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SERRIN BLOW-UP CRITERION FOR STRONG SOLUTIONS TO THE 3-D COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS WITH VACUUM

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Abstract. In this article, we extend the well-known Serrin's blow-up criterion for solutions of the 3-D incompressible Navier-Stokes equations to the 3-D compressible nematic liquid crystal flows where the initial vacuum is allowed. It is proved that for the initial-boundary value problem of the 3-D compressible nematic liquid crystal flows in a bounded domain, the strong solution exists globally if the velocity satisfies the Serrin's condition and $L^1(0,T;L^\infty)$-norm of the gradient of the velocity is bounded.

1. Introduction

The time evolution of the density, the velocity and the orientation of a compressible nematic liquid crystal (NLC) flows occupying a bounded domain Ω of $\mathbb{R}^3$ can be described by the system

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \quad (x,t)\in\Omega\times(0,+\infty), \tag{1.1}$$

$$\begin{aligned}&\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P\\ &= \mu\Delta u - \lambda\nabla\cdot\big(\nabla d\odot\nabla d - (\frac{1}{2}|\nabla d|^2 + F(d))I\big), (x,t)\in\Omega\times(0,+\infty),\end{aligned} \tag{1.2}$$

$$\partial_t d + (u\cdot\nabla)d = \nu(\Delta d - f(d)), \quad (x,t)\in\Omega\times(0,+\infty), \tag{1.3}$$

together with the initial value:

$$\rho(0,x) = \rho_0(x) \ge 0, \quad u(0,x) = u_0(x), \quad d(0,x) = d_0(x), \quad \forall x\in\Omega, \tag{1.4}$$

and the boundary conditions:

$$u(t,x) = 0, \quad d(t,x) = d_0(x), \quad |d_0(x)| = 1, \quad \forall (t,x)\in[0,+\infty)\times\partial\Omega. \tag{1.5}$$

Here we denote by ρ the unknown density, $u = (u_1,u_2,u_3)$ the unknown velocity, $d = (d_1,d_2,d_3)$ the unknown orientation parameter of the nematic liquid crystal material, and $P = P(\rho)$ the pressure function. μ,λ and ν are positive viscosity coefficients. The unusual term $\nabla d\odot\nabla d$ denotes the 3×3 matrix, whose (i,j)-th element is given by $\sum_{k=1}^3 \partial_i d_k\partial_j d_k$. I is the 3×3 unit matrix. $f(d)$ is a polynomial function of d which satisfies $f(d) = \frac{\partial}{\partial_d}F(d)$, where $F(d)$ is the bulk part of the

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elastic energy, usually we choose $F(d)$ to be the Ginzburg-Landau penalization; i.e.,

$$F(d) = \frac{1}{4\sigma^2}(|d|^2 - 1)^2, \quad f(d) = \frac{1}{\sigma^2}(|d|^2 - 1)d,$$

where σ is a positive constant. In what follows, we will assume $\sigma = 1$ since it does not play a special role in our discussion. Throughout this paper, we adopt the following simplified notations for standard Sobolev spaces

$$L^q := L^q(\Omega), \quad W^{k,p} := W^{k,p}(\Omega), \quad H^k := H^k(\Omega) = W^{k,2}, \quad H_0^1 := H_0^1(\Omega),$$

where $1 \le p,\, q \le \infty$ and $k \in \mathbb{N}$.

System (1.1)–(1.5) is a simplified version of Ericksen-Lesile system modeling the flow of compressible nematic liquid crystals materials, and the hydrodynamic theory of liquid crystals was established by Ericksen [5, 6] and Leslie [18] in the 1960's. In [34], Wang and Yu established the global existence and large-time behavior of weak solutions for the initial-boundary value problem (1.1)–(1.5). When the direction d does not appear, the system (1.1)–(1.5) becomes the compressible Navier-Stokes (CNS) equations. Matsumura and Nishida [29] obtained global existence of smooth solutions for the initial data is a small perturbation of a non–vacuum equilibrium. For the existence of solutions for arbitrary initial value, Lions [19] and Feireisl [9, 10] established the global existence of weak solution to the CNS equations. Cho et al [2, 3, 4] proved that the existence and uniqueness of local strong solutions of the CNS equations in the case where initial density need not to be positive and may vanish in an open set. Xin in [36] showed that there is no global smooth solution to the Cauchy problem of the CNS equations with a nontrivial compactly supported initial density. Hence, there are many works [3, 7, 8, 13, 14, 15, 32, 33] trying to establish blow-up criterion for the strong solution to the CNS equations. In particular, it is proved in [15] by Huang, Li and Xin that the Serrin's blow–up criterion (see [30]) for the incompressible Navier–Stokes equations still holds for the CNS equations; i.e., if T^* is the maximal time of existence strong solution, then

$$\lim_{T\to T^*}(\|\operatorname{div} u\|_{L^1(0,T;L^\infty)} + \|\rho^{1/2}u\|_{L^s(0,T;L^r)}) = \infty \tag{1.6}$$

or

$$\lim_{T\to T^*}(\|\rho\|_{L^1(0,T;L^\infty)} + \|\rho^{1/2}u\|_{L^s(0,T;L^r)}) = \infty, \tag{1.7}$$

where r and s satisfy $\frac{2}{s} + \frac{3}{r} \le 1$, $3 < r \le \infty$. Huang et al [13, 14] established that the Beale-Kato-Majda criterion (see [1]) for the ideal incompressible flows still holds for the CNS equations; that is,

$$\lim_{T\to T^*}\int_0^T \|\nabla u\|_{L^\infty}\, \mathrm{d}t = \infty.$$

Sun, Wang and Zhang in [32] (see also [15]) obtained another Beale–Kato–Majda criterion in terms of the density, i.e.,

$$\lim\sup_{T\to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty.$$

Recently, Wen and Zhu in [35] established a blow-up criterion of the strong solution for the CNS equations in terms of the density,

$$\limsup_{T\to T^*} \|\rho\|_{L^\infty(0,T;L^q)} = \infty,$$

for some $1 < q < \infty$ large enough.

When ρ is a positive constant, the system (1.1)-(1.3) becomes the incompressible nematic liquid crystal (INLC) equations, the global-in-time weak solutions and local-in-time strong solution have been studied by Lin and Liu [21]. Later, in [23], they further proved that the one-dimensional spacetime Hausdorff measure of the singular set of the so-called suitable weak solutions is zero. In [12], Hu and Wang established global existence of strong solutions and weak–strong uniqueness for initial data belonging to the Besov spaces of positive order under some smallness assumptions. Liu and Cui in [26] obtained that the blow–up criterion (1.6) or (1.7) still holds for the solution of the INLC equations. We also refer [11, 20, 22, 24, 25, 31] and the references cited therein for other related work on the INLC equations.

Inspired by the above mentioned works on blow-up criteria of the strong solutions to the CNS equations and the INLC equations, particularly the results of Huang et al [13, 14] and Sun et al [32, 33], we want to investigate the similar problem for the compressible nematic liquid crystal flow (1.1)–(1.5).

When the initial vacuum is allowed, the well-posedness and a blow-up criterion for strong solutions to the compressible nematic liquid crystal flows (1.1)-(1.5) were established by Liu et al [27, 28]. More precisely, under the assumption of the pressure P satisfies

$$P = P(\cdot) \in C^1[0,\infty), \quad P(0) = 0. \tag{1.8}$$

They established the following result.

Theorem 1.1 ([27, 28]). *Suppose that the initial value (ρ_0, u_0, d_0) satisfies the regularity conditions*

$$0 \le \rho_0 \in W^{1,6}, \quad u_0 \in H_0^1 \cap H^2, \quad d_0 \in H^3,$$

and the compatibility condition

$$\mu\Delta u_0 - \lambda\,\mathrm{div}(\nabla d_0 \odot \nabla d_0 - (\frac{1}{2}|\nabla d_0|^2 + F(d_0))I) - \nabla P(\rho_0) = \sqrt{\rho} g \tag{1.9}$$

for some function $g \in L^2$. Then there exist a small $T \in (0,\infty)$ and a unique strong solution (ρ, u, d) to the system (1.1)-(1.3) *with initial boundary condition* (1.4)-(1.5) *such that*

$$\begin{gathered}
0 \le \rho \in C([0,T); W^{1,6}), \quad \rho_t \in C([0,T); L^6),\\
u \in C([0,T); H_0^1 \cap H^2) \cap L^2(0,T; W^{2,6}), \quad u_t \in L^2(0,T; H_0^1),\\
d \in C([0,T); H^3), \quad d_t \in C([0,T); H_0^1) \cap L^2(0,T; H^2),\\
d_{tt} \in L^2(0,T; L^2), \quad \sqrt{\rho} u_t \in C([0,T); L^2).
\end{gathered}$$

Moreover, let T^* be the maximal existence time of the solution. If $T^* < \infty$, then there holds

$$\lim_{T\to T^*} \int_0^T (\|\nabla u\|_\alpha^\beta + \|u\|_{W^{1,\infty}})\,\mathrm{d}t = \infty, \tag{1.10}$$

where α, β satisfying $\frac{3}{\alpha} + \frac{2}{\beta} < 2$ and $\beta \ge 4$.

Recently Huang, Wang and Wen [16, 17] considered a similar, but not equivalent, system of partial differential equations modeling compressible nematic liquid crystal flows, they obtained the existence of local in time strong solution and two blow–up criteria under some suitable assumptions on u and d or on ρ and d.

The purpose of this article is to obtain a blow-up criterion for the strong solutions to the compressible liquid crystal flow only in terms of the velocity. Our main result is stated as follows

Theorem 1.2. *Assume that (ρ, u, d) is the strong solution constructed in Theorem 1.1, and T^* be the maximal existence time of the solution. If $T^* < \infty$, then we have*

$$\lim_{T\to T^*}\{\|\nabla u\|_{L^1(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)}\} = \infty, \tag{1.11}$$

for all r, s satisfying

$$\frac{2}{s} + \frac{3}{r} \le 1, \quad 3 < r \le +\infty. \tag{1.12}$$

The remaining of this article is devoted to proving Theorem 1.2. The main idea used here is similar as the papers [14, 15] of Huang et al, who studied the blow-up criterion of strong solutions to the 3D CNS equations. The key issue in our proofs is derive the suitable higher norm estimates of the strong solution (ρ, u, d). Some of the new difficulties appears due to the fact that system (1.1)–(1.3) is the coupling of the CNS equations (1.1)–(1.2) and the liquid crystal equation (1.3). To proceed, some new estimates are needed. In fact, we found that under the assumption of $L^1(0,T;L^\infty)$-norm of the divergence of velocity implies that both the time-independent upper bound for the density and the $L^\infty(0,T;L^q)$-norm with $2 \le q \le \infty$ of d, and when the opposite to (1.11) holds, the bound of the $L^\infty(0,T;L^q)$-norm with $2 \le q \le \infty$ of ∇d can be directly from the liquid crystal equation (1.3). These properties are important for us to establish the higher order norm estimates of ρ, u and d.

2. Proof of Theorem 1.2

In what follows, we assume that (ρ, u, d) is the unique strong solution to the system (1.1)–(1.3) with initial-boundary condition (1.4)–(1.5) constructed in Theorem 1.1, and T^* is the maximal existence time of the solution. We assume that the opposite to (1.11) holds; i.e.,

$$\lim_{T\to T^*}\{\|\nabla u\|_{L^1(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)}\} \le M < \infty.$$

Hence, for all $0 < T < T^*$, there holds

$$\int_0^T \|\nabla u\|_{L^\infty} + \|u\|_{L^r}^s \,\mathrm{d}t \le M < \infty, \tag{2.1}$$

from which we will prove the same regularity at time T^* as the initial value, a contradiction to the maximality of T^*. Throughout the paper, we will use C to denote a generic positive constant depending only on $\mu, \lambda, \nu, M, T, \Omega$ and the initial data. Here, we notice that thanks to the assumptions (2.1) and (1.12), we have

$$\int_0^T \|u\|_{L^\infty}^2 \,\mathrm{d}t \le M < \infty. \tag{2.2}$$

Under assumption (2.1), it is easy to obtain the L^∞-norm bounds of the density ρ and the L^q-norm (for all $q \ge 2$) of the orientation parameter d by using the mass conservation equation (1.1) and the liquid crystal equation (1.3) respectively. More precisely, we have the following Lemma.

Lemma 2.1. *Assume that*

$$\int_0^T \|\operatorname{div} u\|_{L^\infty}\,\mathrm{d}t \le M, \quad 0 \le T < T^*, \tag{2.3}$$

then

$$\sup_{0\le t\le T} \|\rho\|_{L^\infty} \le C \quad \forall 0 < T < T^*; \tag{2.4}$$

$$\sup_{0\le t\le T} \|d\|_{L^q} \le C \quad \textit{for all } q \ge 2 \textit{ and } 0 < T < T^*, \tag{2.5}$$

where the constant C independent of q.

Proof. The proof of estimate (2.4) is essentially due to Huang and Xin [13], for reader's convenience, we sketch it here. Multiplying the mass conservation equation (1.1) by $p\rho^{p-1}$ with $p > 1$, it follows that

$$\partial_t(\rho^p) + \operatorname{div}(\rho^p u) + (p-1)\rho^p \operatorname{div} u = 0.$$

Integrating the above equality over Ω yields

$$\partial_t \|\rho\|_{L^p}^p \le (p-1)\|\operatorname{div} u\|_{L^\infty} \|\rho\|_{L^p}^p;$$

i.e.,

$$\partial_t \|\rho\|_{L^p} \le \frac{(p-1)}{p}\|\operatorname{div} u\|_{L^\infty} \|\rho\|_{L^p}. \tag{2.6}$$

Condition (2.3) and estimate (2.6) imply

$$\sup_{0\le t\le T} \|\rho\|_{L^p} \le C \quad \forall p > 1,$$

where C is a positive constant independent of p, letting $p \to \infty$, we obtain (2.4), and this completes the first part of the proof.

To prove the estimate (2.5), we multiply the liquid crystal equation (1.3) by $q|d|^{q-2}d$ with $q \ge 2$, integrate it over Ω, and make use of the boundary condition (1.5), then there holds

$$\begin{aligned}
&\frac{d}{dt}\|d\|_{L^q}^q + \int_\Omega (q\nu|\nabla d|^2|d|^{q-2} + q(q-2)\nu|d|^{q-2}|\nabla|d||^2)\,\mathrm{d}x + q\nu\int_\Omega |d|^{q+2}\,\mathrm{d}x \\
&= -\sum_{i=1}^3 \int_\Omega u_i\partial_i(|d|^q)\,\mathrm{d}x + q\nu\int_\Omega |d|^q\,\mathrm{d}x \\
&\le \int_\Omega |\operatorname{div} u||d|^q\,\mathrm{d}x + \Big(\frac{q\nu}{2}\int_\Omega |d|^{q+2}\,\mathrm{d}x\Big)^{\frac{q}{q+2}}(2|\Omega|)^{\frac{2}{q+2}} \\
&\le C\|\operatorname{div} u\|_{L^\infty}\|d\|_q^q + \frac{q\nu}{2}\int_\Omega |d|^{q+2}\,\mathrm{d}x + C,
\end{aligned} \tag{2.7}$$

where we have used the fact that $d\nabla d = |d|\nabla|d|$ implies

$$\begin{aligned}
-q\int_\Omega \Delta d|d|^{q-2}d\,\mathrm{d}x &= q\int_\Omega |\nabla d|^2|d|^{q-2}\,\mathrm{d}x + q\int_\Omega \nabla d\cdot d\nabla|d|^{q-2}\,\mathrm{d}x \\
&= q\int_\Omega |\nabla d|^2|d|^{q-2}\,\mathrm{d}x + q\int_\Omega |\nabla d||d|(q-2)|d|^{q-2}\nabla|d|\,\mathrm{d}x \\
&= q\int_\Omega |\nabla d|^2|d|^{q-2}\,\mathrm{d}x + q(q-2)\int_\Omega |d|^{q-1}|\nabla|d||^2\,\mathrm{d}x
\end{aligned}$$

and the fact that $f(d)=(|d|^2-1)d$. It follows from inequality (2.7) that

$$\frac{d}{dt}\|d\|_{L^q}^q \le C\|\operatorname{div} u\|_{L^\infty}\|d\|_{L^q}^q + C,$$

which together with the Gronwall's inequality imply

$$\sup_{0\le t\le T}\|d\|_{L^q}\le C \quad \text{for all } q\ge 2, \tag{2.8}$$

where C is a positive constant independent of q. By letting $q\to\infty$, we notice that the estimate (2.5) still holds. □

By assumption (1.8) on the pressure P and the estimate (2.4), it is easy to obtain

$$\sup_{0\le t\le T}\{\|P(\rho)\|_{L^\infty}, \|P'(\rho)\|_{L^\infty}\}\le C<\infty. \tag{2.9}$$

Now, let us derive the stand energy inequality.

Lemma 2.2. *There holds*

$$\sup_{0\le t\le T}\int_\Omega (\rho|u|^2+|\nabla d|^2+2F(d))\,\mathrm{d}x+\int_0^T\int_\Omega |\nabla u|^2+|\Delta d-f(d)|^2\,\mathrm{d}x\,\mathrm{d}t\le C. \tag{2.10}$$

Proof. Multiplying the momentum equation (1.2) by u, integrating it over Ω, making use of the mass conversation equation (1.1) and the boundary condition (1.5), it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u|^2\,\mathrm{d}x+\mu\int_\Omega|\nabla u|^2\,\mathrm{d}x=-\int_\Omega u\nabla P\,\mathrm{d}x-\lambda\int_\Omega(u\cdot\nabla)d\cdot(\Delta d-f(d))\,\mathrm{d}x, \tag{2.11}$$

where we have used the equality $\operatorname{div}(\nabla d\odot\nabla d)=(\nabla d)^T\Delta d+\nabla\left(\frac{|\nabla d|^2}{2}\right)$. Multiplying the liquid crystal equation (1.3) by $\Delta d-f(d)$ and integrating it over Ω, we obtain

$$\frac{d}{dt}\int_\Omega(\frac{1}{2}|\nabla d|^2+F(d))\,\mathrm{d}x+\nu\int_\Omega|\Delta d-f(d)|^2\,\mathrm{d}x=\int_\Omega(u\cdot\nabla)d\cdot(\Delta d-f(d))\,\mathrm{d}x. \tag{2.12}$$

Combining (2.11) and (2.12)

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega[\frac{1}{2}(\rho|u|^2+\lambda|\nabla d|^2)+\lambda F(d)]\,\mathrm{d}x+\mu\int_\Omega|\nabla u|^2\,\mathrm{d}x+\lambda\nu\int_\Omega|\Delta d-f(d)|^2\,\mathrm{d}x\\
&=-\int_\Omega u\nabla P\,\mathrm{d}x\\
&=\int_\Omega P\operatorname{div} u\,\mathrm{d}x\le\varepsilon\int_\Omega|\nabla u|^2\,\mathrm{d}x+C\varepsilon^{-1},
\end{aligned} \tag{2.13}$$

where we have used the estimates (2.4), (2.9) and the Young inequality. Taking ε small enough in (2.13) and applying the Gronwall's inequality, we can establish the estimate (2.10) immediately. □

In the next lemma, we will derive some estimates of the direction field d.

Lemma 2.3. *Under the assumption* (2.1)*, it holds that for* $0\le T<T^*$

$$\sup_{0\le t\le T}\|\nabla d\|_{L^q}\le C \quad \text{for all } 2\le q\le\infty; \tag{2.14}$$

$$\sup_{0\le t\le T}\|\nabla d\|_{L^2}^2+\int_0^T\|d_t\|_{L^2}^2\,\mathrm{d}t\le C; \tag{2.15}$$

$$\sup_{0\le t\le T}\|d\|_{H^2}^2+\int_0^T\|\nabla d_t\|_{L^2}^2\,\mathrm{d}t\le C; \tag{2.16}$$

$$\int_0^T\|\nabla d\|_{H^2}^2\,\mathrm{d}t\le C. \tag{2.17}$$

Proof. Multiply the gradient of the liquid crystal equation (1.3) by $q|\nabla d|^{q-2}\nabla d$ with $q\ge 2$, integrating it over Ω, and using the boundary condition (1.5), then there holds

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla d\|_{L^q}^q+\int_\Omega(q\nu|\nabla(\nabla d)|^2|\nabla d|^{q-2}+q(q-2)\nu|\nabla|\nabla d||^2|\nabla d|^{q-2})\,\mathrm{d}x\\
&=-\sum_{i=1}^3\int_\Omega u_i\partial_i(|\nabla d|^q)\,\mathrm{d}x-\sum_{i=1}^3 q\int_\Omega\nabla u_i\partial_i d|\nabla d|^{q-2}\nabla d\,\mathrm{d}x\\
&\quad-\nu q\int_\Omega\nabla(|d|^2d)|\nabla d|^{q-2}\nabla d\,\mathrm{d}x+\nu q\int_\Omega|\nabla d|^q\,\mathrm{d}x\\
&=\sum_{i=1}^3\int_\Omega\partial_i u_i|\nabla d|^q\,\mathrm{d}x-\sum_{i=1}^3 q\int_\Omega\nabla u_i\partial_i d|\nabla d|^{q-2}\nabla d\,\mathrm{d}x\\
&\quad-\nu q\int_\Omega\nabla(|d|^2d)|\nabla d|^{q-2}\nabla d\,\mathrm{d}x+\nu q\int_\Omega|\nabla d|^q\,\mathrm{d}x\\
&=\int_\Omega\operatorname{div}u|\nabla d|^q\,\mathrm{d}x+\sum_{i=1}^3 q\int_\Omega u_i\partial_i\nabla d|\nabla d|^{q-2}\nabla d\,\mathrm{d}x\\
&\quad+\sum_{i=1}^3 q\int_\Omega u_i\partial_i d\nabla(|\nabla d|^{q-2}\nabla d)\,\mathrm{d}x-\nu q\int_\Omega\nabla(|d|^2d)|\nabla d|^{q-2}\nabla d\,\mathrm{d}x\\
&\quad+\nu q\int_\Omega|\nabla d|^q\,\mathrm{d}x\\
&\le\int_\Omega|\operatorname{div}u||\nabla d|^q\,\mathrm{d}x+2q\int_\Omega|u||\nabla^2 d||\nabla d|^{q-1}\,\mathrm{d}x\\
&\quad+q(q-1)\int_\Omega|u||\nabla|\nabla d|||\nabla d|^{q-1}\,\mathrm{d}x-\nu q\int_\Omega\nabla(|d|^2d)|\nabla d|^{q-2}\nabla d\,\mathrm{d}x+\nu q\|\nabla d\|_{L^q}^q\\
&\le\|\operatorname{div}u\|_{L^\infty}\|\nabla d\|_{L^q}^q+\varepsilon(\||\nabla^2 d||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2+\||\nabla|\nabla d|||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2)\\
&\quad+C\varepsilon^{-1}\||u||\nabla d|^{\frac{q}{2}}\|_{L^2}^2-\nu q\int_\Omega\nabla(|d|^2d)|\nabla d|^{q-2}\nabla d\,\mathrm{d}x+\nu q\|\nabla d\|_{L^q}^q\\
&=\|\operatorname{div}u\|_{L^\infty}\|\nabla d\|_{L^q}^q+\varepsilon(\||\nabla^2 d||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2+\||\nabla|\nabla d|||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2)\\
&\quad+C\varepsilon^{-1}\||u||\nabla d|^{\frac{q}{2}}\|_{L^2}^2-\nu q\int_\Omega|d|^2\nabla d|\nabla d|^{q-2}\nabla d\,\mathrm{d}x\\
&\quad-\nu q\int_\Omega d\nabla(|d|^2)|\nabla d|^{q-2}\nabla d\,\mathrm{d}x+\nu q\|\nabla d\|_{L^q}^q\\
&=\|\operatorname{div}u\|_{L^\infty}\|\nabla d\|_{L^q}^q+\varepsilon(\||\nabla^2 d||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2+\||\nabla|\nabla d|||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2)\\
&\quad+C\varepsilon^{-1}\|u\|_{L^\infty}^2\|\nabla d\|_{L^q}^q-3\nu q\int_\Omega|d|^2|\nabla d|^q\,\mathrm{d}x+\nu q\|\nabla d\|_{L^q}^q\\
&\le C(\|\operatorname{div}u\|_{L^\infty}+\varepsilon^{-1}\|u\|_{L^\infty}^2+1)\|\nabla d\|_{L^q}^q+\varepsilon(\||\nabla^2 d||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2
\end{aligned}
$$

$$+ \||\nabla|\nabla d|||\nabla d|^{\frac{q-2}{2}}\|_{L^2}^2),$$

where we have used the facts that $\nabla|d|^2 = 2|d|\nabla|d| = 2|d|\frac{d\cdot\nabla d}{|d|} = 2d\nabla d$, and $|\nabla d|\nabla|\nabla d| = \nabla(\nabla d)\cdot\nabla d$ implies

$$\begin{aligned}
&-q\int_\Omega \nabla\Delta d|\nabla d|^{q-2}\nabla d\,\mathrm{d}x\\
&= q\int_\Omega |\nabla(\nabla d)|^2|\nabla d|^{q-2}\,\mathrm{d}x + q\int_\Omega \nabla(\nabla d)\cdot\nabla d\nabla|\nabla d|^{q-2}\,\mathrm{d}x\\
&= q\int_\Omega |\nabla(\nabla d)|^2|\nabla d|^{q-2}\,\mathrm{d}x + q\int_\Omega \nabla|(\nabla d)||\nabla d|(q-2)|\nabla d|^{q-2}\nabla|\nabla d|\,\mathrm{d}x\\
&= q\int_\Omega |\nabla(\nabla d)|^2|\nabla d|^{q-2}\,\mathrm{d}x + q(q-2)\int_\Omega |\nabla d|^{q-1}|\nabla|\nabla d||^2\,\mathrm{d}x.
\end{aligned}$$

Taking ε small enough, it follows form the above inequality that

$$\frac{d}{dt}\|\nabla d\|_{L^q}^q \le C(\|\operatorname{div} u\|_{L^\infty} + \|u\|_{L^\infty}^2 + 1)\|\nabla d\|_{L^q}^q,$$

which, together with the Gronwall's inequality, (2.1) and (2.2), imply

$$\sup_{0\le t\le T}\|\nabla d\|_{L^q} \le C \quad \text{for all } q\ge 2. \tag{2.18}$$

Letting $q\to\infty$, estimate (2.18) still holds. Thus the estimate (2.14) holds.

To prove estimate (2.15), multiplying the liquid crystal equation (1.3) by d_t, integrating it over Ω, and using the boundary condition (1.5), then

$$\begin{aligned}
&\|d_t\|_{L^2}^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla d\|_{L^2}^2\\
&= -\int_\Omega (u\cdot\nabla)dd_t\,\mathrm{d}x - \nu\int_\Omega f(d)d_t\,\mathrm{d}x\\
&= -\int_\Omega (u\cdot\nabla)dd_t\,\mathrm{d}x - \nu\int_\Omega (|d|^2-1)dd_t\,\mathrm{d}x\\
&\le \|u\cdot\nabla d\|_{L^2}\|d_t\|_{L^2} + (\|d\|_{L^\infty}^2+1)\|d\|_{L^2}\|d_t\|_{L^2}\\
&\le \frac14\|d_t\|_{L^2}^2 + C(\|u\cdot\nabla d\|_{L^2}^2+1) \le \frac14\|d_t\|_{L^2}^2 + C\|u\|_{L^r}^2\|\nabla d\|_{L^{\frac{2r}{r-2}}}^2 + C\\
&\le \frac14\|d_t\|_{L^2}^2 + C\|u\|_{L^r}^2\|\nabla d\|_{L^2}^{\frac{2(r-3)}{r}}\|\nabla^2 d\|_{L^2}^{\frac{6}{r}} + C\\
&\le \frac14\|d_t\|_{L^2}^2 + \varepsilon\|\nabla^2 d\|_{L^2}^2 + C\varepsilon^{-1}\|u\|_{L^r}^{\frac{2r}{r-3}}\|\nabla d\|_{L^2}^2 + C,
\end{aligned} \tag{2.19}$$

where we have used the Hölder inequality, estimate (2.5), the interpolation inequality and the Young inequality. By applying the standard elliptic regularity result to the liquid crystal equation (1.3), then we have

$$\begin{aligned}
\|\nabla^2 d\|_{L^2}^2 &\le C(\|\Delta d\|_{L^2}^2 + \|d_0\|_{H^2}^2)\\
&\le C(\|d_t\|_{L^2}^2 + \|u\cdot\nabla d\|_{L^2}^2 + \|f(d)\|_{L^2}^2 + \|d_0\|_{H^2}^2)\\
&\le C(\|d_t\|_{L^2}^2 + \|u\|_{L^r}^2\|\nabla d\|_{L^{\frac{2r}{r-2}}}^2 + \|d\|_{L^\infty}^4\|d\|_{L^2}^2 + \|d\|_{L^\infty}^2\|d\|_{L^2}^2 + C)\\
&\le C(\|d_t\|_{L^2}^2 + \eta\|\nabla^2 d\|_{L^2}^2 + \eta^{-1}\|u\|_{L^r}^{\frac{2r}{r-3}}\|\nabla d\|_{L^2}^2 + C).
\end{aligned}$$

Taking η small enough, the above inequality implies that

$$\|\nabla^2 d\|_{L^2}^2 \le C(\|d_t\|_{L^2}^2 + \|u\|_{L^r}^{\frac{2r}{r-3}} \|\nabla d\|_{L^2}^2 + C). \tag{2.20}$$

Inserting the inequality (2.20) in (2.19) gives

$$\|d_t\|_{L^2}^2 + \nu \frac{d}{dt} \|\nabla d\|_{L^2}^2 \le C\|u\|_{L^r}^{\frac{2r}{r-3}} \|\nabla d\|_{L^2}^2 + C,$$

which, together with the Gronwall's inequality, give estimate (2.15).

To prove the estimate (2.16), we first multiply the liquid crystal equation (1.3) by Δd_t, integrate it over Ω and make use of the boundary condition (1.5), then we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \nu|\Delta d|^2 \,\mathrm{d}x + \int_\Omega |\nabla d_t|^2 \,\mathrm{d}x \\
&= \int_\Omega u\cdot \nabla d \Delta d_t \,\mathrm{d}x + \nu \int_\Omega (|d|^2-1) d\Delta d_t \,\mathrm{d}x \\
&= \sum_{i,j=1}^3 \int_\Omega u_i \partial_i d \partial_j^2 d_t \,\mathrm{d}x - \nu \int_\Omega \nabla(|d|^2 d)\nabla d_t \,\mathrm{d}x + \nu \int_\Omega \nabla d \nabla d_t \,\mathrm{d}x \\
&= -\sum_{i,j=1}^3 \int_\Omega \partial_j u_i \partial_i d \partial_j d_t \,\mathrm{d}x - \sum_{i,j=1}^3 \int_\Omega u_i \partial_i \partial_j d \partial_j d_t \,\mathrm{d}x \\
&\quad - \nu \int_\Omega \nabla(|d|^2 d)\nabla d_t \,\mathrm{d}x + \nu \int_\Omega \nabla d \nabla d_t \,\mathrm{d}x \\
&\le C(\|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2} + \||u||\nabla^2 d|\|_{L^2}\|\nabla d_t\|_{L^2} \\
&\quad + \|d\|_{L^\infty}^2 \|\nabla d\|_{L^2}\|\nabla d_t\|_{L^2} + \|\nabla d\|_{L^2}\|\nabla d_t\|_{L^2}) \\
&\le \varepsilon\|\nabla d_t\|_{L^2}^2 + C\varepsilon^{-1}(\|\nabla u\|_{L^2}^2 + \||u||\nabla^2 d|\|_{L^2}^2 + 1) \\
&\le \varepsilon\|\nabla d_t\|_{L^2}^2 + C\varepsilon^{-1}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^r}^2 \|\nabla^2 d\|_{L^{\frac{2r}{r-2}}}^2 + 1) \\
&\le \varepsilon\|\nabla d_t\|_{L^2}^2 + C\varepsilon^{-1}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^r}^2 \|\nabla^2 d\|_{L^2}^{\frac{2(r-3)}{r}} \|\nabla^3 d\|_{L^2}^{\frac{6}{r}} + 1) \\
&\le \varepsilon\|\nabla d_t\|_{L^2}^2 + C\varepsilon^{-1}(\|\nabla u\|_{L^2}^2 + C\varepsilon^{-2}\|u\|_{L^r}^{\frac{2r}{r-3}} \|\nabla^2 d\|_{L^2}^2 + \varepsilon^2\|\nabla^3 d\|_{L^2}^2 + 1),
\end{aligned} \tag{2.21}$$

where we have used the Hölder inequality, the interpolation inequality, the Young inequality, estimates (2.5), (2.10) and (2.14). Now, differentiating the liquid crystal equation (1.3) with respect to space variable gives us

$$\nu\Delta(\nabla d) = \nabla d_t + (\nabla u \cdot \nabla) d + (u\cdot\nabla)\nabla d + \nu \nabla[(|d|^2-1)d]. \tag{2.22}$$

By applying the standard elliptic regularity result to (2.22), one can estimate the term $\|\nabla^3 d\|_{L^2}$ as follows

$$\begin{aligned}
\|\nabla^3 d\|_{L^2} &\le C(\|\nabla d_t\|_{L^2} + \|(\nabla u\cdot\nabla) d\|_{L^2} + \|(u\cdot\nabla)\nabla d\|_{L^2} \\
&\quad + \|\nabla[(|d|^2-1)d]\|_{L^2} + \|d_0\|_{H^3} + \|\nabla d\|_{L^2}) \\
&\le C\Big(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty} + \|u\|_{L^r}\|\nabla^2 d\|_{L^{\frac{2r}{r-2}}} + \|d\|_{L^\infty}^2\|\nabla d\|_{L^2} \\
&\quad + \|\nabla d\|_{L^2} + 1\Big) \\
&\le C\big(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|u\|_{L^r}\|\nabla^2 d\|_{L^2}^{\frac{r-3}{r}} \|\nabla^3 d\|_{L^2}^{\frac{3}{r}} + C\big)
\end{aligned}$$

$$\le C\big(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \eta\|\nabla^3 d\|_{L^2} + \eta^{-1}\|u\|_{L^r}^{\frac{r}{r-3}}\|\nabla^2 d\|_{L^2} + C\big),$$

where we have used the estimates (2.5), (2.10), (2.14), (2.15) and the Young inequality. Taking η small enough, it follows from the above estimates that

$$\|\nabla^3 d\|_{L^2} \le C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|u\|_{L^r}^{\frac{r}{r-3}}\|\nabla^2 d\|_{L^2} + 1). \tag{2.23}$$

Inserting the estimate (2.23) in (2.21), and taking ε small enough, it follows that

$$\frac{d}{dt}\int_\Omega |\Delta d|^2\,\mathrm{d}x + \int_\Omega |\nabla d_t|^2\,\mathrm{d}x \le C(\|\nabla u\|_{L^2}^2 + \|u\|_{L^r}^{\frac{2r}{r-3}}\|\nabla^2 d\|_{L^2}^2 + 1),$$

which, together with the standard elliptic regularity result $\|\nabla^2 d\|_{L^2} \le C(\|\Delta d\|_{L^2} + \|d_0\|_{H^2})$ and the Gronwall's inequality, give that the estimate (2.16) holds.

Now, we will prove the estimate (2.17). By using the standard elliptic regularity result $\|\nabla d\|_{H^2} \le C(\|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^2} + \|d_0\|_{H^3})$, it follows that

$$\begin{aligned}
\int_0^T \|\nabla d\|_{H^2}^2\,\mathrm{d}t &\le C\int_0^T (\|\nabla^3 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|d_0\|_{H^3}^2)\,\mathrm{d}t \\
&\le C\int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^r}^{\frac{2r}{r-3}}\|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + C)\,\mathrm{d}t \\
&\le C\int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^r}^{\frac{2r}{r-3}}\|d\|_{H^2}^2 + C)\,\mathrm{d}t \\
&\le C\int_0^T (\|\nabla d_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^r}^{\frac{2r}{r-3}} + C)\,\mathrm{d}t \le C,
\end{aligned}$$

where we have used the estimates (2.23), (2.10), (2.15) and (2.16). This completes the proof of Lemma 2.3. □

The following two lemmas give some estimates of the velocity u.

Lemma 2.4. *Under assumption* (2.1), *it holds that for* $0 \le T < T^*$,

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \int_0^T \|\rho^{1/2}u_t\|_{L^2}^2\,\mathrm{d}t \le C, \tag{2.24}$$

$$\int_0^T \|u\|_{H^2}^2\,\mathrm{d}t \le C. \tag{2.25}$$

Proof. Notice that the momentum equation (1.2) can be rewritten as

$$\rho u_t + \rho u\cdot\nabla u + \nabla P = \mu\Delta u - \lambda(\nabla d)^T(\Delta d - f(d)). \tag{2.26}$$

We multiply (2.26) by u_t and integrate it over Ω, to obtains the inequality

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \int_\Omega \rho|u_t|^2\,\mathrm{d}x \\
&= -\int_\Omega \rho u\cdot\nabla u u_t\,\mathrm{d}x + \int_\Omega P\,\mathrm{div}\,u_t\,\mathrm{d}x - \lambda\int_\Omega (u_t\cdot\nabla)d(\Delta d - f(d))\,\mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega \rho|u_t|^2\,\mathrm{d}x + 2\int_\Omega \rho|u|^2|\nabla u|^2\,\mathrm{d}x + \int_\Omega P\,\mathrm{div}\,u_t\,\mathrm{d}x \\
&\quad - \lambda\int_\Omega (u_t\cdot\nabla)d(\Delta d - f(d))\,\mathrm{d}x.
\end{aligned} \tag{2.27}$$

Combining the mass conservation equation (1.1) and the assumption (2.1), it follows that the pressure P satisfies the equation

$$P_t + P'(\rho)\nabla\rho\cdot u + P'(\rho)\rho\operatorname{div} u = 0, \tag{2.28}$$

then we have

$$\begin{aligned}\int_\Omega P\operatorname{div} u_t\,\mathrm{d}x &= \frac{d}{dt}\int_\Omega P\operatorname{div} u\,\mathrm{d}x - \int_\Omega P_t\operatorname{div} u\,\mathrm{d}x\\ &= \frac{d}{dt}\int_\Omega P\operatorname{div} u\,\mathrm{d}x + \int_\Omega P'(\rho)(\nabla\rho\cdot u + \rho\operatorname{div} u)\operatorname{div} u\,\mathrm{d}x.\end{aligned} \tag{2.29}$$

Inserting the above equation (2.29) in (2.27), and integrating it over $[0,T]$ give that

$$\begin{aligned}&\|\nabla u\|_{L^2}^2 + \int_0^T\int_\Omega \rho|u_t|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le C + C\int_0^T\int_\Omega \rho|u|^2|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t + C\int_\Omega P(\rho)\operatorname{div} u\,\mathrm{d}x\\ &+ C\int_0^T\int_\Omega P'(\rho)(\nabla\rho\cdot u + \rho\operatorname{div} u)\operatorname{div} u\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \lambda\int_0^T\int_\Omega (u_t\cdot\nabla)d(\Delta d - f(d))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.30}$$

Here, we use the facts $\|\nabla u_0\|_{L^2} \le C < \infty$ and $\int_\Omega P(\rho_0)\operatorname{div} u_0\,\mathrm{d}x \le C < \infty$. To estimate the term $\int_0^T\int_\Omega \rho|u|^2|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t$, we first have

$$\begin{aligned}\int_\Omega \rho|u|^2|\nabla u|^2\,\mathrm{d}x &\le C\int_\Omega |u|^2|\nabla u|^2\,\mathrm{d}x\\ &\le C\|u\|_{L^r}^2\|\nabla u\|_{L^{\frac{2r}{r-2}}}^2 \le C\|u\|_{L^r}^2\|\nabla u\|_{L^2}^{\frac{2r-6}{r}}\|\nabla u\|_{H^1}^{\frac{6}{r}}\\ &\le C\varepsilon\|\nabla u\|_{H^1}^2 + C\varepsilon^{-1}\|u\|_{L^r}^{\frac{2r}{r-3}}\|\nabla u\|_{L^2}^2,\end{aligned} \tag{2.31}$$

where we have used the Hölder's inequality, the interpolation inequality and the Young inequality. By using the standard elliptic estimate on (2.26) yields

$$\begin{aligned}\|u\|_{H^2} &\le C(\|\rho u_t\|_{L^2} + \|\rho u\cdot\nabla u\|_{L^2} + \|\nabla P\|_{L^2} + \|(\nabla d)^T(\Delta d - f(d))\|_{L^2})\\ &\le C(\|\rho^{1/2}u_t\|_{L^2} + \|\rho\|_{L^\infty}\|u\cdot\nabla u\|_{L^2} + \|(\nabla d)^T(d_t + (u\cdot\nabla)d)\|_{L^2}\\ &\le C(\|\rho^{1/2}u_t\|_{L^2} + \|u\|_{L^r}\|\nabla u\|_{L^r}^{\frac{2r}{r-2}} + \|(\nabla d)^T(d_t + (u\cdot\nabla)d)\|_{L^2}\\ &\le C(\|\rho^{1/2}u_t\|_{L^2} + \|u\|_{L^r}\|\nabla u\|_{L^2}^{1-\frac{3}{r}}\|\nabla u\|_{H^1}^{\frac{3}{r}} + \|(\nabla d)^T(d_t + (u\cdot\nabla)d)\|_{L^2}\\ &\le C\Big(\|\rho^{1/2}u_t\|_{L^2} + \sigma\|\nabla u\|_{H^1} + \sigma^{-1}\|u\|_{L^r}^{\frac{r}{r-3}}\|\nabla u\|_{L^2} + \|\nabla\rho\|_{L^2}\\ &\quad + \|\nabla d\|_{L^3}\|d_t\|_{L^6} + \|\nabla d\|_{L^6}^2\|u\|_{L^6}\Big)\\ &\le C\Big\|\rho^{1/2}u_t\|_{L^2} + \sigma\|\nabla u\|_{H^1} + \sigma^{-1}\|u\|_{L^r}^{\frac{r}{r-3}}\|\nabla u\|_{L^2} + \|\nabla\rho\|_{L^2} + \|d_t\|_{L^6}\\ &\quad + \|\nabla u\|_{L^2} + 1\Big)\\ &\le C\Big(\|\rho^{1/2}u_t\|_{L^2} + \sigma\|\nabla u\|_{H^1} + \sigma^{-1}\|u\|_{L^r}^{\frac{r}{r-3}}\|\nabla u\|_{L^2} + \|\nabla\rho\|_{L^2}\\ &\quad + \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + 1\Big),\end{aligned}$$

where we have used the estimates (2.6), (2.9), (2.10) and (2.14) in the last inequality. Taking σ small enough yields

$$\|u\|_{H^2} \le C(\|\rho^{1/2}u_t\|_{L^2}+\|u\|_{L^r}^{\frac{r}{r-3}}\|\nabla u\|_{L^2}+\|\nabla\rho\|_{L^2}+\|\nabla d_t\|_{L^2}+\|\nabla u\|_{L^2}+1). \quad (2.32)$$

Inserting estimate (2.32) into (2.31), it follows that

$$\begin{aligned}
\int_0^T\int_\Omega \rho|u|^2|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \int_0^T(\varepsilon\|\nabla u\|_{H^1}^2 + C\varepsilon^{-1}\|u\|_{L^r}^{\frac{2r}{r-3}}\|\nabla u\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\varepsilon\int_0^T(\|\rho^{1/2}u_t\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2+1)\,\mathrm{d}t\\
&\quad + C\varepsilon^{-1}\int_0^T(\|u\|_{L^r}^{\frac{2r}{r-3}}+1)\|\nabla u\|_{L^2}^2\,\mathrm{d}t \qquad (2.33)\\
&\le C\varepsilon\int_0^T(\|\rho^{1/2}u_t\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2)\,\mathrm{d}t\\
&\quad + C\varepsilon^{-1}\int_0^T(\|u\|_{L^r}^{\frac{2r}{r-3}}+1)\|\nabla u\|_{L^2}^2\,\mathrm{d}t + C,
\end{aligned}$$

where we have used the estimate (2.16). For the term $-\lambda\int_0^T\int_\Omega(u_t\cdot\nabla)d(\Delta d - f(d))\,\mathrm{d}x\,\mathrm{d}t$, we have

$$\begin{aligned}
&-\int_\Omega(u_t\cdot\nabla)d(\Delta d-f(d))\,\mathrm{d}x\\
&=\sum_{i,j=1}^3\Big(\int_\Omega\partial_j u_{it}\partial_i d\partial_j d\,\mathrm{d}x+\int_\Omega u_{it}\partial_i\partial_j d\partial_j d\,\mathrm{d}x\Big)+\int_\Omega u_t\cdot\nabla d f(d)\,\mathrm{d}x\\
&=\sum_{i,j=1}^3\Big(\int_\Omega\partial_j u_{it}\partial_i d\partial_j d\,\mathrm{d}x-\frac12\int_\Omega\partial_i u_{it}|\partial_j d|^2\,\mathrm{d}x\Big)-\sum_{i=1}^3\int_\Omega\partial_i u_{it}\Big(\frac{|d|^4}{4}-\frac{|d|^2}{2}\Big)\,\mathrm{d}x\\
&=\sum_{i,j=1}^3\Big\{\frac{d}{dt}\int_\Omega\Big(\partial_j u_i\partial_i d\partial_j d-\frac12\partial_i u_i|\partial_j d|^2\Big)\,\mathrm{d}x-\int_\Omega\partial_j u_i\partial_i d_t\partial_j d\,\mathrm{d}x\\
&\quad-\int_\Omega\partial_j u_i\partial_i d\partial_j d_t\,\mathrm{d}x+\int_\Omega\partial_i u_i\partial_j d_t\partial_j d\,\mathrm{d}x\\
&\quad-\frac{d}{dt}\int_\Omega\Big(\frac{\partial_i u_i|d|^4}{4}-\frac{\partial_i u_i|d|^2}{2}\Big)\,\mathrm{d}x+\int_\Omega\partial_i u_i(|d|^3 d_t-|d|d_t)\,\mathrm{d}x\Big\}\\
&\le\sum_{i,j=1}^3\frac{d}{dt}\int_\Omega\Big(\partial_j u_i\partial_i d\partial_j d-\frac12\partial_i u_i|\partial_j d|^2-\frac14\partial_i u_i|d|^4+\frac12\partial_i u_i|d|^2\Big)\,\mathrm{d}x\\
&\quad+C\|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}+C\|\nabla u\|_{L^2}\|d_t\|_{L^2}(\|d\|_{L^\infty}^2+1)\|d\|_{L^\infty}\\
&\le\sum_{i,j=1}^3\frac{d}{dt}\int_\Omega\Big(\partial_j u_i\partial_i d\partial_j d-\frac12\partial_i u_i|\partial_j d|^2-\frac14\partial_i u_i|d|^4+\frac12\partial_i u_i|d|^2\Big)\,\mathrm{d}x\\
&\quad+C\|\nabla u\|_{L^2}^2+C\|\nabla d_t\|_{L^2}^2+C\|d_t\|_{L^2}^2\\
&=\frac{d}{dt}\int_\Omega\Big((\nabla u\cdot\nabla d)\nabla d-\frac12\operatorname{div}u|\nabla d|^2-\frac14\operatorname{div}u|d|^4+\frac12\operatorname{div}u|d|^2\Big)\,\mathrm{d}x\\
&\quad+C(\|\nabla u\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2+\|d_t\|_{L^2}^2), \qquad (2.34)
\end{aligned}$$

where we have used estimate (2.10) in the last inequality. Integrating (2.34) over $[0,T]$, and by using Lemmas 2.1–2.3 give

$$\begin{aligned}
&-\lambda\int_0^T\int_\Omega (u_t\cdot\nabla)d(\Delta d-f(d))\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_\Omega(|\nabla u||\nabla d|^2+|\operatorname{div}u|(|\nabla d|^2+|d|^4+|d|^2))\,\mathrm{d}x\\
&\quad+C\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2+\|d_t\|_{L^2}^2)\,\mathrm{d}t\\
&\le C\|\nabla u\|_{L^2}\|\nabla d\|_{L^4}^2+C\|\nabla u\|_{L^2}(\|\nabla d\|_{L^4}^2+\|d\|_{L^8}^4+\|d\|_{L^4}^2)+C\\
&\le \frac14\|\nabla u\|_{L^2}^2+C
\end{aligned}\tag{2.35}$$

To estimate the remaining term of the right side of estimate (2.30), by using Lemma 2.1, the estimates (2.9) and (2.10), we obtain

$$\int_\Omega P(\rho)\operatorname{div}u\,\mathrm{d}x\le\frac14\|\nabla u\|_{L^2}^2+C\int_\Omega|P(\rho)|^2\,\mathrm{d}x\le\frac14\|\nabla u\|_{L^2}^2+C;\tag{2.36}$$

$$\begin{aligned}
&\int_0^T\int_\Omega P'(\rho)(\nabla\rho\cdot u)\operatorname{div}u\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_0^T\|P'(\rho)\|_{L^3}\|\nabla\rho\|_{L^2}\|u\|_{L^6}\|\operatorname{div}u\|_{L^\infty}\,\mathrm{d}t\\
&\le C\int_0^T\|P'(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^2}\|u\|_{L^6}\|\operatorname{div}u\|_{L^\infty}\,\mathrm{d}t\\
&\le C\int_0^T\|\nabla\rho\|_{L^2}\|\nabla u\|_{L^2}\|\operatorname{div}u\|_{L^\infty}\,\mathrm{d}t\\
&\le C\int_0^T\|\nabla\rho\|_{L^2}^2\|\operatorname{div}u\|_{L^\infty}\,\mathrm{d}t+C\int_0^T\|\nabla u\|_{L^2}^2\|\operatorname{div}u\|_{L^\infty}\,\mathrm{d}t\\
&\le C\int_0^T\|\nabla\rho\|_{L^2}^2\|\nabla u\|_{L^\infty}\,\mathrm{d}t+\int_0^T\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^\infty}\,\mathrm{d}t;
\end{aligned}\tag{2.37}$$

$$\int_0^T\int_\Omega P'(\rho)\rho|\operatorname{div}u|^2\,\mathrm{d}x\,\mathrm{d}t\le C\int_0^T\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^2\,\mathrm{d}t\le C\int_0^T\|\nabla u\|_{L^2}^2\,\mathrm{d}t\le C;\tag{2.38}$$

Inserting estimates (2.33), (2.35)–(2.38) into (2.30), we obtain

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2+\int_0^T\int_\Omega\rho|u_t|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\varepsilon\int_0^T(\|\rho^{1/2}u_t\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2)\,\mathrm{d}t+C\varepsilon^{-1}\int_0^T(\|u\|_{L^r}^{\frac{2r}{r-3}}+1)\|\nabla u\|_{L^2}^2\,\mathrm{d}t\\
&\quad+C\int_0^T\|\nabla\rho\|_{L^2}^2\|\nabla u\|_{L^\infty}\,\mathrm{d}t+C.
\end{aligned}\tag{2.39}$$

Now, applying the operator ∇ to the mass conservation equation (1.1), then multiplying it by $\nabla\rho$ and integrating it over Ω yield

$$\frac{d}{dt}\|\nabla\rho\|_{L^2}^2=-\int_\Omega|\nabla\rho|^2\operatorname{div}u\,\mathrm{d}x-2\int_\Omega\rho\nabla\rho\nabla\operatorname{div}u\,\mathrm{d}x-2\int_\Omega(\nabla\rho\cdot\nabla u)\nabla\rho\,\mathrm{d}x$$

$$\le C\|\nabla\rho\|_{L^2}^2\|\nabla u\|_{L^\infty} + C\|\nabla\rho\|_{L^2}\|\nabla \operatorname{div} u\|_{L^2};$$

that is,

$$\frac{d}{dt}\|\nabla\rho\|_{L^2} \le C\|\nabla\rho\|_{L^2}\|\nabla u\|_{L^\infty} + C\|u\|_{H^2}. \tag{2.40}$$

By applying the Gronwall's inequality, for $t \in (0, T]$, we obtain

$$\begin{aligned}\|\nabla\rho\|_{L^2} &\le (\|\rho_0\|_{H^1} + \int_0^t \|u\|_{H^2}\,\mathrm{d}\tau)\exp(C\int_0^t \|\nabla u\|_{L^\infty}\,\mathrm{d}\tau)\\ &\le C(1 + \int_0^t \|u\|_{H^2}\,\mathrm{d}\tau).\end{aligned} \tag{2.41}$$

Inserting (2.32) into (2.41), for $\xi > 0$, gives

$$\begin{aligned}\xi\|\nabla\rho\|_{L^2}^2 &\le C\xi(1 + \int_0^T (\|\rho^{1/2}u_t\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\,\mathrm{d}t)\\ &\le C\xi(1 + \int_0^T (\|\rho^{1/2}u_t\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2)\,\mathrm{d}t),\end{aligned} \tag{2.42}$$

where we have used the estimate (2.16) again. Combining (2.39) and (2.42), and taking $\varepsilon < \xi$ small enough, it follows that

$$\begin{aligned}&\|\nabla u\|_{L^2}^2 + \frac{\xi}{2}\|\nabla\rho\|_{L^2}^2 + \frac{1}{2}\int_0^T\int_\Omega \rho|u_t|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(1+\xi)\int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2)(\|\nabla u\|_{L^\infty} + \|u\|_{L^r}^{\frac{2r}{r-3}} + \|u\|_{L^\infty}^2 + 1)\,\mathrm{d}t + C.\end{aligned} \tag{2.43}$$

By using the Gronwall's inequality and noticing that the assumption (2.1), we can deduce that

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \int_0^T\int_\Omega \rho|u_t|^2\,\mathrm{d}x\,\mathrm{d}t \le C.$$

For (2.25), it follows from (2.32) that

$$\int_0^T \|u\|_{H^2}^2\,\mathrm{d}t \le C\int_0^T (\|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + 1)\,\mathrm{d}t \le C,$$

where we have used the estimates (2.10), (2.16) and (2.24) in the last inequality above. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Under the assumption* (2.1)*, it holds that for* $0 \le T < T^*$,

$$\begin{aligned}&\sup_{0\le t\le T}(\|\rho^{1/2}u_t\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)\\ &+ \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2)\,\mathrm{d}t \le C.\end{aligned} \tag{2.44}$$

Proof. Differentiating equation (2.26) with respect to time, multiplying the resulting equation by u_t, integrating it over Ω, and using the equation (1.1) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_t|^2\,\mathrm{d}x + \mu\int_\Omega |\nabla u_t|^2\,\mathrm{d}x - \int_\Omega P_t \operatorname{div} u_t\,\mathrm{d}x \\
&= -\int_\Omega \rho u\cdot\nabla(|u_t|^2 + (u\cdot\nabla)uu_t) + \rho(u_t\cdot\nabla)uu_t\,\mathrm{d}x \\
&\quad - \lambda\int_\Omega (u_t\cdot\nabla)d_t(\Delta d - f(d))\,\mathrm{d}t - \lambda\int_\Omega (u_t\cdot\nabla)d(\Delta d - f(d))_t\,\mathrm{d}x \\
&= \int_\Omega \nabla\rho\cdot u|u_t|^2 + \rho\operatorname{div} u|u_t|^2 - \rho u\cdot\nabla((u\cdot\nabla)uu_t) - \rho(u_t\cdot\nabla)uu_t\,\mathrm{d}x \\
&\quad - \lambda\int_\Omega (u_t\cdot\nabla)d_t(\Delta d - f(d))\,\mathrm{d}t - \lambda\int_\Omega (u_t\cdot\nabla)d(\Delta d - f(d))_t\,\mathrm{d}x.
\end{aligned} \tag{2.45}
$$

Differentiating the liquid crystal equation (1.3) with respect to time gives

$$
(u_t\cdot\nabla)d = \nu(\Delta d - f(d))_t - d_{tt} - (u\cdot\nabla)d_t. \tag{2.46}
$$

Multiplying the above equality with $(\Delta d - f(d))_t$ and integrating it over Ω, one obtains the equality

$$
\begin{aligned}
&\int_\Omega (u_t\cdot\nabla)d(\Delta d - f(d))_t\,\mathrm{d}x \\
&= \int_\Omega (\nu|(\Delta d - f(d))_t|^2 - d_{tt}\Delta d_t + d_{tt}f(d)_t - (u\cdot\nabla)d_t(\Delta d - f(d))_t)\,\mathrm{d}x \\
&= \int_\Omega \nu|(\Delta d - f(d))_t|^2\,\mathrm{d}x + \frac{1}{2}\frac{d}{dt}\|\nabla d_t\|_{L^2}^2 - \int_\Omega ((u_t\cdot\nabla)d)f(d)_t\,\mathrm{d}x \\
&\quad + \int_\Omega \nu f(d)_t(\Delta d - f(d))_t\,\mathrm{d}x - \int_\Omega ((u\cdot\nabla)d_t)(\Delta d - f(d))_t\,\mathrm{d}x \\
&\quad - \int_\Omega ((u\cdot\nabla)d_t)f(d)_t\,\mathrm{d}x.
\end{aligned} \tag{2.47}
$$

From (2.28), we have

$$
\int_\Omega P_t \operatorname{div} u_t\,\mathrm{d}x = -\int_\Omega P'(\rho)(\nabla\rho\cdot u + \rho\operatorname{div} u)\operatorname{div} u_t\,\mathrm{d}x. \tag{2.48}
$$

Inserting (2.47) and (2.48) into (2.45) gives

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (\frac{1}{2}\rho|u_t|^2+\lambda|\nabla d_t|^2)\,\mathrm{d}x+\mu\|\nabla u_t\|_{L^2}^2+\lambda\nu\|(\Delta d-f(d))_t\|_{L^2}^2\\
&\le C\int_\Omega \Big(|\nabla\rho||u||u_t|^2+\rho|\operatorname{div}u||u_t|^2+\rho|u||u_t||\nabla u|^2+\rho|u|^2|u_t||\nabla^2 u|\\
&\quad +\rho|u|^2|\nabla u||\nabla u_t|+\rho|u_t|^2|\nabla u|\Big)\,\mathrm{d}x+C\int_\Omega\Big(|(u_t\cdot\nabla)df(d)_t|\\
&\quad +|(\Delta d-f(d))_tf(d)_t|+|(u\cdot\nabla)d_t(\Delta d-f(d))_t|+|((u\cdot\nabla)d_t)f(d)_t|\Big)\,\mathrm{d}x \qquad (2.49)\\
&\quad +C\int_\Omega |(u_t\cdot\nabla)d_t(\Delta d-f(d))|\,\mathrm{d}x\\
&\quad +C\int_\Omega |P'(\rho)||\nabla\rho||u||\operatorname{div}u_t|+\rho|P'(\rho)||\operatorname{div}u||\operatorname{div}u_t|\,\mathrm{d}x\\
&=\sum_{j=1}^{13}J_j.
\end{aligned}
$$

We will estimate J_j term by term. In the following calculations, we will make extensive use of Sobolev embedding, Hölder inequality, Lemmas 2.1–2.4, and estimate (2.5)

$$
J_1\le C\|\nabla\rho\|_{L^2}\|u_t\|_{L^6}^2\|u\|_{L^6}\le C\|\nabla u_t\|_{L^2}^2\|\nabla u\|_{L^2}\le\varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1};
$$

$$
J_2=\int_\Omega\rho|\operatorname{div}u||u_t|^2\,\mathrm{d}x\le C\|\nabla u\|_{L^\infty}\|\rho^{1/2}u_t\|_{L^2}^2;
$$

$$
\begin{aligned}
J_3&=\int_\Omega\rho|u||\nabla u|^2|u_t|\,\mathrm{d}x\le C\|\rho\|_{L^\infty}\|u\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\\
&\le C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|u\|_{H^2}\le\varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}\|u\|_{H^2}^2;
\end{aligned}
$$

$$
\begin{aligned}
J_4&=\int_\Omega\rho|u|^2|\nabla^2u||u_t|\,\mathrm{d}x\le C\|\rho\|_{L^\infty}\|u_t\|_{L^6}\|u\|_{L^6}^2\|\nabla^2u\|_{L^2}\\
&\le C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|u\|_{H^2}\le\varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}\|u\|_{H^2}^2;
\end{aligned}
$$

$$
\begin{aligned}
J_5&=\int_\Omega\rho|u|^2|\nabla u||\nabla u_t|\,\mathrm{d}x\le C\|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\le C\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2}^2\|u\|_{H^2}\le\varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}\|u\|_{H^2}^2;
\end{aligned}
$$

$$
\begin{aligned}
J_6&=\int_\Omega\rho|u_t|^2|\nabla u|\,\mathrm{d}x\\
&\le C\|\rho^{1/2}u_t\|_{L^3}^2\|\nabla u\|_{L^3}\\
&\le C\|\rho^{1/2}u_t\|_{L^2}\|\rho^{1/2}u_t\|_{L^6}\|\nabla u\|_{L^2}^{1/2}\|\nabla^2u\|_{L^2}^{1/2}\\
&\le C\|\rho^{1/2}u_t\|_{L^2}\|\rho^{1/2}\|_{L^\infty}\|u_t\|_{L^6}\|\nabla u\|_{L^2}^{1/2}\|\nabla^2u\|_{L^2}^{1/2}\\
&\le C\|\rho^{1/2}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|\nabla^2u\|_{L^2}^{1/2}\\
&\le\varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}\|\rho^{1/2}u_t\|_{L^2}^2\|u\|_{H^2}\\
&\le\varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}\|\rho^{1/2}u_t\|_{L^2}^2(\|u\|_{H^2}^2+1);
\end{aligned}
$$

$$\begin{aligned}
J_7 &\le C\|u_t\|_{L^2}\|\nabla d\|_{L^3}(\|d\|_{L^\infty}^2+1)\|d_t\|_{L^6}\\
&\le C\|\rho^{1/2}u_t\|_{L^2}\|\nabla d_t\|_{L^2}\le \varepsilon\|\rho^{1/2}u_t\|_{L^2}^2+C\varepsilon^{-1}\|\nabla d_t\|_{L^2}^2;\\
J_8 &\le C\|(\Delta d-f(d))_t\|_{L^2}\|f(d)_t\|_{L^2}\le \varepsilon\|(\Delta d-f(d))_t\|_{L^2}^2+C\varepsilon^{-1}\|d_t\|_{L^2}^2;\\
J_9 &= \int_\Omega |(u\cdot)d_t(\Delta d-f(d))_t|\,\mathrm{d}x\le C\|(\Delta d-f(d))_t\|_{L^2}\||u||\nabla d_t|\|_{L^2}\\
&\le \varepsilon\|(\Delta d-f(d))_t\|_{L^2}^2+C\varepsilon^{-1}\|u\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2;\\
J_{10} &= \int_\Omega |((u\cdot\nabla)d_t)f(d)_t|\,\mathrm{d}x\\
&= \int_\Omega |((u\cdot\nabla)d_t)(|d|^2d-d)_t|\,\mathrm{d}x\\
&\le C\int_\Omega |u||\nabla d_t|(|d|^2|d_t|+|d_t|)\,\mathrm{d}x\\
&\le C(\|\nabla d_t\|_{L^2}\|d\|_{L^\infty}^2\||u||d_t|\|_{L^2}+\|\nabla d_t\|_{L^2}\||u||d_t|\|_{L^2})\\
&\le \frac{\varepsilon}{2}\|\nabla d_t\|_{L^2}^2+C\varepsilon^{-1}\||u||d_t|\|_{L^2}^2\le \frac{\varepsilon}{2}\|\nabla d_t\|_{L^2}^2+C\varepsilon^{-1}\|u\|_{L^r}^2\|d_t\|_{L^{\frac{2r}{r-2}}}^2\\
&\le \frac{\varepsilon}{2}\|\nabla d_t\|_{L^2}^2+C\varepsilon^{-1}\|u\|_{L^r}^2\|d_t\|_{L^2}^{\frac{2(r-3)}{r}}\|\nabla d_t\|_{L^2}^{\frac{6}{r}}\\
&\le \varepsilon\|\nabla d_t\|_{L^2}^2+C\varepsilon^{-2}\|u\|_{L^r}^{\frac{2r}{r-3}}\|d_t\|_{L^2}^2;\\
J_{11} &\le C\|u_t\|_{L^6}\|\nabla d_t\|_{L^2}\|\Delta d\|_{L^3}+C\|u_t\|_{L^2}\|\nabla d_t\|_{L^2}(\|d\|_{L^\infty}^2+1)\|d\|_{L^\infty}\\
&\le C\|\nabla u_t\|_{L^2}\|\nabla d_t\|_{L^2}\|d\|_{H^2}^{1/2}\|\nabla d\|_{H^2}^{1/2}+C\|\rho^{1/2}u_t\|_{L2}\|\nabla d_t\|_{L^2}\\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}\|\nabla d_t\|_{L^2}^2(\|\nabla d\|_{H^2}^2+1)+\varepsilon\|\rho^{1/2}u_t\|_{L^2}^2+C\varepsilon^{-1}\|\nabla d_t\|_{L^2}^2\\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2+\varepsilon\|\rho^{1/2}u_t\|_{L^2}^2+C\varepsilon^{-1}\|\nabla d_t\|_{L^2}^2(\|\nabla d\|_{H^2}^2+1);\\
J_{12} &\le C\|\nabla\rho\|_{L^2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\le C\|\nabla u_t\|_{L^2}\|u\|_{H^2}\\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}\|u\|_{H^2}^2,\\
J_{13} &\le C\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\le \varepsilon\|\nabla u_t\|_{L^2}^2+C\varepsilon^{-1}.
\end{aligned}$$

Substituting all the estimates of J_j into (2.49), and taking ε small enough, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega(\rho|u_t|^2+|\nabla d_t|^2)\,\mathrm{d}x+\|\nabla u_t\|_{L^2}^2+\|(\Delta d-f(d))_t\|_{L^2}^2\\
&\le C\|\rho^{1/2}u_t\|_{L^2}^2(\|\operatorname{div}u\|_{L^\infty}+\|u\|_{H^2}^2+C)+C\|\nabla d_t\|_{L^2}^2(\|u\|_{L^\infty}^2\\
&\quad+\|\nabla d\|_{H^2}^2+C)+C\|d_t\|_{L^2}^2(\|u\|_{L^r}^{\frac{2r}{r-3}}+1)+C\|u\|_{H^2}^2+C\\
&\le C(\|\rho^{1/2}u_t\|_{L^2}^2+\|d_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2)(\|\nabla u\|_{L^\infty}+\|u\|_{L^\infty}^2\\
&\quad+\|u\|_{L^r}^{\frac{2r}{r-3}}+\|u\|_{H^2}^2+\|\nabla d\|_{H^2}^2+C)+C\|u\|_{H^2}^2+C.
\end{aligned}\tag{2.50}$$

To estimate the terms $\|d_t\|_{L^2}$, by multiplying (2.46) with d_t, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |d_t|^2 \mathrm{d}x + \nu\|\nabla d_t\|_{L^2}^2 \\
&= -\int_\Omega (u_t\cdot\nabla)d d_t\,\mathrm{d}x - \int_\Omega (u\cdot\nabla)d_t d_t\,\mathrm{d}x - \nu\int_\Omega f(d)_t d_t\,\mathrm{d}x \\
&= -\int_\Omega (u_t\cdot\nabla)d d_t\,\mathrm{d}x + \int_\Omega \operatorname{div} u\frac{|d_t|^2}{2}\,\mathrm{d}x - \nu\int_\Omega f(d)_t d_t\,\mathrm{d}x \\
&\le C(\|u_t\|_{L^6}\|\nabla d\|_{L^3}\|d_t\|_{L^2} + \|\nabla u\|_{L^\infty}\|d_t\|_{L^2}^2 + \|d\|_{L^\infty}^2\|d_t\|_{L^2}^2 + \|d_t\|_{L^2}^2) \\
&\le \varepsilon\|\nabla u_t\|_{L^2}^2 + C\varepsilon^{-1}\|d_t\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|d_t\|_{L^2}^2,
\end{aligned}
\tag{2.51}
$$

where we have used the estimate (2.14) and the Cauchy inequality. Combining the estimates (2.50) and (2.51) together, and let ε small enough, it follows that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (\rho|u_t|^2 + |d_t|^2 + |\nabla d_t|^2)\,\mathrm{d}x + \|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2 \\
&\le C(\|\rho^{1/2}u_t\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2)(\|\nabla u\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u\|_{L^r}^{\frac{2r}{r-3}} + \|u\|_{H^2}^2 \\
&\quad + \|\nabla d\|_{H^2}^2 + C) + C\|u\|_{H^2}^2 + C.
\end{aligned}
$$

Applying the Gronwall's inequality to the above estimate, we deduce

$$
\begin{aligned}
&\sup_{0\le t\le T}\int_\Omega (\rho|u_t|^2 + |d_t|^2 + |\nabla d_t|^2)\,\mathrm{d}x + \int_0^T \|\nabla u_t\|_{L^2}^2 + \|(\Delta d - f(d))_t\|_{L^2}^2\,\mathrm{d}t \\
&\le C\int_0^T (\|u\|_{H^2}^2 + 1)\,\mathrm{d}t \exp\{C\int_0^T (\|\nabla u\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u\|_{L^r}^{\frac{2r}{r-3}} \\
&\quad + \|u\|_{H^2}^2 + \|\nabla d\|_{H^2}^2 + C)\,\mathrm{d}t\} \le C,
\end{aligned}
\tag{2.52}
$$

where we have used estimates (2.17) and (2.25), and the assumption (2.1) in the last inequality. This completes the proof of Lemma 2.5. □

The following lemma gives the higher order norm estimates of u, d and ρ.

Lemma 2.6. *Under assumption* (2.1)*, it holds that for* $0 \le T < T^*$*,*

$$
\sup_{0\le t\le T}(\|u\|_{H^2} + \|\nabla d\|_{H^2}) \le C; \tag{2.53}
$$

$$
\sup_{0\le t\le T}\|\nabla\rho\|_{L^6} + \int_0^T \|\nabla^2 u\|_{L^6}^2\,\mathrm{d}t \le C. \tag{2.54}
$$

Proof. From estimates (2.6), (2.24) and (2.44), we have

$$
\|u\|_{H^2} \le C(\|\rho^{1/2}u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla\rho\|_{L^2} + \|\nabla d_t\|_{L^2}) \le C. \tag{2.55}
$$

By the standard elliptic regularity result for the liquid crystal equation (1.3), one obtains

$$
\begin{aligned}
\|\nabla^3 d\|_{L^2} &\le C(\|\nabla d_t\|_{L^2} + \|\nabla(u\cdot\nabla d)\|_{L^2} + \|\nabla f(d)\|_{L^2} + \|d_0\|_{H^3}) \\
&\le C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty} + \||u||\nabla^2 d|\|_{L^2} \\
&\quad + \|\nabla d\|_{L^2}(\|d\|_{L^\infty}^2 + 1) + \|d_0\|_{H^3}) \\
&\le C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \||u||\nabla^2 d|\|_{L^2} + C).
\end{aligned}
\tag{2.56}
$$

Notice that

$$\begin{aligned}\||u||\nabla^2 d|\|_{L^2} &\le C\|u\|_{L^6}\|\nabla^2 d\|_{L^3} \le C\|\nabla u\|_{L^2}\|\nabla d\|_{L^6}^{1/2}\|\nabla^3 d\|_{L^2}^{1/2}\\ &\le C\|\nabla u\|_{L^2}\|\nabla^3 d\|_{L^2}^{1/2} \le \eta\|\nabla^3 d\|_{L^2} + C\eta^{-1}\|\nabla u\|_{L^2}^2\\ &\le \eta\|\nabla^3 d\|_{L^2} + C\eta^{-1},\end{aligned}$$

where we have used the estimates (2.14) and (2.24). Taking η small enough, the above estimate and (2.56) imply that

$$\||u||\nabla^2 d|\|_{L^2} \le C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + C).$$

Hence

$$\|\nabla^3 d\|_{L^2} \le C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + C),$$

and

$$\begin{aligned}\|\nabla d\|_{H^2} &\le C(\|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^2})\\ &\le C(\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla d\|_{L^2} + C) \le C,\end{aligned} \tag{2.57}$$

where we have used the estimates (2.14), (2.24) and (2.44) in the last inequality. Combining the estimates (2.55) and (2.57) above gives the estimate (2.53).

To prove (2.54). Applying the operator ∇ to the mass conservation equation (1.1), then multiplying the resulting equation by $6|\nabla\rho|^4\nabla\rho$ and integrating it over Ω give

$$\begin{aligned}&\frac{d}{dt}\|\nabla\rho\|_{L^6}^6\\ &= -6\int_\Omega |\nabla\rho|^6\nabla u\,\mathrm{d}x - \int_\Omega \nabla(|\nabla\rho|^6)\cdot u\,\mathrm{d}x - 6\int_\Omega |\nabla\rho|^6 \operatorname{div} u\,\mathrm{d}x\\ &\quad - 6\int_\Omega \rho|\nabla\rho|^4\nabla\rho\nabla\operatorname{div} u\,\mathrm{d}x\\ &= -6\int_\Omega |\nabla\rho|^6\nabla u\,\mathrm{d}x - 5\int_\Omega |\nabla\rho|^6\operatorname{div} u\,\mathrm{d}x - 6\int_\Omega \rho|\nabla\rho|^4\nabla\rho\nabla\operatorname{div} u\,\mathrm{d}x\\ &\le C\|\operatorname{div} u\|_{L^\infty}\|\nabla\rho\|_{L^6}^6 + C\|\rho\|_{L^\infty}\|\nabla\operatorname{div} u\|_{L^6}\|\nabla\rho\|_{L^6}^5\\ &\le C\|\operatorname{div} u\|_{L^\infty}\|\nabla\rho\|_{L^6}^6 + C\|\nabla\operatorname{div} u\|_{L^6}\|\nabla\rho\|_{L^6}^5;\end{aligned}$$

that is,

$$\frac{d}{dt}\|\nabla\rho\|_{L^6} \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^6} + C\|\nabla\operatorname{div} u\|_{L^6}. \tag{2.58}$$

Using the Gronwall's inequality in the above estimate gives

$$\begin{aligned}\|\nabla\rho\|_{L^6} &\le (\|\rho_0\|_{W^{1,6}} + C\int_0^T \|\nabla\operatorname{div} u\|_{L^6}\,\mathrm{d}t)\exp\{C\int_0^T \|\nabla u\|_{L^\infty}\,\mathrm{d}t\}\\ &\le C(\int_0^T \|\nabla^2 u\|_{L^6}\,\mathrm{d}t + 1).\end{aligned} \tag{2.59}$$

Applying the standard elliptic regularity result $\|\nabla^2 u\|_{L^6} \le C\|\Delta u\|_{L^6}$, Hölder inequality, Sobolev embedding, the estimates (2.10) and (2.46), we have

$$\begin{aligned}
\|\nabla^2 u\|_{L^6} &\le C(\|\rho u_t\|_{L^6} + \|\rho u\cdot\nabla u\|_{L^6} + \|\nabla P\|_{L^6} + \|(\nabla d)^T(\Delta d - f(d))\|_{L^6}) \\
&\le C(\|\nabla u_t\|_{L^2} + \|u\|_{L^\infty}\|\nabla u\|_{L^6} + \|\nabla\rho\|_{L^6} + \|\nabla d\|_{L^\infty}\|\Delta d\|_{L^2} \\
&\quad + \|\nabla d\|_{L^6}\|f(d)\|_{L^\infty}) \\
&\le C(\|\nabla u_t\|_{L^2} + \|u\|_{H^2}^2 + \|\nabla\rho\|_{L^6} + \|d\|_{H^3}^2 + \|d\|_{H^2}) \\
&\le C(\|\nabla u_t\|_{L^2} + \|\nabla\rho\|_{L^6} + 1),
\end{aligned} \tag{2.60}$$

where we have used estimate (2.53) and the fact that $\|d\|_{H^3} \le C(\|\nabla d\|_{H^2} + \|d\|_{L^2})$. Inserting the estimate (2.60) into (2.59) yields

$$\begin{aligned}
\|\nabla\rho\|_{L^6} &\le C\int_0^T (\|\nabla u_t\|_{L^2} + \|\nabla\rho\|_{L^6} + 1)\,\mathrm{d}t \\
&\le C\int_0^T (\|\nabla u_t\|_{L^2}^2 + \|\nabla\rho\|_{L^6} + 1)\,\mathrm{d}t \\
&\le C\int_0^T \|\nabla\rho\|_{L^6}\,\mathrm{d}t + C,
\end{aligned}$$

where we have used the estimate (2.38), then applying the Gronwall's inequality, we have

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^6} \le C. \tag{2.61}$$

From (2.60) and (2.61), we have

$$\int_0^T \|\nabla^2 u\|_{L^6}^2\,\mathrm{d}t \le C\Big(\int_0^T \|\nabla u_t\|_{L^2}^2\,\mathrm{d}t + \sup_{0\le t\le T}\|\nabla\rho\|_{L^6}^2 + C\Big) \le C. \tag{2.62}$$

It is easy to show that the estimate (2.54) follows from (2.59) and (2.62) immediately. This completes the proof of Lemma 2.6. $\square$

Proof of Theorem 1.2. From the existence result of Theorem 1.1, we know $\|u(t)\|_{H^2}$, $\|\rho(t)\|_{W^{1,6}}$, $\|d(t)\|_{H^3}$ and $\|\rho^{1/2}u_t(t)\|_{L^2}$ are all continuous on time interval $[0, T^*)$. From the above Lemmas 2.1–2.6, we known there holds

$$\begin{aligned}
&(\|u\|_{H^2}, \|\rho\|_{W^{1,6}}, \|d\|_{H^3}, \|\rho^{1/2}u_t\|_{L^2})|_{t=T^*} \\
&= \lim_{t\to T^*}(\|u(t)\|_{H^2}, \|\rho(t)\|_{W^{1,6}}, \|d(t)\|_{H^3}, \|\rho^{1/2}u_t(t)\|_{L^2}) \le C.
\end{aligned} \tag{2.63}$$

Furthermore,

$$\rho^{1/2}u_t + \rho^{1/2}u\cdot\nabla u \in L^\infty([0,T^*]; L^2) \tag{2.64}$$

and for all $T \in (0, T^*]$,

$$(\mu\Delta u - \lambda\,\mathrm{div}(\nabla d\odot\nabla d - (\frac{1}{2}|\nabla d|^2 + F(d))I) - \nabla P)(T) = (\rho u_t + \rho u\cdot\nabla u)(T) = \sqrt{\rho}g(T) \tag{2.65}$$

where $g(T) := (\rho^{1/2}u_t + \rho^{1/2}u\cdot\nabla u)(\cdot, T) \in L^2$. Therefore, from (2.64) and (2.65), we can take $(\rho, u, d)|_{t=T^*}$ as the initial data and apply Theorem 1.1 to extend the local strong solution beyond T^*, this contradicts with the maximality of T^*, hence, the assumption (2.1) does not hold. This completes the proof of Theorem 1.2. $\square$

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