u)dx + \int_\Omega A_0(x,u)\Psi(G_k(u))dx \\ &\leq I'_k + I''_k + \int_\Omega |B(x,u,\nabla_2 u)||\Psi(G_k(u))|dx + \int_\Omega |f||\Psi(G_k(u))|dx. \end{aligned}$$

Hence, using (2.5)–(2.7), we obtain

$$\int_\Omega \big\{c_3\Psi'(G_k(u))G'_k(u) - c_4|\Psi(G_k(u))|\big\}\Phi dx + c_0\int_\Omega |u|^{q-1}|\Psi(G_k(u))|dx$$

$$\le I_k' + I_k'' + \int_\Omega g_3\Psi'(G_k(u))G_k'(u)dx + \int_\Omega (g_4 + |f|)|\Psi(G_k(u))|dx.$$

In turn, from this and assertion (A5) it follows that

$$\begin{aligned}&\frac{c_3}{2}\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx\\ &\le I_k' + I_k'' + \int_\Omega g_3\Psi'(G_k(u))G_k'(u)dx + \int_\Omega (g_4 + |f|)|\Psi(G_k(u))|dx.\end{aligned} \tag{5.27}$$

Step 3. Let us obtain suitable estimates for the addends in the right-hand side of this inequality.

First, assume that $\varphi(k) > 0$. We set

$$\varepsilon = [\varphi(k)]^{1/(t-2)}. \tag{5.28}$$

Since $k \ge k_0$, by (5.12) and (5.14) we have $\varphi(k) < 1$. Therefore,

$$0 < \varepsilon < 1. \tag{5.29}$$

We shall prove the inequality

$$\begin{aligned}I_k' \le \frac{c_3}{12}\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx + \frac{1}{2}\int_\Omega g_2|\Psi(G_k(u))|dx\\ + \frac{1}{2}\varepsilon^{-2qp/(q-2p)}\int_\Omega |\Psi(G_k(u))|dx + c_{14}[\varphi(k)]^{1/r'}.\end{aligned} \tag{5.30}$$

Obviously,

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1.$$

Using this equality and the Young's inequality, we establish that if α is an n-dimensional multi-index, $|\alpha| = 2$, and β is an n-dimensional multi-index, $|\beta| = 1$, then

$$|A_\alpha(x,\nabla_2 u)||D^\beta u|^2 \le \varepsilon^2|A_\alpha(x,\nabla_2 u)|^{p/(p-1)} + \varepsilon^2|D^\beta u|^q + \varepsilon^{2-2qp/(q-2p)} \quad \text{on } \Omega.$$

This and relation (2.4) yields

$$\begin{aligned}I_k' \le n(c_2+n)\varepsilon^2\int_\Omega \Phi\Psi'(G_k(u))|h_k''(u)|dx + n\varepsilon^2\int_\Omega g_2\Psi'(G_k(u))|h_k''(u)|dx\\ + n^3\varepsilon^{2-2qp/(q-2p)}\int_\Omega \Psi'(G_k(u))|h_k''(u)|dx.\end{aligned} \tag{5.31}$$

Let us estimate the second integral in the right-hand side of (5.31). By (5.19), (5.20), (5.23) and the inequality $k > 1$, we have

$$\int_\Omega g_2\Psi'(G_k(u))|h_k''(u)|dx \le \lambda t^2\int_\Omega g_2(|\Psi(G_k(u))| + |G_k(u)|^{\mu-1}\exp(\lambda|G_k(u)|))dx. \tag{5.32}$$

Also, it is clear that

$$\begin{aligned}&\int_\Omega g_2|G_k(u)|^{\mu-1}\exp(\lambda|G_k(u)|)dx\\&=\int_{\{|G_k(u)|<1\}} g_2|G_k(u)|^{\mu-1}\exp(\lambda|G_k(u)|)dx\\&\quad+\int_{\{|G_k(u)|\ge 1\}} g_2|G_k(u)|^{\mu-1}\exp(\lambda|G_k(u)|)dx\\&\le e^\lambda\int_{\{|u|\ge k\}} g_2dx+\int_\Omega g_2|\Psi(G_k(u))|dx\\&\le e^\lambda\|g_2\|_r[\varphi(k)]^{1/r'}+\int_\Omega g_2|\Psi(G_k(u))|dx.\end{aligned}\tag{5.33}$$

From (5.32) and (5.33) it follows that

$$\int_\Omega g_2\Psi'(G_k(u))|h_k''(u)|dx\le 2\lambda t^2\int_\Omega g_2|\Psi(G_k(u))|dx+c_{15}\|g_2\|_r[\varphi(k)]^{1/r'}.\tag{5.34}$$

Similar to (5.34) we obtain the following estimate of the third integral in the right-hand side of inequality (5.31):

$$\int_\Omega \Psi'(G_k(u))|h_k''(u)|dx\le 2\lambda t^2\int_\Omega|\Psi(G_k(u))|dx+c_{15}[\varphi(k)].\tag{5.35}$$

Before estimating the first integral in the right-hand side of inequality (5.31), we remark that

$$\Psi'(G_k(s))<2\lambda e^\lambda k\varepsilon^{t(\mu-1)}\quad\text{if } k\le|s|<k(1+\varepsilon k^{-1/t}).\tag{5.36}$$

Indeed, let $k\le|s|<k(1+\varepsilon k^{-1/t})$. Then, by (5.20), (5.23), (5.29), assertion (A4) and the inequality $k>1$, we have

$$\begin{aligned}\Psi'(G_k(s))&=|G_k(s)|^{\mu-1}(\lambda k+\lambda|G_k(s)|)\exp(\lambda|G_k(s)|)\\&<\varepsilon^{t(\mu-1)}e^{\lambda\varepsilon^t}(\lambda k+\lambda)<2\lambda e^\lambda k\varepsilon^{t(\mu-1)}.\end{aligned}$$

Hence, assertion (5.36) is true.

Next, it is clear that

$$\begin{aligned}\int_\Omega\Phi\Psi'(G_k(u))|h_k''(u)|dx&=\int_{\{k\le|u|<k(1+\varepsilon k^{-1/t})\}}\Phi\Psi'(G_k(u))|h_k''(u)|dx\\&\quad+\int_{\{k(1+\varepsilon k^{-1/t})\le|u|\le 2k\}}\Phi\Psi'(G_k(u))|h_k''(u)|dx.\end{aligned}\tag{5.37}$$

From assertions (A1), (5.36) and (5.4) it follows that

$$\int_{\{k\le|u|<k(1+\varepsilon k^{-1/t})\}}\Phi\Psi'(G_k(u))|h_k''(u)|dx\le\frac{2\lambda e^\lambda C_2t^2}{k^{1-2/t}}\varepsilon^{t\mu-2},\tag{5.38}$$

and by assertion (A2), we have

$$\begin{aligned}&\int_{\{k(1+\varepsilon k^{-1/t})\le|u|\le 2k\}}\Phi\Psi'(G_k(u))|h_k''(u)|dx\\&\le\frac{t}{k^{1-1/t}\varepsilon}\int_\Omega\Phi\Psi'(G_k(u))G_k'(u)dx.\end{aligned}\tag{5.39}$$

From (5.37)–(5.39) it follows that

$$\int_\Omega \Phi\Psi'(G_k(u))|h_k''(u)|dx \le \frac{2\lambda e^\lambda C_2 t^2}{k^{1-2/t}}\varepsilon^{t\mu-2} + \frac{t}{k^{1-1/t}\varepsilon}\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx. \tag{5.40}$$

In turn, using (5.31), (5.34), (5.35) and (5.40) and taking into account (5.5), (5.8), (5.9), (5.12), (5.14), (5.28) and (5.29), and the inequality $\mu > 1$, we obtain (5.30).

Step 4. Let us estimate the integral I_k''. We shall establish the inequality

$$\begin{aligned} I_k'' \le \frac{c_3}{12}\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx + \frac{1}{2}\int_\Omega g_2|\Psi(G_k(u))|dx \\ + \frac{1}{2}\varepsilon^{-2qp/(q-2p)}\int_\Omega |\Psi(G_k(u))|dx + c_{16}[\varphi(k)]^{1/r'}. \end{aligned} \tag{5.41}$$

Similar to (5.31), we have

$$\begin{aligned} I_k'' \le n(c_2+n)\varepsilon^2\int_\Omega \Phi|\Psi''(G_k(u))|(G_k'(u))^2dx + n\varepsilon^2\int_\Omega g_2|\Psi''(G_k(u))|(G_k'(u))^2dx \\ + n^3\varepsilon^{2-2qp/(q-2p)}\int_\Omega |\Psi''(G_k(u))|(G_k'(u))^2dx. \end{aligned} \tag{5.42}$$

Let us estimate the first integral in the right-hand side of inequality (5.42). By (5.23), (5.24) and (5.18), for every $s \in \mathbb{R}$,

$$|\Psi''(G_k(s))|(G_k'(s))^2 \le \mu^2|G_k(s)|^{\mu-2}\exp(\lambda|G_k(s)|)(G_k'(s))^2 + 2\lambda\Psi'(G_k(s))G_k'(s).$$

From this it follows that

$$\begin{aligned} \int_\Omega \Phi|\Psi''(G_k(u))|(G_k'(u))^2dx \le \int_\Omega \Phi\mu^2|G_k(u)|^{\mu-2}\exp(\lambda|G_k(u)|)(G_k'(u))^2dx \\ + 2\lambda\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx. \end{aligned} \tag{5.43}$$

Clearly,

$$\begin{aligned} &\int_\Omega \Phi\mu^2|G_k(u)|^{\mu-2}\exp(\lambda|G_k(u)|)(G_k'(u))^2dx \\ &= \int_{\{k\le|u|<k(1+\varepsilon k^{-1/(t-2)})\}} \Phi\mu^2|G_k(u)|^{\mu-2}\exp(\lambda|G_k(u)|)(G_k'(u))^2dx \\ &\quad + \int_{\{|u|\ge k(1+\varepsilon k^{-1/(t-2)})\}} \Phi\mu^2|G_k(u)|^{\mu-2}\exp(\lambda|G_k(u)|)(G_k'(u))^2dx. \end{aligned} \tag{5.44}$$

Now, observe that the following assertions hold:

(A8) if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $k \le |s| \le k(1+\varepsilon k^{-1/(t-2)})$, then

$$\mu^2|G_k(s)|^{\mu-2}\exp(\lambda|G_k(s)|)(G_k'(s))^2 \le (\lambda t)^2 e^\lambda \varepsilon^{\mu(t-2)};$$

(A9) if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $|s| \ge k(1+\varepsilon k^{-1/(t-2)})$, then

$$\mu^2|G_k(s)|^{\mu-2}G_k'(s)\exp(\lambda|G_k(s)|) \le \frac{\lambda t(t+1)k^{1/(t-2)}}{2\varepsilon}\Psi'(G_k(s)).$$

Indeed, let $s \in \mathbb{R}$, $k \le |s| \le k(1+\varepsilon k^{-1/(t-2)})$ and $y = (|s|-k)/k$. Using assertions (5.16), (5.17) and (A4), equality (5.20) and the inequalities $0 \le y \le \varepsilon k^{-1/(t-2)} < 1$, $k > 1$ and $\mu > 1$, we obtain

$$
\begin{aligned}
&\mu^2 |G_k(s)|^{\mu-2} \exp(\lambda |G_k(s)|)(G_k'(s))^2 \\
&= \lambda^2 (t-(t-1)y)^2 k^\mu y^{t\mu-2} \Big(1 - \frac{t-1}{t+1} y\Big)^{\mu-2} \exp(\lambda |G_k(s)|) \\
&\le (\lambda t)^2 (k y^{t-2})^\mu e^\lambda \\
&\le (\lambda t)^2 e^\lambda \varepsilon^{\mu(t-2)}.
\end{aligned}
$$

Consequently, assertion (A8) is true.

Now let $s \in \mathbb{R}$, $k(1+\varepsilon k^{-1/(t-2)}) \le |s| \le 2k$ and $y = (|s|-k)/k$. Using assertions (5.16) and (5.17), equalities (5.20) and (5.23) and the inequality $\varepsilon k^{-1/(t-2)} \le y \le 1$, we obtain

$$
\begin{aligned}
&\mu^2 |G_k(s)|^{\mu-2} G_k'(s) \exp(\lambda |G_k(s)|) \\
&= \lambda \mu k^{\mu-1} y^{t(\mu-1)} \Big(1 - \frac{t-1}{t+1} y\Big)^{\mu-1} \frac{(t-(t-1)y)}{y(1-\frac{t-1}{t+1}y)} \exp(\lambda |G_k(s)|) \\
&\le \frac{\lambda t(t+1)}{2y} \mu |G_k(s)|^{\mu-1} \exp(\lambda |G_k(s)|) \\
&\le \frac{\lambda t(t+1) k^{1/(t-2)}}{2\varepsilon} \Psi'(G_k(s)).
\end{aligned}
$$

Finally, suppose that $s \in \mathbb{R}$ and $|s| \ge 2k$. Then, by the definitions of the functions h_k and G_k, we have

$$
|G_k(s)| = |s| - \frac{2kt}{t+1} \ge \frac{2k}{t+1}.
$$

Therefore,

$$
k \le (t+1)|G_k(s)|/2. \tag{5.45}
$$

Using (5.20), (5.23), (5.45), the equality $G_k'(s) = 1$ and taking into account the inequalities $t > 2$, $k > 1$ and (5.29), we obtain

$$
\begin{aligned}
&\mu^2 |G_k(s)|^{\mu-2} G_k'(s) \exp(\lambda |G_k(s)|) \\
&= \lambda \mu k |G_k(s)|^{\mu-2} \exp(\lambda |G_k(s)|) \\
&\le \frac{\lambda(t+1)}{2} \mu |G_k(s)|^{\mu-1} \exp(\lambda |G_k(s)|) \\
&\le \frac{\lambda t(t+1) k^{1/(t-2)}}{2\varepsilon} \Psi'(G_k(s)).
\end{aligned}
$$

Thus, assertion (A9) holds.

From assertion (A8) and (5.4) it follows that

$$
\begin{aligned}
&\int_{\{k \le |u| < k(1+\varepsilon k^{-1/(t-2)})\}} \Phi \mu^2 |G_k(u)|^{\mu-2} \exp(\lambda |G_k(u)|)(G_k'(u))^2 dx \\
&\le C_2 (\lambda t)^2 e^\lambda \varepsilon^{\mu(t-2)},
\end{aligned} \tag{5.46}
$$

and by assertion (A9), we have

$$\begin{aligned}&\int_{\{|u|\ge k(1+\varepsilon k^{-1/(t-2)})\}} \Phi\mu^2|G_k(u)|^{\mu-2}\exp(\lambda|G_k(u)|)(G_k'(u))^2dx\\&\le \frac{\lambda t(t+1)k^{1/(t-2)}}{2\varepsilon}\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx,\end{aligned} \tag{5.47}$$

From (5.9), (5.44), (5.46) and (5.47) we deduce the inequality

$$\begin{aligned}&\int_\Omega \Phi\mu^2|G_k(u)|^{\mu-2}\exp(\lambda|G_k(u)|)(G_k'(u))^2dx\\&\le c_{17}\varepsilon^{\mu(t-2)}+\frac{\lambda t(t+1)k^{1/(t-2)}}{2\varepsilon}\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx.\end{aligned}$$

In turn, from this inequality and (5.43) we obtain the following estimate for the first integral in the right-hand side of inequality (5.42),

$$\begin{aligned}&\int_\Omega \Phi|\Psi''(G_k(u))|(G_k'(u))^2dx\\&\le c_{17}\varepsilon^{\mu(t-2)}+\frac{\lambda t(t+1)k^{1/(t-2)}}{\varepsilon}\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx.\end{aligned} \tag{5.48}$$

Now, let us estimate the second integral in the right-hand side of inequality (5.42). By (5.21) and (5.24), for every $s\in\mathbb{R}$, we have

$$|\Psi''(s)|\le \mu^2|s|^{\mu-2}\exp(\lambda|s|)+2\lambda\mu|s|^{\mu-1}\exp(\lambda|s|)+\lambda^2|\Psi(s)|.$$

Hence,

$$\begin{aligned}&\int_\Omega g_2|\Psi''(G_k(u))|(G_k'(u))^2dx\\&\le \int_\Omega \mu^2|G_k(u)|^{\mu-2}(G_k'(u))^2\exp(\lambda|G_k(u)|)g_2dx\\&\quad+2\lambda\int_\Omega \mu|G_k(u)|^{\mu-1}(G_k'(u))^2\exp(\lambda|G_k(u)|)g_2dx\\&\quad+\lambda^2\int_\Omega g_2|\Psi(G_k(u))|(G_k'(u))^2dx.\end{aligned} \tag{5.49}$$

Clearly,

$$\begin{aligned}&\int_\Omega \mu^2|G_k(u)|^{\mu-2}(G_k'(u))^2\exp(\lambda|G_k(u)|)g_2dx\\&=\int_{\{k\le|u|<k(1+\varepsilon^{1/2}k^{-1/(t-2)})\}}\mu^2|G_k(u)|^{\mu-2}(G_k'(u))^2\exp(\lambda|G_k(u)|)g_2dx\\&\quad+\int_{\{|u|\ge k(1+\varepsilon^{1/2}k^{-1/(t-2)})\}}\mu^2|G_k(u)|^{\mu-2}(G_k'(u))^2\exp(\lambda|G_k(u)|)g_2dx.\end{aligned} \tag{5.50}$$

Using assertion (A8), the Hölder inequality and (5.28), we obtain

$$\begin{aligned}&\int_{\{k\le|u|<k(1+\varepsilon^{1/2}k^{-1/(t-2)})\}}\mu^2|G_k(u)|^{\mu-2}(G_k'(u))^2\exp(\lambda|G_k(u)|)g_2dx\\&\le(\lambda t)^2e^\lambda\varepsilon^{\mu(t-2)/2}\int_{\{|u|\ge k\}}g_2dx\\&\le(\lambda t)^2e^\lambda\varepsilon^{\mu(t-2)/2}\|g_2\|_r[\varphi(k)]^{1/r'}\le c_{18}[\varphi(k)]^{\mu/2+1/r'}.\end{aligned} \tag{5.51}$$

For estimating the second integral in the right-hand side of equality (5.50), at first we observe that the following assertion holds:

(A10) if $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $|s| \ge k(1+\varepsilon^{1/2}k^{-1/(t-2)})$, then

$$\mu^2 |G_k(s)|^{\mu-2}(G_k'(s))^2 \exp(\lambda |G_k(s)|) \le \frac{(\lambda t)^2 (t+1)^2 k^{2/(t-2)}}{4\varepsilon} |\Psi(G_k(s))|.$$

Indeed, let $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ and $k(1+\varepsilon^{1/2}k^{-1/(t-2)}) \le |s| \le 2k$. Then, setting $y = (|s|-k)/k$ and using (5.16), (5.17), (5.20) and the inequality $\varepsilon^{1/2}/k^{1/(t-2)} \le y \le 1$, we obtain

$$\begin{aligned}
\mu^2 |G_k(s)|^{\mu-2}(G_k'(s))^2 &= \lambda^2 k^\mu y^{t\mu} \Big(1 - \frac{t-1}{t+1} y\Big)^\mu \frac{(t-(t-1)y)^2}{y^2 (1-\frac{t-1}{t+1}y)^2} \\
&\le \lambda^2 |G_k(s)|^\mu \frac{t^2 k^{2/(t-2)}}{\varepsilon (2/(t+1))^2} \\
&= \frac{(\lambda t)^2 (t+1)^2 k^{2/(t-2)}}{4\varepsilon} |G_k(s)|^\mu.
\end{aligned}$$

From this and (5.21) it follows that assertion (A10) is valid.

Now, let $s \in \mathbb{R}$ and $|s| \ge 2k$. Then, by (5.20), (5.21), (5.45), the equality $G_k'(s) = 1$ and the inequalities $t > 2$, $k > 1$ and (5.29), we have

$$\begin{aligned}
\mu^2 |G_k(s)|^{\mu-2}(G_k'(s))^2 \exp(\lambda |G_k(s)|) &= \lambda^2 k^2 |G_k(s)|^{\mu-2} \exp(\lambda |G_k(s)|) \\
&\le \frac{\lambda^2 (t+1)^2}{4} |G_k(s)|^\mu \exp(\lambda |G_k(s)|) \\
&= \frac{(\lambda t)^2 (t+1)^2 k^{2/(t-2)}}{4\varepsilon} |\Psi(G_k(s))|.
\end{aligned}$$

Thus, assertion (A10) holds.

From assertion (A10) it follows that

$$\begin{aligned}
&\int_{\{|u| \ge k(1+\varepsilon^{1/2}k^{-1/(t-2)})\}} \mu^2 |G_k(u)|^{\mu-2}(G_k'(u))^2 \exp(\lambda |G_k(u)|) g_2 \, dx \\
&\le \frac{(\lambda t)^2 (t+1)^2 k^{2/(t-2)}}{4\varepsilon} \int_\Omega g_2 |\Psi(G_k(u))| \, dx.
\end{aligned} \tag{5.52}$$

Using (5.50)–(5.52), we obtain

$$\begin{aligned}
&\int_\Omega \mu^2 |G_k(u)|^{\mu-2}(G_k'(u))^2 \exp(\lambda |G_k(u)|) g_2 \, dx \\
&\le c_{18} [\varphi(k)]^{\mu/2+1/r'} + \frac{(\lambda t)^2 (t+1)^2 k^{2/(t-2)}}{4\varepsilon} \int_\Omega g_2 |\Psi(G_k(u))| \, dx.
\end{aligned} \tag{5.53}$$

Before estimating the second integral in the right-hand side of inequality (5.49), we note that for every $s \in \mathbb{R}$ the following inequality holds:

$$\mu |G_k(s)|^{\mu-1}(G_k'(s))^2 \exp(\lambda |G_k(s)|) \le \frac{\lambda t^2 (t+1)}{2} |\Psi(G_k(s))|. \tag{5.54}$$

Indeed, if $s \in \mathbb{R}$ and $|s| \le k$, then both sides of inequality (5.54) are equal zero and therefore, this inequality is true.

Now, let $k < |s| < 2k$ and $y = (|s| - k)/k$. Using (5.16), (5.17), (5.20) and the inequalities $0 < y < 1$ and $t > 2$, we obtain

$$\begin{aligned} \mu|G_k(s)|^{\mu-1}(G_k'(s))^2 &= \lambda k^\mu y^{t\mu+t-2}\Big(1 - \frac{t-1}{t+1}y\Big)^\mu \frac{(t-(t-1)y)^2}{(1-\frac{t-1}{t+1}y)} \\ &< \lambda k^\mu y^{t\mu}\Big(1 - \frac{t-1}{t+1}y\Big)^\mu \frac{t^2(t+1)}{2} \\ &= \frac{\lambda t^2(t+1)}{2}|G_k(s)|^\mu. \end{aligned}$$

These relations and (5.21) imply that inequality (5.54) holds.

Finally, let $|s| \geq 2k$. Then, by (5.20), (5.21), (5.45), the equality $G_k'(s) = 1$ and the inequality $t > 2$, we obtain

$$\begin{aligned} \mu|G_k(s)|^{\mu-1}(G_k'(s))^2\exp(\lambda|G_k(s)|) &= \lambda k|G_k(s)|^{\mu-1}\exp(\lambda|G_k(s)|) \\ &\leq \frac{\lambda(t+1)}{2}|G_k(s)|^\mu\exp(\lambda|G_k(s)|) \\ &= \frac{\lambda t^2(t+1)}{2}|\Psi(G_k(s))|. \end{aligned}$$

Therefore, inequality (5.54) holds. Thus, inequality (5.54) holds for every $s \in \mathbb{R}$.

From (5.54) it follows that

$$\int_\Omega \mu|G_k(u)|^{\mu-1}(G_k'(u))^2\exp(\lambda|G_k(u)|)g_2dx \leq \frac{\lambda t^2(t+1)}{2}\int_\Omega g_2|\Psi(G_k(u))|dx. \tag{5.55}$$

In turn, using (5.49), (5.53), (5.55), (5.18), (5.10) and (5.29) along with the inequalities $t > 2$ and $k > 1$, we deduce that

$$\begin{aligned} &\int_\Omega g_2|\Psi''(G_k(u))|(G_k'(u))^2dx \\ &\leq c_{18}[\varphi(k)]^{\mu/2+1/r'} + \frac{3(\lambda t)^2(t+1)^2k^{2/(t-2)}}{4\varepsilon}\int_\Omega g_2|\Psi(G_k(u))|dx. \end{aligned} \tag{5.56}$$

Similar to (5.56) we have

$$\begin{aligned} &\int_\Omega |\Psi''(G_k(u))|(G_k'(u))^2dx \\ &\leq c_{19}[\varphi(k)]^{1+\mu/2} + \frac{3(\lambda t)^2(t+1)^2k^{2/(t-2)}}{4\varepsilon}\int_\Omega |\Psi(G_k(u))|dx. \end{aligned} \tag{5.57}$$

Now, using (5.42), (5.48), (5.56) and (5.57) and taking into account (5.5), (5.8), (5.12), (5.28) and (5.29), we obtain (5.41).

Step 5. Let us prove that for the third integral in the right-hand side of inequality (5.27) the following inequality holds:

$$\int_\Omega g_3\Psi'(G_k(u))G_k'(u)dx \leq c_{20}\varphi(k) + \frac{\lambda t(t+1)k^{1/(t-1)}}{\varepsilon}\int_\Omega g_3|\Psi(G_k(u))|dx. \tag{5.58}$$

In fact, by (5.18) and (5.23), we have

$$\begin{aligned} &\int_\Omega g_3\Psi'(G_k(u))G_k'(u)dx \\ &\leq \int_\Omega \mu|G_k(u)|^{\mu-1}G_k'(u)\exp(\lambda|G_k(u)|)g_3dx + \lambda\int_\Omega g_3|\Psi(G_k(u))|dx. \end{aligned} \tag{5.59}$$

It is clear that

$$
\begin{aligned}
&\int_\Omega \mu|G_k(u)|^{\mu-1}G_k'(u)\exp(\lambda|G_k(u)|)g_3dx\\
&=\int_{\{k\le|u|<k(1+\varepsilon k^{-1/(t-1)})\}}\mu|G_k(u)|^{\mu-1}G_k'(u)\exp(\lambda|G_k(u)|)g_3dx\\
&\quad+\int_{\{|u|\ge k(1+\varepsilon k^{-1/(t-1)})\}}\mu|G_k(u)|^{\mu-1}G_k'(u)\exp(\lambda|G_k(u)|)g_3dx.
\end{aligned}
\tag{5.60}
$$

Similar to assertions (A8) and (A10) we establish that the following assertions hold:

(A11) if $\varepsilon\in(0,1)$, $s\in\mathbb{R}$ and $k\le|s|\le k(1+\varepsilon/k^{1/(t-1)})$, then
$$
\mu|G_k(s)|^{\mu-1}G_k'(s)\exp(\lambda|G_k(s)|)\le\lambda e^\lambda t\varepsilon^{\mu(t-1)};
$$

(A12) if $\varepsilon\in(0,1)$, $s\in\mathbb{R}$ and $|s|\ge k(1+\varepsilon/k^{1/(t-1)})$, then
$$
\mu|G_k(s)|^{\mu-1}G_k'(s)\exp(\lambda|G_k(s)|)\le\frac{\lambda t(t+1)k^{1/(t-1)}}{2\varepsilon}|\Psi(G_k(s))|.
$$

Taking into account (5.28) and (5.29) and the inequalities $t>1$, $k>1$ and $\mu>1$, from (5.59), (5.60) and assertions (A11) and (A12) we deduce (5.58).

Step 6. Using (5.3), (5.12), (5.27)–(5.30), (5.41) and (5.58), and taking into account that $k\ge k_*$ and $t>2$, we obtain that

$$
\frac{c_3}{3}\int_\Omega\Phi\Psi'(G_k(u))G_k'(u)dx\le\varepsilon^{-2qp/(q-2p)}\int_\Omega\tilde F|\Psi(G_k(u))|dx+c_{21}[\varphi(k)]^{1/r'}.
\tag{5.61}
$$

For the integral in the right-hand side of this inequality we shall establish the estimate

$$
\begin{aligned}
&\int_\Omega\tilde F|\Psi(G_k(u))|dx\\
&\le c_{22}k^{(1-\theta)\mu}e^{\lambda k}[\varphi(k)]^{1/r'}\\
&\quad+\frac{c^q(\lambda(t+1))^{q-1}}{q^q}\|\tilde F\|_r k^{\theta(q-1)}[\varphi(k)]^{q/n-1/r}\int_\Omega\Phi\Psi'(G_k(u))G_k'(u)dx.
\end{aligned}
\tag{5.62}
$$

Using Hölder's inequality and the definition of the function Ψ, we obtain

$$
\begin{aligned}
&\int_\Omega\tilde F|\Psi(G_k(u))|dx\\
&=\int_{\{|G_k(u)|<k^{1-\theta}\}}\tilde F|\Psi(G_k(u))|dx+\int_{\{|G_k(u)|\ge k^{1-\theta}\}}\tilde F|\Psi(G_k(u))|dx\\
&\le\|\tilde F\|_r\cdot k^{(1-\theta)\mu}e^{\lambda k^{1-\theta}}[\varphi(k)]^{1/r'}+\int_{\{|G_k(u)|\ge k^{1-\theta}\}}\tilde F|\Psi(G_k(u))|dx.
\end{aligned}
\tag{5.63}
$$

To estimate the integral in the right-hand side of inequality (5.63), we define the function $w:\mathbb{R}\to\mathbb{R}$ by

$$
w(s)=\begin{cases}(|s|^{\mu/q}e^{\lambda|s|/q}-k^{(1-\theta)\mu/q}e^{\lambda k^{1-\theta}/q})\operatorname{sign}s & \text{if } |s|>k^{1-\theta},\\ 0 & \text{if } |s|\le k^{1-\theta}.\end{cases}
$$

Using the definitions of the functions Ψ and w and Hölder's inequality, we establish that

$$\begin{aligned}&\int_{\{|G_k(u)|\geq k^{1-\theta}\}} \tilde{F}|\Psi(G_k(u))|dx \\ &\leq 2^{q-1}\|\tilde{F}\|_r k^{(1-\theta)\mu} e^{\lambda k^{1-\theta}}[\varphi(k)]^{1/r'} + \int_\Omega \tilde{F}|w(G_k(u))|^q dx.\end{aligned} \tag{5.64}$$

Taking into account (5.2), (2.2) and (5.18) and using Hölder's inequality, we obtain

$$\begin{aligned}&\int_\Omega \tilde{F}|w(G_k(u))|^q dx \\ &\leq \|\tilde{F}\|_r \|w(G_k(u))\|^q_{qr'} \\ &\leq \|\tilde{F}\|_r \|w(G_k(u))\|^q_{q^*}[\varphi(k)]^{q/n-1/r} \\ &\leq \frac{c^q 2^{q-1}}{q^q}\|\tilde{F}\|_r[\varphi(k)]^{q/n-1/r} \\ &\quad \times \int_{\{|G_k(u)|\geq k^{1-\theta}\}} (\mu^q|G_k(u)|^{\mu-q} + \lambda^q|G_k(u)|^\mu)\exp(\lambda|G_k(u)|)G'_k(u)\Phi dx.\end{aligned} \tag{5.65}$$

To proceed estimating the integral in the left-hand side of (5.65), we observe that the following assertion holds:

If $s\in\mathbb{R}$ and $|G_k(s)|\geq k^{1-\theta}$, then

$$\mu^q|G_k(s)|^{\mu-q} \leq (\lambda(t+1)/2)^{q-1}k^{\theta(q-1)}\mu|G_k(s)|^{\mu-1}. \tag{5.66}$$

Indeed, let $s\in\mathbb{R}$, $|G_k(s)|\geq k^{1-\theta}$ and $k<|s|<2k$. Then, setting $y=(|s|-k)/k$ and taking into account the inequality $0<y<1$, from the inequality $|G_k(s)|\geq k^{1-\theta}$ and assertion (5.16) we deduce that $y^t>k^{-\theta}$. Using this inequality and assertion (5.16), we obtain

$$\mu^q|G_k(s)|^{\mu-q} = \frac{\lambda^{q-1}\mu|G_k(s)|^{\mu-1}}{y^{t(q-1)}(1-\frac{t-1}{t+1}y)^{q-1}} \leq (\lambda(t+1)/2)^{q-1}k^{\theta(q-1)}\mu|G_k(s)|^{\mu-1}.$$

Now, let $|s|\geq 2k$. Then, by (5.20), (5.45) and the inequality $k^{\theta(q-1)}\geq 1$, we have

$$\mu^q|G_k(s)|^{\mu-q} = \lambda^{q-1}\mu k^{q-1}|G_k(s)|^{\mu-q} \leq (\lambda(t+1)/2)^{q-1}k^{\theta(q-1)}\mu|G_k(s)|^{\mu-1}.$$

Thus, assertion (5.66) holds.

From (5.65) and assertion (5.66), taking into account the definition of the function Ψ and the inequalities $t>1$ and $k\geq 1$, we deduce that

$$\begin{aligned}&\int_\Omega \tilde{F}|w(G_k(u))|^q dx \\ &\leq \frac{c^q(\lambda(t+1))^{q-1}}{q^q}\|\tilde{F}\|_r k^{\theta(q-1)}[\varphi(k)]^{q/n-1/r}\int_\Omega \Phi\Psi'(G_k(u))G'_k(u)dx.\end{aligned} \tag{5.67}$$

In turn, using (5.63), (5.64) and (5.67) along with the inequalities $k>1$ and $\theta>0$, we obtain (5.62).

Inequalities (5.61) and (5.62) along with the inequalities $0<\varphi(k)<1$, $k>1$, $\theta<1$, (5.6), (5.8), (5.13) and (5.28) imply that

$$\frac{c_3}{6}\int_\Omega \Phi\Psi'(G_k(u))G'_k(u)dx \leq (c_{21}+c_{22})k^{(1-\theta)\mu}e^{\lambda k}[\varphi(k)]^{\vartheta+q/q^*}. \tag{5.68}$$

Step 7. Let us estimate from below the integral in the left-hand side of inequality (5.68). This will allow us to apply Lemma 3.3 and to obtain the conclusion of the theorem.

We fix $l \in (k, 2k]$. Using (2.2), (5.10), (5.18), (5.20) and (5.23) and the inequality $\mu \geq q$, we obtain

$$\begin{aligned} &\int_\Omega \Phi\Psi'(G_k(u))G_k'(u)dx \\ &\geq \frac{(q/2\lambda)^q}{k^q}\sum_{|\alpha|=1}\int_\Omega |D^\alpha(|G_k(u)|^{\mu/q+1}\operatorname{sign} G_k(u))|^q dx \\ &\geq \frac{(q/2\lambda c)^q}{k^q}\Big(\int_{\{|u|\geq l\}} |G_k(u)|^{(\mu/q+1)q^*}dx\Big)^{q/q^*}. \end{aligned} \tag{5.69}$$

From assertion (A3) it follows that

$$\int_{\{|u|\geq l\}} |G_k(u)|^{(\mu/q+1)q^*}dx \geq \Big(\frac{2}{t+1}\Big)^{(\mu/q+1)q^*}\frac{(l-k)^{t(\mu/q+1)q^*}}{k^{(t-1)(\mu/q+1)q^*}}\varphi(l). \tag{5.70}$$

From (5.68)–(5.70), taking into account the equality $\mu = \lambda k$, we deduce that

$$\varphi(l) \leq c_{23}\Big[\frac{e(t+1)}{2}\Big]^{\lambda q^* k/q}\frac{k^{\lambda q^*(t-\theta)k/q+tq^*}}{(l-k)^{\lambda q^* tk/q+tq^*}}[\varphi(k)]^{1+\vartheta q^*/q}.$$

This and the inequality $(e(t+1)k^{-\theta/2}/2)^{\lambda q^* k/q} \leq c_{24}$ allow us to conclude that the following assertion holds:

If $k_0 \leq k < l \leq 2k$, then

$$\varphi(l) \leq \frac{c_{25}k^{\lambda q^*(t-\theta/2)k/q+tq^*}}{(l-k)^{\lambda q^* tk/q+tq^*}}[\varphi(k)]^{1+\vartheta q^*/q}.$$

Using this assertion and Lemma 3.3, we establish inequality (2.16). The theorem is proved.

6. Proof of Theorem 2.2

Let $r > n/q$, let the functions g_2, g_3, g_4 and f belong to $L^r(\Omega)$. Let for every $i \in \mathbb{N}$, $T_i : \mathbb{R} \to \mathbb{R}$ be the function such that

$$T_i(s) = \begin{cases} s & \text{if } |s| \leq i, \\ i \operatorname{sign} s & \text{if } |s| > i. \end{cases}$$

Now, for every $i \in \mathbb{N}$ we define the function $B_i : \Omega \times \mathbb{R} \times \mathbb{R}^{n,2} \to \mathbb{R}$ by

$$B_i(x,s,\xi) = T_i(B(x,s,\xi)), \quad (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2}.$$

Obviously, for every $i \in \mathbb{N}$ and for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{n,2}$,

$$|B_i(x,s,\xi)| \leq i, \tag{6.1}$$

$$|B_i(x,s,\xi)| \leq c_4\Big\{\sum_{|\alpha|=1}|\xi_\alpha|^q + \sum_{|\alpha|=2}|\xi_\alpha|^p\Big\} + g_4(x). \tag{6.2}$$

From (2.2)–(2.5), (2.8), (2.9), (2.13) and (6.1) and the results of [9] on solvability of equations with pseudomonotone operators it follows that if $i \in \mathbb{N}$, then there exists a function $u_i \in \mathring{W}^{1,q}_{2,p}(\Omega)$ such that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_i) D^\alpha v + A_0(x, u_i)v + B_i(x, u_i, \nabla_2 u_i)v \Big\} dx = \int_\Omega f v dx. \quad (6.3)$$

Hence, on the basis of the inclusions g_2, g_3, $f \in L^r(\Omega)$ and $B_i(x, u_i, \nabla_2 u_i) \in L^\infty(\Omega)$ and a slight modification (due to the presence in (6.3) of the term A_0 satisfying conditions (2.7) and (2.8)) of the proof of assertion (iii) of [8, Theorem 1] we establish that for every $i \in \mathbb{N}$,

$$u_i \in L^\infty(\Omega).$$

Using this inclusion, inequality (6.2) and Theorems 2.4 and 2.5, we obtain that for every $i \in \mathbb{N}$,

$$\int_\Omega \Big(\sum_{|\alpha|=1} |D^\alpha u_i|^q + \sum_{|\alpha|=2} |D^\alpha u_i|^p \Big) dx \le C_2, \quad (6.4)$$

$$\|u_i\|_\infty \le C_1. \quad (6.5)$$

By (6.4), (2.2) and in view of the compactness of the embedding $\mathring{W}^{1,q}(\Omega)$ into $L^\lambda(\Omega)$ with $\lambda < q^*$, there exist an increasing sequence $\{i_j\} \subset \mathbb{N}$ and a function $u_0 \in \mathring{W}^{1,q}_{2,p}(\Omega)$ such that

$$\begin{gathered} u_{i_j} \to u_0 \quad \text{weakly in } \mathring{W}^{1,q}_{2,p}(\Omega), \\ u_{i_j} \to u_0 \quad \text{a. e. in } \Omega. \end{gathered} \quad (6.6)$$

From (6.5) and (6.6) we deduce estimate (2.14).

Using (2.8), (6.5) and (6.6) along with Dominated Convergence Theorem, we establish the following assertion:

For every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega A_0(x, u_{i_j}) v dx = \int_\Omega A_0(x, u_0) v dx. \quad (6.7)$$

Moreover, using arguments analogous to those contained in the proof of [15, Theorem 2.1], we establish the following assertions:

$$u_{i_j} \to u_0 \quad \text{strongly in } \mathring{W}^{1,q}_{2,p}(\Omega);$$

for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_{i_j}) D^\alpha v \Big\} dx = \int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_0) D^\alpha v \Big\} dx; \quad (6.8)$$

for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\lim_{j\to\infty} \int_\Omega B_{i_j}(x, u_{i_j}, \nabla_2 u_{i_j}) v \, dx = \int_\Omega B(x, u_0, \nabla_2 u_0) v \, dx. \quad (6.9)$$

From (6.3) and assertions (6.7)–(6.9) it follows that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha\in\Lambda} A_\alpha(x, \nabla_2 u_0) D^\alpha v + A_0(x, u_0)v + B(x, u_0, \nabla_2 u_0)v \Big\} dx = \int_\Omega f v dx.$$

The obtained properties of the function u_0 allow us to conclude that u_0 is a generalized solution of problem (2.10), (2.11). This completes the proof of the theorem.

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