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EXISTENCE OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS ARISING FROM THE FRACTIONAL ADVECTION DISPERSION EQUATION

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Abstract. We study the existence of multiple solutions to the boundary value problem

$$\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big)+\lambda\nabla F(t,u(t))=0,\quad t\in[0,T],$$
$$u(0)=u(T)=0,$$

where $T>0$, $\lambda>0$ is a parameter, $0\le\beta<1$, ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are, respectively, the left and right Riemann-Liouville fractional integrals of order β, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is a given function. Our interest in the above system arises from studying the steady fractional advection dispersion equation. By applying variational methods, we obtain sufficient conditions under which the above equation has at least three solutions. Our results are new even for the special case when $\beta=0$. Examples are provided to illustrate the applicability of our results.

1. Introduction

In recent years, the subject of fractional calculus has gained considerable popularity and importance due mainly to its applications in numerous seemingly diverse and widespread fields of science and engineering. The monographs [15, 16, 17] are excellent sources for the theory and applications of fractional calculus. In this article, we study the existence of three solutions to fractional boundary value problems (BVPs) of the form

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big)+\lambda\nabla F(t,u(t))=0,\quad t\in[0,T],\\ u(0)=u(T)=0,\end{gathered}\tag{1.1}$$

where $T>0$, $\lambda>0$ is a parameter, $0\le\beta<1$, ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order β, respectively, $N\ge1$ is an integer, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is a given function such that $F(t,\mathbf{x})$ is measurable in t for each $\mathbf{x}=(x_1,\dots,x_N)\in\mathbb{R}^N$ and continuously differentiable in $\mathbf{x}$ for a.e. $t\in[0,T]$, $F(t,0,\dots,0)\equiv0$ on $[0,T]$, and $\nabla F(t,\mathbf{x})=(\partial F/\partial x_1,\dots,\partial F/\partial x_N)$ is

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the gradient of F at $\mathbf{x}$. By a solution of (1.1), we mean an absolutely continuous function $u : [0,T] \to \mathbb{R}^N$ such that $u(t)$ satisfies both equation for a.e. $t \in [0,T]$ and the boundary conditions in (1.1). We notice that when $\beta = 0$, problem (1.1) has the form

$$\begin{gathered} u''(t) + \lambda \nabla F(t, u(t)) = 0, \quad t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{1.2}$$

which has been extensively studied.

The equation in (1.1) is motivated by the steady fractional advection dispersion equation studied in [10],

$$-D\,a\,(p\,{}_0D_t^{-\beta} + q\,{}_tD_T^{-\beta})Du + b(t)Du + c(t)u = f, \tag{1.3}$$

where D represents a single spatial derivative, $0 \le p, q \le 1$ satisfying $p + q = 1$, $a > 0$ is a constant, and b, c, f are functions satisfying some suitable conditions. The interest in (1.3) arises from its application as a model for physical phenomena exhibiting anomalous diffusion; i.e., diffusion not accurately modeled by the usual advection dispersion equation. Anomalous diffusion has been used in modeling turbulent flow [8, 20], and chaotic dynamics of classical conservative systems [21]. The reader may find more background information and applications on (1.3) in [4, 10].

Remark 1.1. When $N = 1$, problem (1.1) reduces to the scalar BVP

$$\begin{gathered} \frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \lambda f(t, u(t)) = 0, \quad t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{1.4}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is such that $f(t,x)$ is measurable in t for each $x \in \mathbb{R}$ and continuous in x for a.e. $t \in [0,T]$.

It is clear that the equation in (1.4) is of the form of (1.3) with $D = d/dt$, $a = 1$, $p = q = 1/2$, $b(t) = c(t) = 0$, and $f = \lambda f(t,u)$.

We also notice that since (1.3) is the steady fractional advection dispersion equation, it has no dependence on the time variable and it just depends on the space variable t (here, the notation t stands for the space variable in (1.3)). Since the space we studied is one dimensional and has the form of an interval, say $[0,T]$, the boundary conditions in the space reduce to the conditions at the two endpoints $t = 0$ and $t = T$ of the interval. In our system, we study the Dirichlet type boundary conditions.

In recent years, the existence of solutions of various BVPs of fractional differential equations is under strong research. For a small sample of the work on this subject, we refer the reader to [1, 3, 9, 11, 12, 13, 14, 22]. We remark that most existing results on fractional BVPs were obtained by using various fixed point theorems and that few results were established by using variational methods. This is because that it is often very difficult to establish a suitable space and variational functional for fractional BVPs. As pointed out in [10, 14], these difficulties are mainly caused by the following properties of fractional differential operators: (i) fractional differential operators are not local operators, and (ii) the adjoint of a fractional differential operator is not the negative of itself.

Recently, in [10, 14] suitable fractional derivative spaces and variational structures for the system (1.1) were developed. Moreover, the existence of at least one solution for the system (1.1) was established in [14] by using the minimax methods in critical point theory. Our goal in this paper is to obtain some sufficient conditions to guarantee that the system (1.1) has at least three solutions. Our analysis is mainly based on a recent three critical points theorem appeared in [2, 7], see Lemma 4.1 below. This lemma and its various variations have been frequently used to obtain multiplicity theorems for nonlinear problems of variational nature. See, for example, [2, 5, 6, 7, 18, 19] and the references therein.

The rest of this article is organized as follows. Section 2 contains some preliminaries on fractional calculus, Section 3 contains the main results of this paper and two illustrative examples, and the proofs of the main results are presented in Section 4.

2. Preliminaries on fractional calculus

To make this paper self-contained, in this section, we recall some basic definitions and properties of the fractional calculus. The presentation here and more information on fractional calculus can be found in, for example, [15, 17].

Definition 2.1. Let f be a function defined on $[a,b]$ and $\gamma > 0$. The left and right Riemann-Liouville fractional integrals of order γ for the function f, denoted respectively by ${}_aD_t^{-\gamma}$ and ${}_tD_b^{-\gamma}$, are defined by

$$\begin{aligned} {}_aD_t^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)ds, \quad t \in [a,b], \\ {}_tD_b^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s)ds, \quad t \in [a,b], \end{aligned}$$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma > 0$ is the gamma function.

Remark 2.2. When $\gamma = n \in \mathbb{N}$, ${}_aD_t^{-\gamma} f(t)$ and ${}_tD_b^{-\gamma} f(t)$ coincide with the nth integrals of the form

$$\begin{aligned} {}_aD_t^{-n} f(t) &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)ds, \quad t \in [a,b], \\ {}_tD_b^{-n} f(t) &= \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} f(s)ds, \quad t \in [a,b]. \end{aligned}$$

Definition 2.3. Let f be a function defined on $[a,b]$ and $\gamma > 0$. The left and right Riemann-Liouville fractional derivatives of order γ for the function f, denoted respectively by ${}_aD_t^{\gamma}$ and ${}_tD_b^{\gamma}$, are defined by

$${}_aD_t^{\gamma} f(t) = \frac{d^n}{dt^n} {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \Big(\int_a^t (t-s)^{n-\gamma-1} f(s)ds \Big),$$

and

$${}_tD_b^{\gamma} f(t) = (-1)^n \frac{d^n}{dt^n} {}_tD_b^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} (-1)^n \frac{d^n}{dt^n} \Big(\int_t^b (s-t)^{n-\gamma-1} f(s)ds \Big),$$

where $t \in [a,b]$, $n-1 \le \gamma < n$, and $n \in \mathbb{N}$.

Remark 2.4. If $\gamma = n-1$ for some $n \in \mathbb{N}$, then

$${}_aD_t^{\gamma} f(t) = f^{(n-1)}(t) \quad \text{and} \quad {}_tD_b^{\gamma} f(t) = (-1)^{n-1} f^{(n-1)}(t), \quad t \in [a,b],$$

where $f^{(n-1)}(t)$ is the usual derivative of order $n-1$.

Remark 2.5. Let $AC([a,b],\mathbb{R})$ be the space of real-valued functions $f(x)$ which are absolutely continuous on $[a,b]$, and for $n\in\mathbb{N}$, let $AC^n([a,b],\mathbb{R})$ be the space of real-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x)\in AC([a,b],\mathbb{R})$. By [15, Lemma 2.2], the Riemann-Liouville fractional derivatives ${}_aD_t^{\gamma}f(t)$ and ${}_tD_b^{\gamma}f(t)$ exist a.e. on $[a,b]$ if $f\in AC^n([a,b],\mathbb{R})$, where $n-1\le\gamma<n$.

Definition 2.6. Let $\gamma\ge 0$ and $n\in\mathbb{N}$.

(a) If $\gamma\in(n-1,n)$ and $f\in AC^n([a,b],\mathbb{R})$, then the left and right Caputo fractional derivatives of order γ for the function f, denoted respectively by ${}_a^cD_t^{\gamma}$ and ${}_t^cD_b^{\gamma}$, and are defined by
$$
{}_a^cD_t^{\gamma}f(t)={}_aD_t^{\gamma-n}f^{(n)}(t)=\frac{1}{\Gamma(n-\gamma)}\int_a^t(t-s)^{n-\gamma-1}f^{(n)}(s)ds
$$
and
$$
{}_t^cD_b^{\gamma}f(t)=(-1)^n{}_tD_b^{\gamma-n}f^{(n)}(t)=\frac{(-1)^n}{\Gamma(n-\gamma)}\int_t^b(s-t)^{n-\gamma-1}f^{(n)}(s)ds,
$$
for a.e. $t\in[a,b]$.

(b) If $\gamma=n-1$ and $f\in AC^{n-1}([a,b],\mathbb{R})$, then
$$
{}_a^cD_t^{n-1}f(t)=f^{(n-1)}(t)\quad\text{and}\quad {}_t^cD_b^{n-1}f(t)=(-1)^nf^{(n-1)}(t),\quad t\in[0,T].
$$
In particular, ${}_a^cD_t^{0}f(t)={}_t^cD_b^{0}f(t)=f(t)$, $t\in[a,b]$.

3. Main results

For $0\le\beta<1$ given in (1.1), let $\alpha=1-\beta/2\in(1/2,1]$ and define
$$
\rho_\alpha=\frac{16N}{T^2\Gamma^2(2-\alpha)}\Big(\frac{1}{3-2\alpha}\big(\frac{T}{4}\big)^{3-2\alpha}+\int_{T/4}^{3T/4}g^2(t)dt+\int_{3T/4}^{T}h^2(t)dt\Big), \tag{3.1}
$$
where
$$
g(t)=t^{1-\alpha}-(t-T/4)^{1-\alpha}, \tag{3.2}
$$
$$
h(t)=t^{1-\alpha}-(t-T/4)^{1-\alpha}-(t-3T/4)^{1-\alpha}. \tag{3.3}
$$

In the remainder of this article, for some $c,d,l,m,p\in\mathbb{R}$, let the bold letters $\mathbf{c}$, $\mathbf{d}$, $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{p}$ be the constant vectors in $\mathbb{R}^N$ defined by
$$
\mathbf{c}=(c,\dots,c),\quad \mathbf{d}=(d,\dots,d),\quad \mathbf{l}=(l,\dots,l),\quad \mathbf{m}=(m,\dots,m),\quad \mathbf{p}=(p,\dots,p),
$$
and any other bold letter, such as $\mathbf{x}$, is used to denote an arbitrary vector in $\mathbb{R}^N$. We now state the results of this paper.

Theorem 3.1. *Assume that there exist four positive constants c, d, l, and m, with*
$$
d<m \text{ and } c<\frac{T^{\alpha-1/2}\rho_\alpha^{1/2}d}{\Gamma(\alpha)(2\alpha-1)^{1/2}}<|\cos(\pi\alpha)|l<|\cos(\pi\alpha)|m, \tag{3.4}
$$
such that
$$
F(t,\mathbf{x})\ge 0\quad \text{for } (t,\mathbf{x})\in[0,T]\times[-m,m]^N, \tag{3.5}
$$
$$
\max_{|\mathbf{x}|\le c}F(t,\mathbf{x})\le F(t,\mathbf{c}),\quad \max_{|\mathbf{x}|\le l}F(t,\mathbf{x})\le F(t,\mathbf{l}),\quad \max_{|\mathbf{x}|\le m}F(t,\mathbf{x})\le F(t,\mathbf{m}), \tag{3.6}
$$
$$
\frac{\int_0^T F(t,\mathbf{c})dt}{c^2}<\frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4}F(t,\mathbf{d})dt-\int_0^T F(t,\mathbf{c})dt\Big), \tag{3.7}
$$

$$\frac{\int_0^T F(t,\mathbf{l})dt}{l^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big), \quad (3.8)$$

$$\frac{\int_0^T F(t,\mathbf{m})dt}{m^2-l^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big). \quad (3.9)$$

Then, for each $\lambda \in (\underline{\lambda}, \overline{\lambda})$*, the system* (1.1) *has at least three solutions* u_1*,* u_2*, and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)| < c$*,* $\max_{t\in[0,T]}|u_2(t)| < l$*, and* $\max_{t\in[0,T]}|u_3(t)| < m$*, where*

$$\underline{\lambda} = \frac{\rho_\alpha d^2}{2|\cos(\pi\alpha)|\big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\big)} \quad (3.10)$$

and

$$\begin{aligned}\overline{\lambda} = \min\Big\{&\frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|c^2}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{c})dt}, \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|l^2}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{l})dt}, \\ &\frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|(m^2-l^2)}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{m})dt}\Big\}.\end{aligned} \quad (3.11)$$

The following results are consequences of Theorem 3.1. In particular, Corollaries 3.2 and 3.4 give some conditions for the system (1.2) to have at least three solutions, and Corollary 3.3 provide some relatively simpler existence criteria for the system (1.1).

Corollary 3.2. *Assume that there exist four positive constants* c*,* d*,* l*, and* m*, with*

$$c < (8N)^{1/2}d < l < m,$$

such that (3.5) *and* (3.6) *hold, and*

$$\frac{\int_0^T F(t,\mathbf{c})dt}{c^2} < \frac{1}{8Nd^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big),$$

$$\frac{\int_0^T F(t,\mathbf{l})dt}{l^2} < \frac{1}{8Nd^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big),$$

$$\frac{\int_0^T F(t,\mathbf{m})dt}{m^2-l^2} < \frac{1}{8Nd^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big).$$

Then, for each $\lambda \in (\underline{\lambda}_1, \overline{\lambda}_1)$*, system* (1.2) *has at least three solutions* u_1*,* u_2*, and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)| < c$*,* $\max_{t\in[0,T]}|u_2(t)| < l$*, and* $\max_{t\in[0,T]}|u_3(t)| < m$*, where*

$$\underline{\lambda}_1 = \frac{4Nd^2}{T\big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\big)},$$

$$\overline{\lambda}_1 = \min\Big\{\frac{c^2}{2T\int_0^T F(t,\mathbf{c})dt}, \frac{l^2}{2T\int_0^T F(t,\mathbf{l})dt}, \frac{m^2-l^2}{2T\int_0^T F(t,\mathbf{m})dt}\Big\}.$$

Corollary 3.3. *Assume that there exist three positive constants* c*,* d*, and* p*, with*

$$d < p \text{ and } c < \frac{T^{\alpha-1/2}\rho_\alpha^{1/2} d}{\Gamma(\alpha)(2\alpha-1)^{1/2}} < \frac{|\cos(\pi\alpha)|p}{\sqrt{2}}, \quad (3.12)$$

such that

$$F(t,\mathbf{x}) \geq 0 \quad \textit{for } (t,\mathbf{x}) \in [0,T] \times [-p,p]^N, \tag{3.13}$$

$$\max_{|\mathbf{x}|\leq c} F(t,\mathbf{x}) \leq F(t,\mathbf{c}), \quad \max_{|\mathbf{x}|\leq p/\sqrt{2}} F(t,\mathbf{x}) \leq F(t,\mathbf{p}/\sqrt{2}), \quad \max_{|\mathbf{x}|\leq p} F(t,\mathbf{x}) \leq F(t,\mathbf{p}), \tag{3.14}$$

$$\frac{\int_0^T F(t,\mathbf{c})dt}{c^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))} \int_{T/4}^{3T/4} F(t,\mathbf{d})dt, \tag{3.15}$$

$$\frac{\int_0^T F(t,\mathbf{p})dt}{p^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{2T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))} \int_{T/4}^{3T/4} F(t,\mathbf{d})dt. \tag{3.16}$$

Then, for each $\lambda \in (\underline{\lambda}_2, \overline{\lambda}_2)$, *system* (1.1) *has at least three solutions* u_1, u_2, *and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)| < c$, $\max_{t\in[0,T]}|u_2(t)| < p/\sqrt{2}$, *and* $\max_{t\in[0,T]}|u_3(t)| < p$, *where*

$$\underline{\lambda}_2 = \frac{\rho_\alpha d^2(1+\cos^2(\pi\alpha))}{2|\cos(\pi\alpha)|\int_{T/4}^{3T/4} F(t,\mathbf{d})dt}, \tag{3.17}$$

$$\overline{\lambda}_2 = \min\Big\{\frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|c^2}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{c})dt}, \ \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|p^2}{4T^{2\alpha-1}\int_0^T F(t,\mathbf{p})dt}\Big\}. \tag{3.18}$$

Corollary 3.4. *Assume that there exist three positive constants* c, d, *and* p, *with*

$$c < (8N)^{1/2} d < \frac{p}{\sqrt{2}}, \tag{3.19}$$

such that (3.13) *and* (3.14) *hold, and*

$$\frac{\int_0^T F(t,\mathbf{c})dt}{c^2} < \frac{1}{16Nd^2}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt, \tag{3.20}$$

and

$$\frac{\int_0^T F(t,\mathbf{p})dt}{p^2} < \frac{1}{32Nd^2}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt. \tag{3.21}$$

Then, for each $\lambda \in (\underline{\lambda}_3, \overline{\lambda}_3)$, *system* (1.2) *has at least three solutions* u_1, u_2, *and* u_3 *such that* $\max_{t\in[0,T]}|u_1(t)| < c$, $\max_{t\in[0,T]}|u_2(t)| < p/\sqrt{2}$, *and* $\max_{t\in[0,T]}|u_3(t)| < p$, *where*

$$\underline{\lambda}_3 = \frac{8Nd^2}{T\int_{T/4}^{3T/4} F(t,\mathbf{d})dt},$$

$$\overline{\lambda}_3 = \min\Big\{\frac{c^2}{2T\int_0^T F(t,\mathbf{c})dt}, \ \frac{p^2}{4T\int_0^T F(t,\mathbf{p})dt}\Big\}.$$

Remark 3.5. We want to point out that when F does not depend on t, (3.20) and (3.21) reduce to

$$\frac{F(\mathbf{c})}{c^2} < \frac{F(\mathbf{d})}{32Nd^2} \quad \text{and} \quad \frac{F(\mathbf{p})}{p^2} < \frac{F(\mathbf{d})}{64Nd^2}, \tag{3.22}$$

and $\underline{\lambda}_3$ and $\overline{\lambda}_3$ become

$$\underline{\lambda}_3 = \frac{16Nd^2}{T^2F(\mathbf{d})} \quad \text{and} \quad \overline{\lambda}_3 = \min\Big\{\frac{c^2}{2T^2F(\mathbf{c})}, \ \frac{p^2}{4T^2F(\mathbf{p})}\Big\}. \tag{3.23}$$

Remark 3.6. We observe that, in our results, no asymptotic condition on F is needed and only local conditions on F are imposed to guarantee the existence of solutions. Moreover, in the conclusions of the above results, one of the three solutions may be trivial since $\nabla F(t,0,\dots,0)$ may be zero.

In the remainder of this section, we give two examples to illustrate the applicability of our results.

Example 3.7. Let $T>0$. For $(t,x,y)\in[0,T]\times\mathbb{R}^2$, let $F(t,x,y)=tG(x,y)$, where $G:\mathbb{R}^2\to\mathbb{R}$ satisfies that $G(-x,-y)=G(x,y)$, and that for $x\in[0,\infty)$ and $y\in\mathbb{R}$,

$$G(x,y)=\begin{cases} x^3+|y|^3, & 0\le x\le 1,\ 0\le|y|\le 1,\\ x^3+2|y|^{3/2}-1, & 0\le x\le 1,\ |y|>1,\\ 2x^{3/2}+|y|^3-1, & x>1,\ 0\le|y|\le 1,\\ 2x^{3/2}+2|y|^{3/2}-2, & x>1,\ |y|>1. \end{cases} \tag{3.24}$$

It is easy to verify that $F:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ is measurable in t for $(x,y)\in\mathbb{R}^2$ and continuously differentiable in x and y for $t\in[0,T]$, and $F(t,0,0)\equiv 0$ on $[0,T]$.

Let $0\le\beta<1$, $\alpha=1-\beta/2\in(1/2,1]$, ρ_α be defined by (3.1), and $u(t)=(u_1(t),u_2(t))$. We claim that for each

$$\lambda\in\Big(\frac{\rho_\alpha(1+\cos^2(\pi\alpha))}{T^2|\cos(\pi\alpha)|},\infty\Big),$$

the system

$$\begin{gathered} \frac{d}{dt}\Big(\frac{1}{2}{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big)+\lambda\nabla F(t,u(t))=0,\quad t\in[0,T],\\ u(0)=u(T)=0, \end{gathered} \tag{3.25}$$

has at least three solutions.

In fact, system (3.25) is a special case of system (1.1) with $N=2$. For $0<c<1$ and $p>1$, in view of (3.24), we have

$$\frac{\int_0^T F(t,c,c)dt}{c^2}=\frac{2c^3\int_0^T tdt}{c^2}=T^2c, \tag{3.26}$$

$$\frac{\int_0^T F(t,p,p)dt}{p^2}=\frac{(4p^{3/2}-2)\int_0^T tdt}{p^2}=\frac{T^2(2p^{3/2}-1)}{p^2}. \tag{3.27}$$

Choose $d=1$. Then,

$$\int_{T/4}^{3T/4}F(t,d,d)dt=2\int_{T/4}^{3T/4}tdt=\frac{1}{2}T^2. \tag{3.28}$$

By (3.26)–(3.28), we see that there exist $0<c^*<1$ and $p^*>1$ such that (3.12), (3.15), and (3.16) hold for any $0<c<c^*$ and $p>p^*$. Moreover, (3.13) and (3.14) hold for any $c,p>0$. Finally, note from (3.17) and (3.18) that

$$\underline{\lambda}_2=\frac{\rho_\alpha(1+\cos^2(\pi\alpha))}{T^2|\cos(\pi\alpha)|},$$
$$\overline{\lambda}_2\to\infty\quad\text{as } c\to 0^+ \text{ and } p\to\infty.$$

Then, the claim follows from Corollary 3.3.

Example 3.8. Let $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies that $F(-x, -y) = F(x, y)$, and that for $x \in [0, \infty)$ and $y \in \mathbb{R}$,

$$F(x,y) = \begin{cases} x^3, & 0 \le x \le 1,\ 0 \le |y| \le 1, \\ x^3 + 2|y|^{3/2} - 3|y| + 1, & 0 \le x \le 1,\ |y| > 1, \\ 2x^{3/2} - 1, & x > 1,\ 0 \le |y| \le 1, \\ 2x^{3/2} + 2|y|^{3/2} - 3|y|, & x > 1,\ |y| > 1. \end{cases} \tag{3.29}$$

It is easy to verify that $F : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable in x and y and $F(0, 0) = 0$.

Let $T > 0$ and $u(t) = (u_1(t), u_2(t))$. We claim that for each $\lambda \in (32/T^2, \infty)$, the system

$$\begin{gathered} u''(t) + \lambda \nabla F(u(t)) = 0, \quad t \in [0, T], \\ u(0) = u(T) = 0 \end{gathered} \tag{3.30}$$

has at least three solutions. In fact, the system (3.30) is a special case of the system (1.2) with $N = 2$. For $0 < c < 1$ and $p > 1$, from (3.29), we have

$$\frac{F(c,c)}{c^2} = \frac{c^3}{c^2} = c, \tag{3.31}$$

$$\frac{F(p,p)}{p^2} = \frac{4p^{3/2} - 3p}{p^2} = \frac{4p^{1/2} - 3}{p}. \tag{3.32}$$

Choose $d = 1$. Then

$$\frac{F(d,d)}{32Nd^2} = \frac{1}{64} \quad \text{and} \quad \frac{F(d,d)}{64Nd^2} = \frac{1}{128}. \tag{3.33}$$

By (3.31)–(3.33), we see that there exist $0 < c^* < 1$ and $p^* > 1$ such that (3.19) and (3.22) hold for any $0 < c < c^*$ and $p > p^*$. Moreover, (3.13) and (3.14) hold for any $c, p > 0$. Finally, note from (3.23) that

$$\underline{\lambda}_3 = \frac{32}{T^2} \quad \text{and} \quad \overline{\lambda}_3 \to \infty \quad \text{as } c \to 0^+ \text{ and } p \to \infty$$

Then, the claim follows from Corollary 3.4 and Remark 3.5.

Remark 3.9. As noted in Remark 3.6, one of the three solutions in the conclusions of the above examples may be trivial.

4. Proofs of the main results

Let X be nonempty set and $\Phi, \tilde{\Psi} : X \to \mathbb{R}$ be two functionals. For $r, r_1, r_2, r_3 \in \mathbb{R}$ with $r_1 < \sup_X \Phi$, $r_2 > \inf_X \Phi$, $r_2 > r_1$, and $r_3 > 0$, we define

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\left(\sup_{u \in \Phi^{-1}(-\infty, r)} \tilde{\Psi}(u)\right) - \tilde{\Psi}(u)}{r - \Phi(u)}, \tag{4.1}$$

$$\beta(r_1, r_2) := \inf_{u \in \Phi^{-1}(-\infty, r_1)} \sup_{v \in \Phi^{-1}[r_1, r_2)} \frac{\tilde{\Psi}(v) - \tilde{\Psi}(u)}{\Phi(v) - \Phi(u)}, \tag{4.2}$$

$$\gamma(r_2, r_3) := \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \tilde{\Psi}(u)}{r_3}, \tag{4.3}$$

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}. \tag{4.4}$$

The following lemma is fundamental in our proofs. The reader may refer to [2, Theorem 5.2] or [7, Theorem 3.3] for its proof.

Lemma 4.1. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\tilde{\Psi} : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

(a) $\inf_X \Phi = \Phi(0) = \tilde{\Psi}(0) = 0$,

(b) *for every u_1, u_2 satisfying $\tilde{\Psi}(u_1) \geq 0$ and $\tilde{\Psi}(u_2) \geq 0$, one has*

$$\inf_{t\in[0,1]} \tilde{\Psi}\left(tu_1 + (1-t)u_2\right) \geq 0.$$

Assume further that there exist three positive constants r_1, r_2, and r_3, with $r_1 < r_2$, such that

(c) $\alpha(r_1, r_2, r_3)) < \beta(r_1, r_2)$.

Then, for each $\lambda \in \big(1/\beta(r_1, r_2), 1/\alpha(r_1, r_2, r_3)\big)$, the functional $\Phi - \lambda\tilde{\Psi}$ has three distinct critical points u_1, u_2, and u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$, and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

Let E^α be the space of functions $u \in L^2([0, T], \mathbb{R}^N)$ having an α-order Caputo fractional derivatives ${}_0^cD_t^\alpha u \in L^2([0, T], \mathbb{R}^N)$ and $u(0) = u(T) = 0$. Then, by [14, Remark 3.1 (i) and Proposition 3.1], E^α is a reflexive and separable Banach space with the norm

$$\|u\| = \Big(\int_0^T |u(t)|^2 dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2} \quad \text{for any } u \in E^\alpha. \tag{4.5}$$

We recall the norms

$$\|u\|_{L^2} = \Big(\int_0^T |u(t)|^2 dt\Big)^{1/2} \quad \text{and} \quad \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

Lemmas 4.2 and 4.4 are special cases of [14, Propositions 3.2 and 3.3] with $p = 2$, respectively, and Lemma 4.5 corresponds to [14, Proposition 4.1].

Lemma 4.2. *For $u \in E^\alpha$, we have*

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha u\|_{L^2}, \tag{4.6}$$

$$\|u\|_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} \|{}_0^cD_t^\alpha u\|_{L^2}. \tag{4.7}$$

Remark 4.3. From (4.6), we see that the norm $\|\cdot\|$ in (4.5) is equivalent to the norm $\|\cdot\|_\alpha$ defined by

$$\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{1/2} \quad \text{for any } u \in E^\alpha. \tag{4.8}$$

Lemma 4.4. *Assume that a sequence $\{u_n\}$ converges weakly to u in E^α ($u_n \rightharpoonup u$). Then, $u_n \to u$ in $C([0, T], \mathbb{R}^N)$; i.e., $\|u_n - u\|_\infty \to 0$.*

Lemma 4.5. *For any $u \in E^\alpha$, we have*

$$|\cos(\pi\alpha)|\, \|u\|_\alpha^2 \leq -\int_0^T \big({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\big) dt \leq \frac{1}{\cos(\pi\alpha)} \|u\|_\alpha^2.$$

For $u \in E^\alpha$, let the functionals Φ and Ψ be defined as follows

$$\Phi(u) = -\frac{1}{2}\int_0^T \big({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\big)dt, \tag{4.9}$$

$$\Psi(u) = \int_0^T F(t, u(t))dt. \tag{4.10}$$

Then, by [14, Theorem 4.1], we see that Φ and Ψ are continuously differentiable, and for any $u, v \in E^\alpha$, we have

$$\langle \Phi'(u), v\rangle = -\frac{1}{2}\int_0^T \big[\big({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)\big) + \big({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t)\big)\big]dt, \tag{4.11}$$

$$\langle \Psi'(u), v\rangle = \int_0^T \big(\nabla F(t, u(t)), v(t)\big)dt. \tag{4.12}$$

Parts (a) and (b) of Lemma 4.6 below are taken from [14, Lemma 5.1 and Theorem 4.2], respectively.

Lemma 4.6. *We have that*

- (a) *The functional Φ is convex and continuous on E^α.*
- (b) *If $u \in E^\alpha$ is a critical point of the functional $\Phi - \lambda\Psi$, then u is a solution of BVP* (1.1)*.*

We are now in a position to prove our results.

Proof of Theorem 3.1. For any $x \in \mathbb{R}$, let $p(x) = \max\{-m, \ \min\{x, m\}\}$. For any $\mathbf{x} = (x_1, \dots, x_N) \in E^\alpha$, let $\tilde{F}(t, \mathbf{x}) = F(t, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}} = (p(x_1), \dots, p(x_N))$. Then, $\tilde{F}(t, \mathbf{x})$ is measurable in t for each $\mathbf{x} \in \mathbb{R}^N$ and continuously differentiable in $\mathbf{x}$ for a.e. $t \in [0, T]$, and $\tilde{F}(t, 0, \dots, 0) = 0$ on $[0, T]$. Note that $-m \le p(u_i) \le m$ for any $u = (u_1, \dots, u_N) \in E^\alpha$ and $i = 1, \dots, N$. Then, (3.5) implies that

$$\tilde{F}(t, u) \ge 0 \quad \text{for } (t, u) \in [0, T] \times E^\alpha. \tag{4.13}$$

Note that $d < m$ and $c < l < m$ by (3.4). Then, we have

$$\begin{gathered}\tilde{F}(t, \mathbf{x}) = F(t, \mathbf{x}) \quad \text{for } (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^N \text{ with } |\mathbf{x}| < m, \\ \tilde{F}(t, \mathbf{c}) = F(t, \mathbf{c}), \quad \tilde{F}(t, \mathbf{d}) = F(t, \mathbf{d}), \quad \tilde{F}(t, \mathbf{l}) = F(t, \mathbf{l}), \quad \tilde{F}(t, \mathbf{m}) = F(t, \mathbf{m}).\end{gathered} \tag{4.14}$$

Let the continuously differentiable functional Φ be given by (4.9) and the functional $\tilde{\Psi}$ be defined by

$$\tilde{\Psi}(u) = \int_0^T \tilde{F}(t, u(t))dt \quad \text{for } u \in E^\alpha. \tag{4.15}$$

Then, by Lemma 4.5 and (4.9), we have

$$\frac{1}{2}|\cos(\pi\alpha)|\, \|u\|_\alpha^2 \le \Phi(u) \le \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 \quad \text{for } u \in E^\alpha. \tag{4.16}$$

Moreover, $\tilde{\Psi}$ is continuously differentiable, and for any $u, v \in E^\alpha$, in view of (4.13), we have

$$\tilde{\Psi}(u) \ge 0 \quad \text{and} \quad \langle \tilde{\Psi}'(u), v\rangle = \int_0^T \big(\nabla\tilde{F}(t, u(t)), v(t)\big)dt. \tag{4.17}$$

In the following, we will apply Lemma 4.1 with $X = E^\alpha$ to the functionals Φ and $\tilde{\Psi}$.

We first show that some basic assumptions of Lemma 4.1 are satisfied. The convexity and coercivity of Φ follow from Lemma 4.6 (a) and (4.16), respectively. For any $u, v \in E^\alpha$, from Lemma 4.5 and (4.11),

$$\begin{aligned}
&\langle \Phi'(u) - \Phi'(v), u - v\rangle \\
&= -\frac{1}{2}\int_0^T \big[\big({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha(u(t)-v(t))\big) + \big({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha(u(t)-v(t))\big)\big]dt \\
&\quad + \frac{1}{2}\int_0^T \big[\big({}_0^cD_t^\alpha v(t), {}_t^cD_T^\alpha(u(t)-v(t))\big) + \big({}_t^cD_T^\alpha v(t), {}_0^cD_t^\alpha(u(t)-v(t))\big)\big]dt \\
&= -\int_0^T \big[\big({}_0^cD_t^\alpha (u(t)-v(t)), {}_t^cD_T^\alpha(u(t)-v(t))\big)dt \\
&\geq |\cos(\pi\alpha)|\, \|u-v\|_\alpha^2.
\end{aligned}$$

Thus, Φ' is uniformly monotone. Hence, by [23, Theorem 26.A (d)], $(\Phi')^{-1} : (E^\alpha)^* \to E^\alpha$ exists and is continuous. Suppose that $u_n \rightharpoonup u \in E^\alpha$. Then, by Lemma 4.4, $u_n \to u$ in $C([0,T], \mathbb{R}^N)$. Since $\tilde{F}(t, \mathbf{x})$ is continuously differentiable in $\mathbf{x}$ for a.e. $t \in [0,1]$, from the derivative formula in (4.17), we have $\tilde{\Psi}'(u_n) \to \tilde{\Psi}'(u)$, i.e., $\tilde{\Psi}'$ is strongly continuous. Therefore, $\tilde{\Psi}'$ is a compact operator by [23, Proposition 26.2].

Next, note that the facts that $\tilde{F}(t, 0, \dots, 0) = 0$ on $[0,T]$ and the inequality in (4.17), from Lemma 4.5, (4.9), and (4.15), we see that conditions (a) and (b) of Lemma 4.1 are satisfied.

Now, we show that condition (c) of Lemma 4.1 holds. For $i = 1, \dots, N$, let

$$w_i(t) = \begin{cases} \frac{4d}{T}t, & t \in [0, T/4), \\ d, & t \in [T/4, 3T/4], \\ \frac{4d}{T}(T-t), & t \in (3T/4, T], \end{cases}$$

and $w(t) = (w_1(t), \dots, w_N(t))$. Then, $w \in E^\alpha$ and

$${}_0^cD_t^\alpha w_i(t) = \frac{4d}{T\Gamma(2-\alpha)} \begin{cases} t^{1-\alpha}, & t \in [0, T/4), \\ g(t), & t \in [T/4, 3T/4], \\ h(t), & t \in (3T/4, T], \end{cases} \tag{4.18}$$

where $g(t)$ and $h(t)$ are defined by (3.2) and (3.3). From (3.1) and (4.18),

$$\begin{aligned}
&\int_0^T |{}_0^cD_t^\alpha w(t)|^2 dt \\
&= N\Big(\int_0^T |{}_0^cD_t^\alpha w_1(t)|^2 dt + \int_{T/4}^{3T/4} |{}_0^cD_t^\alpha w_1(t)|^2 dt + \int_{3T/4}^T |{}_0^cD_t^\alpha w_1(t)|^2 dt\Big) \\
&= \frac{16Nd^2}{T^2\Gamma^2(2-\alpha)}\Big(\int_0^{T/4} t^{2-2\alpha} dt + \int_{T/4}^{3T/4} g^2(t) dt + \int_{3T/4}^T |h(t)|^2 dt\Big) \\
&= \frac{16Nd^2}{T^2\Gamma^2(2-\alpha)}\Big(\frac{1}{3-2\alpha}\big(\frac{T}{4}\big)^{3-2\alpha} + \int_{T/4}^{3T/4} g^2(t) dt + \int_{3T/4}^T h^2(t) dt\Big) \\
&= \rho_\alpha d^2.
\end{aligned}$$

Then, $\|w\|_\alpha^2 = \rho_\alpha d^2$. Thus, from (4.16) with $u = w$,

$$\frac{1}{2}|\cos(\pi\alpha)|\rho_\alpha d^2 \le \Phi(w) \le \frac{1}{2|\cos(\pi\alpha)|}\rho_\alpha d^2. \tag{4.19}$$

Let

$$r_1 = \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}{2T^{2\alpha-1}}c^2, \quad r_2 = \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}{2T^{2\alpha-1}}l^2, \tag{4.20}$$

$$r_3 = \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}{2T^{2\alpha-1}}(m^2 - l^2). \tag{4.21}$$

Then, from (3.4) and (4.19), we have $r_1 < \Phi(w) < r_2$ and $r_3 > 0$. For any $u \in E^\alpha$, from the first inequality in (4.16), we see that $\|u\|_\alpha^2 \le 2\Phi(u)/|\cos(\pi\alpha)|$. Then, by (4.7) and (4.8), we have

$$\|u\|_\infty^2 \le \frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\|u\|_\alpha^2 \le \frac{2T^{2\alpha-1}\Phi(u)}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}.$$

Thus, by (4.20) and (4.21), we have the following implications

$$\begin{gathered}\Phi(u) < r_1 \Rightarrow \|u\|_\infty < c,\\ \Phi(u) < r_2 \Rightarrow \|u\|_\infty < l,\\ \Phi(u) < r_2 + r_3 \Rightarrow \|u\|_\infty < m.\end{gathered} \tag{4.22}$$

This, together with (3.6) and (4.14), implies

$$\begin{gathered}\sup_{u\in\Phi^{-1}(-\infty,r_1)}\int_0^T \tilde F(t,u(t))dt \le \int_0^T \max_{|\mathbf{x}|\le c} F(t,\mathbf{x})dt \le \int_0^T F(t,\mathbf{c})dt, \\ \sup_{u\in\Phi^{-1}(-\infty,r_2)}\int_0^T \tilde F(t,u(t))dt \le \int_0^T \max_{|\mathbf{x}|\le l} F(t,\mathbf{x})dt \le \int_0^T F(t,\mathbf{l})dt, \\ \sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\int_0^T \tilde F(t,u(t))dt \le \int_0^T \max_{|\mathbf{x}|\le m} F(t,\mathbf{x})dt \le \int_0^T F(t,\mathbf{m})dt.\end{gathered} \tag{4.23}$$

Then, taking into account the fact that $0 \in \Phi^{-1}(-\infty, r_i)$, $i = 1, 2$, from (4.1), (4.3), (4.15), (4.20), and (4.21), it follows that

$$\varphi(r_1) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\tilde\Psi(u)}{r_1} \le \frac{2T^{2\alpha-1}\int_0^T F(t,\mathbf{c})dt}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|c^2}, \tag{4.24}$$

$$\varphi(r_2) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\tilde\Psi(u)}{r_2} \le \frac{2T^{2\alpha-1}\int_0^T F(t,\mathbf{l})dt}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|l^2}, \tag{4.25}$$

$$\gamma(r_2,r_3) = \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\tilde\Psi(u)}{r_3} \le \frac{2T^{2\alpha-1}\int_0^T F(t,\mathbf{m})dt}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|(m^2-l^2)}. \tag{4.26}$$

On the other hand, in view of the fact that $w(t) = \mathbf{d} < \mathbf{m}$ on $[T/4, 3T/4]$ and from (4.13) and (4.14),

$$\int_0^T \tilde F(t,w(t))dt \ge \int_{T/4}^{3T/4} \tilde F(t,w(t))dt = \int_{T/4}^{3T/4} \tilde F(t,\mathbf{d})dt.$$

Note that $w \in \Phi^{-1}[r_1, r_2)$, from (4.2) and (4.23), we obtain

$$\beta(r_1,r_2) \ge \inf_{u\in\Phi^{-1}(-\infty,r_1)} \frac{\tilde\Psi(w)-\tilde\Psi(u)}{\Phi(w)-\Phi(u)} \ge \inf_{u\in\Phi^{-1}(-\infty,r_1)} \frac{\tilde\Psi(w)-\tilde\Psi(u)}{\Phi(w)}$$

$$\geq \frac{\int_{T/4}^{3T/4} \tilde{F}(t,\mathbf{d})dt - \int_0^T \tilde{F}(t,\mathbf{c})dt}{\Phi(w)}.$$

By (4.19), $1/\Phi(w) \geq 2|\cos(\pi\alpha)|/(\rho_\alpha d^2)$. Then

$$\beta(r_1, r_2) \geq \frac{2|\cos(\pi\alpha)|}{\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} \tilde{F}(t,\mathbf{d})dt - \int_0^T \tilde{F}(t,\mathbf{c})dt\Big). \tag{4.27}$$

For $\underline{\lambda}$ and $\overline{\lambda}$ defined by (3.10) and (3.11), from (3.7)–(3.9) and (4.24)–(4.27), we have

$$\varphi(r_1) < \frac{1}{\overline{\lambda}} < \frac{1}{\underline{\lambda}} < \beta(r_1, r_2), \quad \varphi(r_2) < \frac{1}{\overline{\lambda}} < \frac{1}{\underline{\lambda}} < \beta(r_1, r_2),$$
$$\gamma(r_2, r_3) < \frac{1}{\overline{\lambda}} < \frac{1}{\underline{\lambda}} < \beta(r_1, r_2).$$

In view of (4.4), $\alpha(r_1, r_2, r_3) < 1/\overline{\lambda} < 1/\underline{\lambda} < \beta(r_1, r_2)$; i.e., condition (c) of Lemma 4.1 holds. Hence, all the assumptions of Lemma 4.1 are satisfied. Then, by Lemma 4.1, for each $\lambda \in (\underline{\lambda}, \overline{\lambda})$, the functional $\Phi - \lambda\tilde{\Psi}$ has three distinct critical points u_1, u_2. and u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$, and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$. From (4.22), we have

$$\|u_1\|_\infty < c, \quad \|u_2\|_\infty < l, \quad \|u_3\|_\infty < m.$$

Then, in view of (4.10), (4.14), and (4.15), we have $\tilde{\Psi}(u) = \Psi(u)$. Therefore, u_1, u_2, and u_3 are three distinct critical points of the functional $\Phi - \lambda\Psi$. Thus, by Lemma 4.5 (b), u_1, u_2, and u_3 are three distinct solutions of (1.1). This completes the proof of the theorem. □

Proof of Corollary 3.2. When $\alpha = 1$, from (3.1), we have $\rho_\alpha = 8N/T$. Then, under the assumptions of Corollary 3.1, it is easy to see that all the conditions of Theorem 3.1 hold for $\alpha = 1$. Note that the system (1.2) is a special case of the system (1.1) with $\alpha = 1$. The conclusion then follows directly from Theorem 3.1. □

Proof of Corollary 3.3. Let $l = p/\sqrt{2}$ and $m = p$. Then, from (3.12)–(3.14), we see that (3.4)–(3.6) hold. By (3.14) and (3.16), we have

$$\begin{aligned}\frac{\int_0^T F(t,\mathbf{l})dt}{l^2} &= \frac{2\int_0^T F(t,\mathbf{p}/\sqrt{2})dt}{p^2} \leq \frac{2\int_0^T F(t,\mathbf{p})dt}{p^2}\\ &< \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt,\end{aligned} \tag{4.28}$$

and

$$\frac{\int_0^T F(t,\mathbf{m})dt}{m^2 - l^2} = \frac{2\int_0^T F(t,\mathbf{p})dt}{p^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt. \tag{4.29}$$

Note from (3.12) it follows that

$$\frac{\Gamma^2(\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2} < \frac{1}{c^2}.$$

Combing this inequality with (3.15), we obtain

$$
\begin{aligned}
&\frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big)\\
&> \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \frac{\cos^2(\pi\alpha)}{c^2}\int_0^T F(t,\mathbf{c})dt\\
&> \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt \\
&\quad - \frac{\Gamma^2(\alpha)\cos^4(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{c})dt\\
&= \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt.
\end{aligned} \tag{4.30}
$$

By (3.15) and (4.28)–(4.30), we see that (3.7)–(3.8) hold. From (3.10), (3.11), (3.17), (3.18), and (4.30), we have $\underline{\lambda} < \underline{\lambda}_2$ and $\overline{\lambda} = \overline{\lambda}_2$. Therefore, the conclusion now follows from Theorem 3.1. □

Proof of Corollary 3.4. When $\alpha = 1$, from (3.1), we have $\rho_\alpha = 8N/T$. Under the assumptions of Corollary 3.4, it is easy to see that all the conditions of Corollary 3.3 hold for $\alpha = 1$. Note that system (1.2) is a special case of system (1.1) with $\alpha = 1$. The conclusion then follows directly from Corollary 3.3. □

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