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EXISTENCE OF PSEUDO ALMOST PERIODIC SOLUTIONS FOR A CLASS OF PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we first introduce a new class of pseudo almost periodic type functions and investigate some properties of pseudo almost periodic type functions; and then we discuss the existence of pseudo almost periodic solutions to the class of abstract partial functional differential equations $x'(t) = Ax(t) + f(t, x_t)$ with finite delay in a Banach space X.

1. Introduction

Let $(X, \|\cdot\|))$ be a Banach space. The main goal in this paper is to study the existence of pseudo almost periodic type solutions to the abstract partial functional differential equation

$$x'(t) = Ax(t) + f(t, x_t), \quad t \in \mathbb{R} \tag{1.1}$$

in X, where A is the infinitesimal generator of an exponential stable C_0-semigroup, f is some class of pseudo almost periodic type solution, and $x_t(s) = x(t + s)$, $s \in [-\delta, 0]$ with $\delta > 0$ be a fixed constant.

The study of the existence of periodic type solutions, almost periodic type solutions, pseudo almost periodic type solutions to equation (1.1) and its variants has been of great interest for many authors. We refer the reader to [1, 2, 3, 4, 9, 10, 11, 12, 13, 14, 15] and references therein for some of recent developments on these topics. In addition, we would like to note that in a recent work [18], in the sense of category, the "amount" of almost periodic functions (not periodic) is far more than the "amount" of continuous periodic functions. Thus, studying the existence of almost periodic solutions for differential equations is necessary.

The direct impetus of this paper comes from three sources. The first source is a paper by Cuevas and Hernández [1], in which the authors studied the existence and uniqueness of pseudo almost periodic solutions for (1.1) with a linear part dominated by a Hill-Yosida type operator with a non-dense domain. The second source is a paper by Diagana and Hernández [3], in which the authors investigate the existence and uniqueness of pseudo almost periodic solutions for the following

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neutral functional-differential equation:

$$\frac{d}{dt}[u(t) + f(t, u_t)] = Au(t) + g(t, u_t), \quad t \in \mathbb{R}. \tag{1.2}$$

The third source is a paper by Diagana [4], in which the author discuss the existence of almost automorphic solutions for (1.2) with S^p-almost automorphic coefficients.

Motivated by the above three papers, in this work, we discuss the existence of pseudo almost periodic solutions for (1.1) with S^p-pseudo almost periodic type coefficients. Throughout the rest of this paper, if there is no special statement, then $\mathbb{N}$ will be the set of positive integers, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers, $BC(\mathbb{R}, X)$ the Banach space of all bounded continuous functions from $\mathbb{R}$ to X with the supremum norm, and Y the Banach space $C([-\delta, 0]; X)$ with the supremum norm.

2. Pseudo almost periodic type functions

In this section, we will recall several important and interesting pseudo almost periodic type functions; and we will also introduce a new class of pseudo almost periodic type functions. Throughout this section, let Z and W be two arbitrary Banach spaces.

2.1. **Classical pseudo almost periodic functions.**

First, let us recall some notions about almost periodic functions and pseudo almost periodic functions (for more details, see [6]).

Definition 2.1. A set $E \subset \mathbb{R}$ is said to be relatively dense if there exists a number $l > 0$ such that

$$(a, a + l) \cap E \neq \emptyset$$

for every $a \in \mathbb{R}$.

Definition 2.2. A continuous function $f : \mathbb{R} \to Z$ is said to be almost periodic if for every $\varepsilon > 0$ there exists a relatively dense set $P(\varepsilon, f) \subset \mathbb{R}$ such that

$$\sup_{t \in \mathbb{R}} \|f(t + \tau) - f(t)\| < \varepsilon,$$

for every $\tau \in P(\varepsilon, f)$. We denote the set of all such functions by $AP(\mathbb{R}, Z)$ or $AP(Z)$.

Definition 2.3. A continuous function $f : \mathbb{R} \times W \to Z$ is called almost periodic in t uniformly for $x \in W$ if for every $\varepsilon > 0$ and every compact subset $K \subset W$, there exists a relatively dense set $P(\varepsilon, f, K) \subset \mathbb{R}$

$$\sup_{t \in \mathbb{R}} \|f(t + \tau, x) - f(t, x)\| < \varepsilon,$$

for every $\tau \in P(\varepsilon, f, K)$ and $x \in K$. We denote by $AP(\mathbb{R} \times W, Z)$ the set of all such functions.

Denote

$$PAP_0(Z) := \{f \in BC(\mathbb{R}, Z) : \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t)\| dt = 0\}.$$

Definition 2.4. A continuous function $f : \mathbb{R} \to Z$ is called pseudo almost periodic if it can be expressed as $f = g + h$, where $g \in AP(Z)$ and $h \in PAP_0(Z)$. The set of such functions will be denoted by $PAP(Z)$.

Definition 2.5. A continuous function $f : \mathbb{R} \times W \to Z$ is called pseudo almost periodic if

$$f(t,x) = g(t,x) + h(t,x), \quad t \in \mathbb{R},\ x \in W,$$

where $g \in AP(\mathbb{R} \times W, Z)$ and for each $x \in W$, $h(\cdot, x) \in PAP_0(Z)$. We denote by $PAP(\mathbb{R} \times W, Z)$ the set of all such functions.

Remark 2.6. Note that Definition 2.5 is slightly different from the notion used in many previous papers.

2.2. **Stepanov-like pseudo almost periodic functions.** In this subsection, if there is no special statement, we assume that $p \geq 1$. Next, let us recall some notions about Stepanov-like pseudo almost periodicity (for more details, see [5, 17]).

Definition 2.7. The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in Z, is defined by

$$f^b(t,s) := f(t+s).$$

Definition 2.8. The space $BS^p(Z)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in Z such that

$$\|f\|_{S^p} := \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \Big)^{1/p} < +\infty.$$

It is obvious that $L^p(\mathbb{R}; Z) \subset BS^p(Z) \subset L^p_{loc}(\mathbb{R}; Z)$ and $BS^p(Z) \subset BS^q(Z)$ whenever $p \geq q \geq 1$.

Definition 2.9. A function $f \in BS^p(Z)$ is said to be Stepanov almost periodic if $f^b \in AP\big(L^p(0,1;Z)\big)$; that is, for every $\varepsilon > 0$, there exists a relatively dense set $P(\varepsilon, f) \subset \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_0^1 \|f(t+s+\tau) - f(t+s)\|^p ds \Big)^{1/p} < \varepsilon,$$

for every $\tau \in P(\varepsilon, f)$. We denote the set of all such functions by $S^pAP(\mathbb{R}, X)$ or $S^pAP(X)$.

Remark 2.10. It is clear that $AP(X) \subset S^pAP(X) \subset S^qAP(X)$ for $p \geq q \geq 1$.

Definition 2.11. A function $f : \mathbb{R}\times W \to Z$, $(t,u) \mapsto f(t,u)$ with $f(\cdot,u) \in BS^p(Z)$ for every $u \in W$, is said to be Stepanov almost periodic in $t \in \mathbb{R}$ uniformly for $u \in W$, if for every $\varepsilon > 0$ and every compact set $K \subset W$, there exists a relatively dense set $P(\varepsilon, f, K) \subset \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_0^1 \|f(t+s+\tau, u) - f(t+s, u)\|^p ds \Big)^{1/p} < \varepsilon,$$

for every $\tau \in P(\varepsilon, f, K)$ and every $u \in K$. We denote by $S^pAP(\mathbb{R} \times W, Z)$ the set of all such functions.

It is also easy to show that $S^pAP(\mathbb{R} \times W, Z) \subset S^qAP(\mathbb{R} \times W, Z)$ for $p \geq q \geq 1$.

Next, we denote $S^pPAP_0(Z)$ be the set of all functions $f \in BS^p(Z)$ with $f^b \in PAP_0(L^p(0,1;Z))$; i.e.,

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} \Big(\int_0^1 \|f(t+s)\|^p ds \Big)^{1/p} dt = 0.$$

Definition 2.12. A function $f \in BS^p(Z)$ is said to be Stepanov-like pseudo almost periodic if it can be decomposed as $f = g + h$ with $g \in S^pAP(Z)$ and $h \in S^pPAP_0(Z)$. We denote the set of all such functions by $S^pPAP(\mathbb{R}, Z)$ or $S^pPAP(Z)$.

Definition 2.13. A function $F : \mathbb{R} \times W \to Z, (t, u) \mapsto F(t, u)$ with $F(\cdot, u) \in BS^p(Z)$ for each $u \in W$, is called Stepanov-like pseudo almost periodic in $t \in \mathbb{R}$ uniformly for $u \in W$ if it can be decomposed as $F = G + H$ with $G \in S^pAP(\mathbb{R} \times W, Z)$ and $H(\cdot, u) \in S^pPAP_0(Z)$ for each $u \in W$. We denote by $S^pPAP(\mathbb{R}\times W, Z)$ the set of all such functions.

2.3. **Pseudo almost periodic type functions of class η.** In this subsection, if there is no special statement, we assume that $\eta \ge 0$ and $p \ge 1$. First, let us recall the notion of pseudo almost periodic type functions of class η (for more details, see [3]). Denote

$$PAP_0(Z, \eta) := \{f \in BC(\mathbb{R}, Z) : \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} \Big(\sup_{s\in[-\eta,0]} \|f(t + s)\|\Big)dt = 0\}.$$

Definition 2.14. A function $f \in BC(\mathbb{R}, Z)$ is called pseudo almost periodic of class η if it can be expressed as $f = g + h$, where $g \in AP(Z)$ and $h \in PAP_0(Z, \eta)$. The set of such functions will be denoted by $PAP(Z, \eta)$.

Definition 2.15. A continuous function $f : \mathbb{R} \times W \to Z$ is called pseudo almost periodic of class η if

$$f(t, x) = g(t, x) + h(t, x), \quad t \in \mathbb{R},\ x \in W,$$

where $g \in AP(\mathbb{R} \times W, Z)$ and for each $x \in W$, $h(\cdot, x) \in PAP_0(Z, \eta)$. We denote by $PAP(\mathbb{R} \times W, Z, \eta)$ the set of all such functions.

Next, to study pseudo almost periodicity of equation (1.1), we introduce a new class of pseudo almost periodic type functions; i.e., *Stepanov-like pseudo almost periodic functions of class η*. We denote $S^pPAP_0(Z, \eta)$ be the set of all functions $f \in BS^p(Z)$ with $f^b \in PAP_0(L^p(0, 1; Z), \eta)$; i.e.,

$$\lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} \Big[\sup_{\theta\in[-\eta,0]} \Big(\int_0^1 \|f(t + \theta + s)\|^p ds\Big)^{1/p}\Big]dt = 0.$$

Definition 2.16. A function $f \in BS^p(Z)$ is called S^p-pseudo almost periodic of class η if it can be expressed as $f = g + h$, where $g \in S^pAP(Z)$ and $h \in S^pPAP_0(Z, \eta)$. The set of such functions will be denoted by $S^pPAP(Z, \eta)$.

Definition 2.17. A function $f : \mathbb{R} \times W \to Z$ is called S^p-pseudo almost periodic of class η if

$$f(t, x) = g(t, x) + h(t, x), \quad t \in \mathbb{R},\ x \in W,$$

where $g \in S^pAP(\mathbb{R} \times W, Z)$ and for each $x \in W$, $h(\cdot, x) \in S^pPAP_0(Z, \eta)$. We denote by $S^pPAP(\mathbb{R} \times W, Z, \eta)$ the set of all such functions.

2.4. **Some properties of pseudo almost periodic type functions.** In this subsection, we investigate some properties and relationships of the above pseudo almost periodic type functions.

Lemma 2.18. *Let $f \in BC(\mathbb{R}, Z)$. Then $f \in PAP_0(Z)$ if and only if for every $\varepsilon > 0$,*

$$\lim_{r\to+\infty} \frac{\operatorname{meas} M_{r,\varepsilon}(f)}{2r} = 0,$$

where $M_{r,\varepsilon}(f) := \{t \in [-r, r] : \|f(t)\| \geq \varepsilon\}$.

Proof. It can be deduced from some earlier results (e.g, see [7, Lemma 3.2]). □

Define $PAP_0(\mathbb{R}^+) := \{f \in PAP_0(\mathbb{R}) : f(t) \geq 0, \forall t \in \mathbb{R}\}$.

Lemma 2.19. *Let $\alpha > 0$. Then $f \in PAP_0(\mathbb{R}^+)$ if and only if $f^\alpha \in PAP_0(\mathbb{R}^+)$, where $f^\alpha(t) := [f(t)]^\alpha$.*

Proof. By using Lemma 2.18, we have $f \in PAP_0(\mathbb{R}^+)$ if and only if for every $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{\operatorname{meas}\{t \in [-r, r] : f(t) \geq \varepsilon\}}{2r} = 0,$$

which is equivalent to: for every $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{\operatorname{meas}\{t \in [-r, r] : f^\alpha(t) \geq \varepsilon\}}{2r} = 0;$$

i.e., $f^\alpha \in PAP_0(\mathbb{R}^+)$. □

Lemma 2.20. *Let $p \geq 1$. Then*

$$PAP_0(Z) \subsetneqq S^p PAP_0(Z),$$

and thus

$$PAP(Z) \subsetneqq S^p PAP(Z).$$

Proof. It suffices to show that $PAP_0(Z) \subsetneqq S^p PAP_0(Z)$. First, let us show that $PAP_0(Z) \subset S^p PAP_0(Z)$. Note that the proof of $PAP_0(Z) \subset S^p PAP_0(Z)$ is given in [3]. But, here we give a different proof.

Let $f \in PAP_0(Z)$. Then $\|f(\cdot)\| \in PAP_0(\mathbb{R})$. By Lemma 2.19, $\|f(\cdot)\|^p \in PAP_0(\mathbb{R})$. Then, by using Lebesgue's dominated convergence theorem, noting that $\|f(\cdot + s)\|^p \in PAP_0(\mathbb{R})$ for each $s \in [0, 1]$, we obtain

$$\int_0^1 \Big[\frac{1}{2r} \int_{-r}^r \|f(t + s)\|^p dt\Big] ds \to 0, \quad r \to +\infty;$$

i.e.,

$$\frac{1}{2r} \int_{-r}^r \Big[\int_0^1 \|f(t + s)\|^p ds\Big] dt \to 0, \quad r \to +\infty,$$

which means that $g \in PAP_0(\mathbb{R})$, where

$$g(t) = \int_0^1 \|f(t + s)\|^p ds, \quad t \in \mathbb{R}.$$

Again by Lemma 2.19, $g^{1/p} \in PAP_0(\mathbb{R})$; i.e.,

$$\frac{1}{2r} \int_{-r}^r \Big[\int_0^1 \|f(t + s)\|^p ds\Big]^{1/p} dt \to 0, \quad r \to +\infty,$$

which means that $f \in S^p PAP_0(Z)$. In addition, it is easy to see that $PAP_0(Z) \neq S^p PAP_0(Z)$. □

Lemma 2.21. *Let $p \geq 1$. Then*
$$PAP_0(Z,1) \subsetneqq S^pPAP_0(Z,1),$$
and thus
$$PAP(Z,1) \subsetneqq S^pPAP(Z,1).$$

Proof. It suffices to show that $PAP_0(Z,1) \subset S^pPAP_0(Z,1)$. Let $f \in PAP_0(Z,1)$. Then $g \in PAP_0(\mathbb{R})$, where
$$g(t) = \sup_{\theta\in[-1,0]} \|f(t+\theta)\|, \quad t \in \mathbb{R}.$$
It follows from Lemma 2.20 that $g \in S^pPAP_0(\mathbb{R})$. Thus, we have
$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\Big(\int_0^1 \|f(t+\theta+s)\|^p ds\Big)^{1/p}\Big]dt \\
&\leq \frac{1}{2r}\int_{-r}^{r}\Big[\int_0^1\Big(\sup_{\theta\in[-1,0]}\|f(t+\theta+s)\|\Big)^p ds\Big]^{1/p}dt \\
&\leq \frac{1}{2r}\int_{-r}^{r}\Big[\int_0^1 |g(t+s)|^p ds\Big]^{1/p}dt \to 0, \quad r \to +\infty,
\end{aligned}$$
which means that $f \in S^pPAP_0(Z,1)$. □

Next, we present an example of function, which belongs to $S^pPAP_0(\mathbb{R},1)$.

Example 2.22. Let f be defined on $\mathbb{R}$ by
$$f(t) = \begin{cases} 1 & t \in [2^n-1, 2^n],\ n \in \mathbb{N}, \\ 0 & \text{otherwise.}\end{cases}$$
Then
$$g(t) := \sup_{\theta\in[-1,0]} |f(t+\theta)| = \begin{cases} 1 & t \in [2^n-1, 2^n+1],\ n \in \mathbb{N}, \\ 0 & \text{otherwise.}\end{cases}$$
For each $r > 0$, denote $n_r = [\log_2 r] + 1$. Then
$$2^{n_r-1} \leq r \leq 2^{n_r}.$$
By a direct calculation, we obtain
$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\Big[\int_0^1 |g(t+s)|^p ds\Big]^{1/p}dt \\
&\leq \frac{1}{2r}\int_{-2^{n_r}}^{2^{n_r}}\Big[\int_0^1 |g(t+s)|^p ds\Big]^{1/p}dt \\
&\leq \frac{3n_r}{2r} \to 0, \quad r \to +\infty.
\end{aligned}$$
Then, it follows from the proof of Lemma 2.21 that $f \in S^pPAP_0(\mathbb{R},1)$. In addition, it is obvious that $f \notin PAP_0(\mathbb{R},1)$.

Lemma 2.23. *Let $\eta > 0$ and $p \geq 1$. The following properties hold:*

(a) *$PAP_0(Z,\eta)$ is translation invariant; i.e., for each $\alpha \in \mathbb{R}$, $f \in PAP_0(Z,\eta)$ implies that $f_\alpha \in PAP_0(Z,\eta)$, where $f_\alpha(t) = f(t+\alpha)$, $t \in \mathbb{R}$;*
(b) *$PAP_0(Z,\eta) = PAP_0(Z,1)$;*
(c) *$S^pPAP_0(Z,\eta)$ is translation invariant, $S^pPAP_0(Z,\eta) = S^pPAP_0(Z,1)$;*
(d) *$S^pPAP(Z,\eta)$ is translation invariant, $S^pPAP(Z,\eta) = S^pPAP_0(Z,1)$;*

(e) $S^pPAP_0(Z,1)$ *is a closed linear subspace of* $BS^p(Z)$.

Proof. Let $\alpha \in \mathbb{R}$ and $f \in PAP_0(Z,\eta)$. Noting that

$$\frac{1}{2r}\int_{-r}^{r}\Big(\sup_{s\in[-\eta,0]}\|f_\alpha(t+s)\|\Big)dt \le \frac{1}{2r}\int_{-r-\alpha}^{r+\alpha}\Big(\sup_{s\in[-\eta,0]}\|f(t+s)\|\Big)dt,$$

we know that $f_\alpha \in PAP_0(Z,\eta)$.

To prove (b), it suffices to show that $PAP_0(Z,1) \subset PAP_0(Z,n)$ for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and $f \in PAP_0(Z,1)$. In view of (a) and

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\Big(\sup_{s\in[-n,0]}\|f(t+s)\|\Big)dt\\
&\le \sum_{k=1}^{n}\frac{1}{2r}\int_{-r}^{r}\Big(\sup_{s\in[-k,-k+1]}\|f(t+s)\|\Big)dt\\
&= \sum_{k=1}^{n}\frac{1}{2r}\int_{-r}^{r}\Big(\sup_{s\in[-1,0]}\|f_{1-k}(t+s)\|\Big)dt,
\end{aligned}$$

we conclude that $f \in PAP_0(Z,n)$.

By noting the definition of $S^pPAP_0(Z,\eta)$, one can deduce (c) from (a) and (b). In addition, (d) follows from (c) and translation invariance of $S^pAP(Z)$.

It remains to show (e). Let $f_n \to f$ in $BS^p(Z)$ and $f_n \in S^pPAP_0(Z,1)$. Then

$$\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\Big(\int_0^1\|f(t+\theta+s)\|^p ds\Big)^{1/p}\Big]dt\\
&= \frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\|f(t+\theta+\cdot)\|_{L^p(0,1;Z)}\Big]dt\\
&\le \frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\|f_n(t+\theta+\cdot)\|_{L^p(0,1;Z)}\Big]dt\\
&\quad + \frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\|f_n(t+\theta+\cdot)-f(t+\theta+\cdot)\|_{L^p(0,1;Z)}\Big]dt\\
&\le \frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\|f_n(t+\theta+\cdot)\|_{L^p(0,1;Z)}\Big]dt + \|f_n-f\|_{BS^p(Z)},
\end{aligned}$$

which yields $f \in S^pPAP_0(Z,1)$. In addition, it is easy to show that $S^pPAP_0(Z,1)$ is a linear subspace of $BS^p(Z)$. This completes the proof. $\square$

Remark 2.24. It has been noted in [3] that $PAP_0(Z,1)$ is a closed subspace of $BC(\mathbb{R},Z)$ and $PAP(Z,1)$ is a Banach space under the supremum norm.

3. Existence of pseudo almost periodic solutions

In this section, we discuss the existence of pseudo almost periodic solutions to equation (1.1). For convenience, we first list some assumptions:

(A1) A generates a C_0-semigroup $T(t)$ in X satisfying $\|T(t)\| \le Me^{-\omega t}$ for all $t \ge 0$, where $M,\omega > 0$ are fixed constants.

(A2) $f \in S^1PAP(\mathbb{R}\times Y, X, 1)$ with $f = g + h$, where $g \in S^1AP(\mathbb{R}\times Y, X)$ and $h \in S^1PAP_0(\mathbb{R}\times Y, X, 1)$; Moreover, there exists a constant $L > 0$ such that for all $t \in \mathbb{R}$ and $u, v \in Y$,
$$\|f(t,u) - f(t,v)\| \le L\|u - v\|, \quad \|g(t,u) - g(t,v)\| \le L\|u - v\|.$$

If (A1) holds, one can define the mild solution of equation (1.1) as follows:

Definition 3.1. A continuous function $u : \mathbb{R} \to X$ is called a mild solution of (1.1) if
$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau, u_\tau)d\tau, \quad t \ge s.$$

Before establishing our main results, we need to prove two important lemmas.

Lemma 3.2. *Assume that $x \in PAP(X, 1)$ and (A2) holds. Then $t \mapsto f(t, x_t)$ belongs to $S^1PAP(X, 1)$.*

Proof. Let $x = y + z$, where $y \in AP(X)$, $z \in PAP_0(X, 1)$. It follows from the proof of [3, Theorem 3.3] that $t \mapsto y_t$ belongs to $AP(Y)$, $t \mapsto z_t$ belongs to $PAP_0(Y, 1)$, and $x_t = y_t + z_t$, $t \in \mathbb{R}$. Let
$$I(t) := g(t, y_t), \quad J(t) := f(t, x_t) - f(t, y_t), \quad H(t) := h(t, y_t).$$
Then $f(t, x_t) = I(t) + J(t) + H(t)$. Next, we give the proof in three steps.

Step 1. $I \in S^1AP(X)$.

First, it is easy to prove that $I \in BS^1(X)$. Let $K = \overline{\{y_t : t \in \mathbb{R}\}}$. Then K is a compact subset of Y. Thus, for each $\varepsilon > 0$, there exists $u_1, \dots, u_k \in K$ such that
$$K \subset \cup_{i=1}^k B(u_i, \varepsilon),$$
where k is a positive integer dependent on ε. Denote
$$i(t) = \min\{i = 1, 2, \dots, k : \|y_t - u_i\| < \varepsilon\}, \quad t \in \mathbb{R}.$$
In addition, since $t \mapsto y_t$ belongs to $AP(Y)$ and $g(\cdot, u_i) \in S^1AP(X)$, $i = 1, 2, \dots, k$, for the above $\varepsilon > 0$, there exists a relatively dense set $P(\varepsilon) \subset \mathbb{R}$ such that
$$\int_0^1 \|g(t+s+\tau, u_i) - g(t+s, u_i)\|ds < \frac{\varepsilon}{k}, \quad i = 1, 2, \dots, k, \tag{3.1}$$
and
$$\|y_{t+\tau} - y_t\| < \varepsilon \tag{3.2}$$
for all $\tau \in P(\varepsilon)$ and $t \in \mathbb{R}$. Now, by using (3.1), (3.2), and $\|g(t,u) - g(t,v)\| \le L\|u - v\|$, we obtain
$$\begin{aligned}
&\int_0^1 \|I(t+s+\tau) - I(t+s)\|ds \\
&= \int_0^1 \|g(t+s+\tau, y_{t+\tau+s}) - g(t+s, y_{t+s})\|ds \\
&\le \int_0^1 \|g(t+s+\tau, y_{t+\tau+s}) - g(t+s+\tau, y_{t+s})\|ds \\
&\quad + \int_0^1 \|g(t+s+\tau, y_{t+s}) - g(t+s, y_{t+s})\|ds \\
&\le L\varepsilon + \int_0^1 \|g(t+s+\tau, y_{t+s}) - g(t+s, y_{t+s})\|ds
\end{aligned}$$

$$
\begin{aligned}
&\le L\varepsilon + \int_0^1 \|g(t+s+\tau, y_{t+s}) - g(t+s+\tau, u_{i(t+s)})\|ds \\
&\quad + \int_0^1 \|g(t+s+\tau, u_{i(t+s)}) - g(t+s, u_{i(t+s)})\|ds \\
&\quad + \int_0^1 \|g(t+s, u_{i(t+s)}) - g(t+s, y_{t+s})\|ds \\
&\le 3L\varepsilon + \int_0^1 \sum_{i=1}^k \|g(t+s+\tau, u_i) - g(t+s, u_i)\|ds \\
&= 3L\varepsilon + \sum_{i=1}^k \int_0^1 \|g(t+s+\tau, u_i) - g(t+s, u_i)\|ds \le (3L+1)\varepsilon
\end{aligned}
$$

for all $\tau \in P(\varepsilon)$ and $t \in \mathbb{R}$.

Step 2. $J \in S^1PAP_0(X,1)$. Noting that

$$
\|J(t)\| = \|f(t,x_t) - f(t,y_t)\| \le L\|z_t\|,
$$

we conclude that J is bounded, and thus $J \in BS^1(X)$. On the other hand, we have

$$
\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[-1,0]} \|J^b(t+\theta)\|dt \\
&= \frac{1}{2r}\int_{-r}^{r} \Big[\sup_{\theta\in[-1,0]} \int_0^1 \|J(t+\theta+s)\|ds\Big]dt \\
&= \frac{L}{2r}\int_{-r}^{r} \Big[\sup_{\theta\in[-1,0]} \int_0^1 \|z_{t+\theta+s}\|ds\Big]dt \\
&= \frac{L}{2r}\int_{-r}^{r} \Big[\sup_{\theta\in[-1,0]} \int_0^1 \sup_{\alpha\in[-\delta,0]} \|z(t+\theta+s+\alpha)\|ds\Big]dt \\
&\le \frac{L}{2r}\int_{-r}^{r} \Big[\int_0^1 \sup_{\theta\in[-1-\delta,0]} \|z(t+\theta+s)\|ds\Big]dt \\
&= L\cdot \int_0^1 \Big[\frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[-1-\delta,0]} \|z(t+\theta+s)\|dt\Big]ds,
\end{aligned}
$$

combining this with the fact that $z(\cdot+s) \in PAP_0(X,1) = PAP_0(X,1+\delta)$ for each $s \in [0,1]$ and z is bounded, by using the Lebesgue's dominated convergence theorem, we obtain that

$$
\lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} \sup_{\theta\in[-1,0]} \|J^b(t+\theta)\|dt = 0.
$$

Step 3. $H \in S^1PAP_0(X,1)$. Take an arbitrary $\varepsilon > 0$. Let K and u_i, $i = 1,2,\dots,k$, be as in the proof of Step 1. Then, for all $t \in \mathbb{R}$, there holds

$$
\begin{aligned}
\|H(t)\| = \|h(t,y_t)\| &= \|h(t,y_t) - h(t,u_{i(t)})\| + \|h(t,u_{i(t)})\| \\
&\le 2L\varepsilon + \sum_{i=1}^k \|h(t,u_i)\|,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[-1,0]}\|H^b(t+\theta)\|dt\\
&=\frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\int_0^1\|H(t+\theta+s)\|ds\Big]dt\\
&\le 2L\varepsilon+\frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\int_0^1\sum_{i=1}^{k}\|h(t+\theta+s,u_i)\|ds\Big]dt\\
&\le 2L\varepsilon+\sum_{i=1}^{k}\frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\int_0^1\|h(t+\theta+s,u_i)\|ds\Big]dt.
\end{aligned}
\tag{3.3}
$$

On the other hand, since $h\in S^1PAP_0(\mathbb{R}\times Y,X,1)$, $h(\cdot,u_i)\in S^1PAP_0(X,1)$; i.e.,

$$
\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\Big[\sup_{\theta\in[-1,0]}\int_0^1\|h(t+\theta+s,u_i)\|ds\Big]dt=0,
$$

which and (3.3) yields

$$
\limsup_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[-1,0]}\|H^b(t+\theta)\|dt\le 2L\varepsilon.
$$

Then, by the arbitrariness of ε, we conclude that

$$
\lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\sup_{\theta\in[-1,0]}\|H^b(t+\theta)\|dt=0.
$$

In addition, it is also easy to see that $H\in BS^1(X)$. This completes the proof. □

Remark 3.3. It is needed to note that we establish a composition theorem of S^p-almost periodic functions in Step 1 of Lemma 3.2. In fact, in some previous work (cf. [8, 16]), we established several composition theorems of S^p-almost periodic functions. However, in the related results of [8, 16], f is assumed to be S^p-almost periodic with $p>1$.

Lemma 3.4. *Let $f\in S^1PAP(X,1)$ and $T(t)$ be a C_0-semigroup on X satisfying $\|T(t)\|\le Me^{-\omega t}$ for all $t\ge 0$, where $M,\omega>0$ are fixed constants. Then*

$$
t\mapsto\int_{-\infty}^{t}T(t-s)f(s)ds
$$

belongs to $PAP(X,1)$.

Proof. Let $f=g+h$, where $g\in S^1AP(X)$ and $h\in S^1PAP_0(X,1)$. Denote

$$
\Phi(t)=\int_{-\infty}^{t}T(t-s)g(s)ds,\quad \Psi(t)=\int_{-\infty}^{t}T(t-s)h(s)ds,\quad t\in\mathbb{R}.
$$

We will give the proof by two steps.

Step 1. $\Phi\in AP(X)$. Let

$$
\Phi_k(t)=\int_{k-1}^{k}T(s)g(t-s)ds,\quad k\in\mathbb{N},\ t\in\mathbb{R}.
$$

Then $\Phi(t) = \sum_{k=1}^{\infty} \Phi_k(t)$, and $\sum_{k=1}^{\infty} \Phi_k(t)$ is uniformly convergent on $\mathbb{R}$ since

$$\|\Phi_k(t)\| \le Me^{-\omega(k-1)} \int_{k-1}^{k} \|g(t-s)\| ds \le Me^{-\omega(k-1)} \|g\|_{S^1}.$$

On the other hand, noting that $g \in S^1AP(X)$ and

$$\begin{aligned}\|\Phi_k(t_1) - \Phi_k(t_2)\| &\le M \int_{k-1}^{k} \|g(t_1 - s) - g(t_2 - s)\| \\ &= M \int_0^1 \|g(t_1 - k + s) - g(t_2 - k + s)\|\end{aligned}$$

for all $t_1, t_2 \in \mathbb{R}$, we deduce that $\Phi_k \in AP(X)$ for each $k \in \mathbb{N}$. Thus, $\Phi \in AP(X)$ since $AP(X)$ is a closed subspace of $BC(\mathbb{R}, X)$.

Step 2. $\Psi \in PAP_0(X, 1)$. Similar to step 1, we denote

$$\Psi_k(t) = \int_{k-1}^{k} T(s)h(t-s)ds, \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

Then $\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t)$, and $\sum_{k=1}^{\infty} \Psi_k(t)$ is uniformly convergent on $\mathbb{R}$. Since $PAP_0(X, 1)$ is a closed subspace of $BC(\mathbb{R}, X)$, it remains to show that $\Psi_k \in PAP_0(X, 1)$ for each $k \in \mathbb{N}$. Now, fix $k \in \mathbb{N}$. We have

$$\begin{aligned}&\frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [-1,0]} \|\Psi_k(t + \theta)\| dt \\ &= \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [-1,0]} \Big\| \int_{k-1}^{k} T(s)h(t + \theta - s)ds \Big\| dt \\ &\le \frac{M}{2r} \int_{-r}^{r} \Big[\sup_{\theta \in [-1,0]} \int_{k-1}^{k} \|h(t + \theta - s)\| ds \Big] dt \\ &= \frac{M}{2r} \int_{-r}^{r} \Big[\sup_{\theta \in [-1,0]} \int_0^1 \|h(t - k + \theta + s)\| ds \Big] dt \\ &= \frac{M}{2r} \int_{-r}^{r} \sup_{\theta \in [-1,0]} \|h^b(t - k + \theta)\| dt.\end{aligned} \tag{3.4}$$

On the other hand, since $h \in S^1PAP_0(X, 1)$, $h^b \in PAP_0(L^1(0, 1; X), 1)$. Then, by the translation invariance of $PAP_0(L^1(0, 1; X), 1)$, we know that $h^b(\cdot - k) \in PAP_0(L^1(0, 1; X), 1)$, which and (3.4) yields that

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \sup_{\theta \in [-1,0]} \|\Psi_k(t + \theta)\| dt = 0;$$

i.e., $\Psi_k \in PAP_0(X, 1)$. This completes the proof. $\square$

Now, we are ready to present our main result.

Theorem 3.5. *Assume that* (A1)–(A2) *hold. Then* (1.1) *has a unique pseudo almost periodic mild solution in* $PAP(X, 1)$ *provided that* $L_f < \frac{\omega}{M}$, *where*

$$L_f := \sup_{t \in \mathbb{R}} \sup_{u, v \in Y, u \neq v} \frac{\|f(t, u) - f(t, v)\|}{\|u - v\|}.$$

Proof. Let
$$(\mathfrak{F}x)(t) = \int_{-\infty}^{t} T(t-s)f(s,x_s)ds, \quad t \in \mathbb{R},\ x \in PAP(X,1).$$
By Lemma 3.2, we obtain that $s \mapsto f(s,x_s)$ belongs to $S^1PAP(X,1)$ for each $x \in PAP(X,1)$. Then, by Lemma 3.4, we conclude that $\mathfrak{F}x \in PAP(X,1)$ for each $x \in PAP(X,1)$, which means that $\mathfrak{F}$ maps $PAP(X,1)$ into $PAP(X,1)$.

On the other hand, for all $u,v \in PAP(X,1)$, by a direct calculation, one can get
$$\|\mathfrak{F}u - \mathfrak{F}v\| \le \frac{\omega L_f}{M}\|u-v\|.$$
Noting $L_f < \frac{\omega}{M}$ and $PAP(X,1)$ is a Banach space, there exists a unique fixed point $x^* \in PAP(X,1)$ of $\mathfrak{F}$. At last, it is not difficult to show that x^* is just the unique pseudo almost periodic mild solution in $PAP(X,1)$ of equation (1.1). □

Example 3.6. Consider the partial functional differential equation
$$u_t(t,x) = u_{xx}(t,x) + a(t)\int_{-1}^{0} \frac{e^s \sin[u(t+s,x)]}{3}ds,\ t \in \mathbb{R},\ x \in (0,\pi), \tag{3.5}$$
where $a(t) = b(t) + f(t)$, $f(t)$ is as in Example 2.22, and
$$b(t) = \begin{cases} \sin k + \sin \pi k & t \in [k, k+\frac{1}{2}],\ k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$
Let $X = L^2(0,\pi)$; $(Au)(x) = u''(x)$ for $x \in (0,\pi)$ and $u \in D(A)$, where
$$\begin{aligned} D(A) = \big\{&u \in C^1[0,1] : u' \text{ is absolutely continuous on } [0,1], \\ &u'' \in X, u(0) = u(\pi) = 0\big\}; \end{aligned}$$
and
$$f(t,\varphi)(x) = a(t)\int_{-1}^{0} \frac{e^s \sin[\varphi(s)(x)]}{3}ds, \quad t \in \mathbb{R},\ \varphi \in C([-1,0];X),\ x \in (0,\pi).$$
It is well-known that A generates a C_0 semigroup $T(t)$ satisfying
$$\|T(t)\| \le e^{-t}, \quad t \ge 0,$$
which means that (A1) holds with $M = \omega = 1$.

It is not difficult to show that $b \in S^1AP(\mathbb{R})$. Combining this with Example 2.22, we obtain $a \in S^1PAP(\mathbb{R},1)$, which yields that $f \in S^1PAP(\mathbb{R}\times C([-1,0];X),X,1)$. In addition, by a direct calculation, we obtain
$$\|f(t,\varphi) - f(t,\phi)\| \le \frac{\sqrt{2}}{2}\|\varphi - \phi\|$$
for all $\varphi, \phi \in C([-1,0];X)$. Then, we conclude that (A2) holds. Moreover, we have
$$L_f \le \frac{\sqrt{2}}{2} < 1.$$
Then, by using Theorem 3.5, equation (3.5) has a unique pseudo almost periodic mild solution in $PAP(X,1)$.

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