Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 110, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND UNIQUENESS OF STATIONARY SOLUTIONS TO BIOCONVECTIVE FLOW EQUATIONS

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Abstract. We analyze a system of nonlinear partial differential equations modeling the stationary flow induced by the upward swimming of certain microorganisms in a fluid. We consider the realistic case in which the effective viscosity of the fluid depends on the concentration of such microorganisms. Under certain conditions, we prove the existence and uniqueness of solutions for such generalized bioconvective flow equations

1. Introduction

In this work we perform a mathematical analysis of the system of partial differential equations

$$\begin{gathered} -2\operatorname{div}(\nu(m)D(\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla q = -m\cdot\chi + \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \\ -\theta\Delta m + \mathbf{u}\cdot\nabla m + U\frac{\partial m}{\partial x_3} = 0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

subject to the boundary and total amount conditions

$$\begin{gathered} \mathbf{u} = \mathbf{0} \quad \text{on } S, \\ \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma, \\ \nu(m)[D(\mathbf{u})\mathbf{n} - \mathbf{n}\cdot(D(\mathbf{u})\mathbf{n})\mathbf{n}] = \mathbf{b}_1 \quad \text{on } \Gamma, \\ \theta\frac{\partial m}{\partial \mathbf{n}} - Un_3 m = 0 \quad \text{on } \partial\Omega, \\ \int_\Omega m\,dx = \alpha. \end{gathered} \tag{1.2}$$

This system is a mathematical model for the stationary flow induced by the upward swimming of certain microorganisms in a fluid in the realistic situation that the concentration of such microorganisms may affect the effective viscosity of the fluid. The last condition in (1.2) fix the total mass of microorganism as α, which is a given positive constant.

2000 *Mathematics Subject Classification.* 35Q80, 76Z10, 92B05.
Key words and phrases. Bioconvective flow; stationary solutions.

Submitted January 17, 2013. Published April 29, 2013.

The flow occurs in a set $\Omega \subset \mathbb{R}^3$, which is assumed to be a bounded domain with smooth boundary $\partial\Omega$; S and Γ are disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = S \cup \overline{\Gamma}$, and the superficial measure of S is strictly positive.

The unknowns in the problem are the fluid velocity $\mathbf{u}$, its associated pressure p and the concentration of microorganisms m. In the previous equations, the following are given: the fluid viscosity function $\nu(\cdot) > 0$; the constant rate of diffusion of microorganisms $\theta > 0$; the constant vector $\chi = (0, 0, 1)^t$, meaning that the coordinate system is placed such that the gravitational force acts along the vertical. Other given data are the following: $\mathbf{f}$, the external force field; U, the average upward speed of swimming of the microorganisms; α, a positive constant given total mass of microorganisms.

As usual, the symbols ∇, Δ and div denote respectively the gradient, Laplacian and divergence operators; $\mathbf{u}\cdot\nabla\mathbf{u}$ denotes the convection operator, whose component are i-th in cartesian coordinates is given by $(\mathbf{u}\cdot\nabla\mathbf{u})_i = \sum_{j=1}^{3} u_j u_{i,x_j}$. Also, $D(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^t)/2$ is the symmetric part of the deformation rate tensor.

Bioconvective flows have been studied by many autors along the years; here we just mention the book by Levandowsky, Childress, Hunter and Spiegel [5], and the articles by Moribe [8] and Kan-On, Narukawa and Teramoto [4]; the interested reader can consult also the references mentioned in these works. Next, we briefly comment on previously published articles that are directly related to the present one.

Kan-on, Narukawa and Teramoto in [4] analyzed the classical bioconvective equations (i.e., the case of constant fluid viscosity) with Dirichlet boundary conditions for the fluid velocity. By using fixed point arguments, they proved the existence of generalized solutions; with the help of classical regularity results, they also prove the existence of strong solutions in the case that of small enough upward swimming speed U.

On the other hand, Lorca and Boldrini in [6] obtained results on existence and uniqueness of weak solutions for the generalized Boussinesq equations, which are equations governing thermally driven flows in the case that the viscosity and thermal conductivity coefficients may depend on the temperature. They also considered Dirichlet boundary conditions for the velocity and used Galerkin approximations and fixed point arguments, together with estimates for the "pressure" associated to a Helmholtz decomposition of a L^2-field, to prove their results.

The present work generalizes the results of Kan-on, Narukawa and Teramoto in [4] to the generalized bioconvective equations, that is, to the case where the fluid viscosity may depend on the concentration of microorganisms. As in [4], we prove results on existence of weak and strong solutions of problem when U is small; we also give a result on uniqueness of weak solutions. For the proofs, we have to adapt some of the techniques presented in Lorca and Boldrini [6] to the case of our boundary conditions, including the estimates for the pressure associated to a Helmholtz decomposition. We remark that one major difficulty to obtain more regular solutions of our system of equations, as well as in the case of the generalized Boussinesq equations, is to estimate the nonlinear terms; we are able to do this with the help of the previously mentioned estimates for the pressure associated to a Helmholtz decomposition.

We finally observe that we are also analyzing the associated evolution problem; the results of such analysis will appear elsewhere; we also remark that Climent-Ezquerra *et al.* recently proved in [3] the existence of time reproductive solutions for this same evolution problem.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain of class C^3. We consider the Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, with the usual norms $|u|_p$; for simplicity, we just denote $|\cdot|_2 = |\cdot|$ and the usual inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$. For $m \geq 0$ and $1 \leq p < \infty$, we consider the usual Sobolev spaces $W^{m,p}(\Omega) = \{u \in L^p(\Omega); D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq m\}$, with the norm $\|u\|_{m,p} = [\sum_{|\alpha|\leq m} \|D^\alpha u(t)\|^p_{L^p(\Omega)}]^{1/p}$; when $p = 2$, we denote, as usual, $W^{m,p}(\Omega) = H^m(\Omega)$. Also, $W^{1-\frac{1}{p},p}(\partial\Omega)$ is the space of traces on $\partial\Omega$ of functions in $W^{1,p}(\Omega)$, equipped with the norm $|\gamma|_{W^{1-\frac{1}{p},p(\partial\Omega)}} = \inf\{\|v\|_{W^{k,p}(\Omega)}; v \in W^{1,p}(\Omega), v = \gamma \text{ on } \partial\Omega\}$; when $p = 2$, we denote $W^{1/2,2}(\partial\Omega) = H^{1/2}(\partial\Omega)$. For details and properties of such spaces, see Adams [1].

We will also need the following classical results.

Lemma 2.1 (Poincaré-Friedrichs inequality)**.** *Let $\Sigma \subseteq \partial\Omega$ a portion of boundary with strictly positive superficial measure; then there exists a positive constant C_P depending only on Ω and Σ such that $|u| \leq C_P|\nabla u|$, for all $u \in H^1(\Omega)$ such that $u|_\Sigma = 0$.*

Lemma 2.2. *There exists a constant C_Ω, depending only on Ω, such that $|\phi| \leq C_\Omega|\nabla\phi|$, for all $\phi \in B = H^1(\Omega) \cap Y$.*

To treat the unknown velocity, we will need the following functional spaces: being S and Γ as described in the Introduction, we define the following functional spaces

$$\dot{\mathbf{H}}(\Omega) = \{\mathbf{u} \in (C^\infty(\overline{\Omega}))^3 : \mathbf{u}|_S = 0,\ \mathbf{u}\cdot\mathbf{n}|_\Gamma = 0\},$$

$$\mathbf{H}(\Omega) \text{ closure of } \dot{\mathbf{H}}(\Omega) \text{ with respect to norm } \|\cdot\|_{\mathbf{H}(\Omega)},$$

where

$$\|\mathbf{u}\|_{\mathbf{H}(\Omega)} = \Big[\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u}dx\Big]^{1/2} = [(\nabla\mathbf{u},\nabla\mathbf{u})]^{1/2} = |\nabla\mathbf{u}|. \tag{2.1}$$

Given $\mathbf{u} : \Omega \to \mathbb{R}^3$ with suitable regularity, the rate of strain tensor is defined as $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$. Let us also define

$$(D(\mathbf{u}), D(\mathbf{v})) = \int_\Omega D(\mathbf{u}) : D(\mathbf{v})dx = \int_\Omega \sum_{i,j=1}^3 (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})dx,$$

and thus, $(D(\mathbf{u}), D(\mathbf{u})) = \int_\Omega D(\mathbf{u}) : D(\mathbf{u})dx \equiv |D(\mathbf{u})|^2$.

We will also need the following results:

Lemma 2.3 (Korn inequality [10, p. 191])**.** *There exists a positive constant $\overline{c}$, such that*

$$\|\mathbf{u}\|_{\mathbf{H}(\Omega)} = |\nabla\mathbf{u}| \leq \overline{c}|D(\mathbf{u})|, \quad \forall \mathbf{u} \in \mathbf{H}(\Omega).$$

As a consequence of this lemma, we have the following result.

Lemma 2.4. *There exists a positive constant γ such that $|\mathbf{u}|^2 \leq \gamma|D(\mathbf{u})|^2$, for all $\mathbf{u} \in \mathbf{H}(\Omega)$.*

The previous results imply that the norms $|\nabla \mathbf{u}|$ and $|D(\mathbf{u})|$ are equivalent in $\mathbf{H}(\Omega)$. Next, we consider $\mathbf{C}^\infty_{0,\sigma}(\Omega) = \{\mathbf{f} \in (C^\infty_0(\Omega))^3 : \operatorname{div} \mathbf{f} = 0\}$, and then we take

$$\mathbf{X}(\Omega) = \text{closure of } \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ in } (L^2(\Omega))^3.$$

It is well known [11] that

$$(L^2(\Omega))^3 = \mathbf{X}(\Omega) \oplus \mathbf{G}(\Omega),$$

with $\mathbf{G}(\Omega) = \{\varphi \in (L^2(\Omega))^3, \varphi = \nabla q,\ q \in H^1(\Omega)\}$.

We will also need the following functional spaces:

$$\dot{\mathbf{J}}(\Omega) = \{\mathbf{u} \in \dot{\mathbf{H}}(\Omega), \operatorname{div} \mathbf{u} = 0\},$$
$$\mathbf{J}_0(\Omega) = \text{ closure of } \dot{\mathbf{J}}(\Omega) \text{ in the norm (2.1).}$$

Next, we consider the following applications:

$$B_0 : \mathbf{J}_0(\Omega) \times \mathbf{J}_0(\Omega) \times \mathbf{J}_0(\Omega) \to \mathbb{R},$$
$$B_1 : \mathbf{J}_0(\Omega) \times (H^1(\Omega))^3 \times (H^1(\Omega))^3 \to \mathbb{R}$$

given by

$$\begin{gathered} B_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = \int_\Omega \sum_{i,j=1}^N u_j(x)((\partial v_i)/(\partial x_j))(x) w_i(x) dx, \\ B_1(\mathbf{u}, \phi, \psi) = (\mathbf{u} \cdot \nabla \phi, \psi) = \int_\Omega \sum_{j=1}^N u_j(x)((\partial \phi)/(\partial x_j))(x) \psi(x) dx. \end{gathered} \tag{2.2}$$

They are well defined trilinear forms with the following properties:

$$\begin{gathered} B_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -B_0(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad B_1(\mathbf{u}, \phi, \psi) = -B_1(\mathbf{u}, \psi, \phi), \\ B_0(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad B_1(\mathbf{u}, \phi, \phi) = 0. \end{gathered} \tag{2.3}$$

Let P be the orthogonal projection from $(L^2(\Omega))^3$ onto $\mathbf{X}(\Omega)$. Then, we define the operator A as the Friedrichs extension of the symmetric operator $P\Delta$, with $D(A) = \{\mathbf{u} \in \mathbf{J}_0(\Omega) \cap (H^2(\Omega))^3; D(\mathbf{u})\mathbf{n} - \mathbf{n} \cdot (D(\mathbf{u})\mathbf{n})\mathbf{n}|_\Gamma = 0\}$.

The proofs of the following results concerning this operator A can be found in Rionero and Mulone [9, pp. 478-481].

Lemma 2.5. *The Stokes operator $A : \mathbf{X}(\Omega) \to \mathbf{X}(\Omega)$ defined as the Friedrichs extension of $-P\Delta$, with domain $D(A) = \{\mathbf{u} \in \mathbf{J}_0(\Omega) \cap (H^2(\Omega))^3; D(\mathbf{u})\mathbf{n} - \mathbf{n} \cdot (D(\mathbf{u})\mathbf{n})\mathbf{n}|_\Gamma = 0\}$, is a selfadjoint, positive definite operator with compact inverse.*

Thus (see Brezis [2]), A has a sequence $\{\alpha_i\}_{i=1}^\infty$ of eigenvalues satisfying $0 < \alpha_1 \leq \alpha_2 \leq \dots$ and $\lim_{i\to+\infty} \alpha_i = +\infty$, whose associated eigenfunctions $\{\overline{\mathbf{w}}^i\}_{i=1}^\infty$ form a complete orthogonal system in $\mathbf{X}(\Omega)$, $\mathbf{J}_0(\Omega)$ and $D(A)$, with their natural inner products.

The following result concerning the Helmholtz decomposition is analogous to the one in Lemma 3.4 of Lorca and Boldrini [7]; its proof can be done similarly as in [7].

Lemma 2.6. *Let $\mathbf{v} \in J_0(\Omega) \cap (H^2(\Omega))^3$ and consider the Helmholtz decomposition of $-\Delta \mathbf{v}$:*

$$-\Delta \mathbf{v} = A\mathbf{v} + \nabla \overline{q},$$

where $\overline{q} \in H^1(\Omega)$ and $\int_\Omega \overline{q}dx = 0$. Then, there exists a positive constants $C > 0$ and, for any $\varepsilon > 0$, a associated positive constant C_ε, such that

$$\|\overline{q}\|_1 \le C|A\mathbf{v}| \quad \text{and} \quad |\overline{q}| \le C_\varepsilon|\nabla \mathbf{v}| + \varepsilon|A\mathbf{v}|, \quad \forall \mathbf{v} \in D(A).$$

To treat the unknown microorganism concentration, we will need the following functional spaces:

Y is the closed subspace of $L^2(\Omega)$ consisting of functions that are orthogonal to the constants; i.e.,

$$Y = \{f \in L^2(\Omega) : \int_\Omega f(x)dx = 0\}.$$

We then define

$$B = H^1(\Omega) \cap Y.$$

Next, let $\overline{P}$ be the orthogonal projection from $L^2(\Omega)$ onto Y. As before, an operator A_1 can be defined as the Friedrichs extension of the symmetric operator $\overline{P}(-\theta\Delta)$, with domain $D(A_1) = \{\varphi \in Y \cap H^2(\Omega); \theta\frac{\partial\varphi}{\partial\mathbf{n}} - Un_3\varphi = 0 \text{ on } \partial\Omega\}$. The proofs of the following results, which are similar to the ones for the operator A, can be found in Kan-On, Narukawa and Teramoto [4, pp. 150-152].

Lemma 2.7. *The operator $A_1 : Y \to Y$, defined as the Friedrichs extension of $\overline{P}(-\theta\Delta)$, with domain $D(A_1) = \{\varphi \in Y \cap H^2(\Omega); \theta\frac{\partial\varphi}{\partial\mathbf{n}} - Un_3\varphi = 0$ on $\partial\Omega\}$, is a selfadjoint, positive definite operator with compact inverse.*

From the definition of A_1, it follows that $D(A_1^{1/2}) = B$ and $(\theta - 2UC_P)^{1/2}|\nabla\varphi| \le |A_1^{1/2}\varphi| \le (\theta + 2UC_P)^{1/2}|\nabla\varphi|$ for all $\varphi \in B$, for this, see again Kan-On, Narukawa and Teramoto [4, pp. 145],

Operator A_1 has a sequence $\{\beta_i\}_{i=1}^\infty$ of eigenvalues satisfying $0 < \beta_1 \le \beta_2 \le \dots$ and $\lim_{i\to+\infty}\beta_i = +\infty$, and whose corresponding eigenfunctions $\{\overline{\phi}^i\}$ form a complete orthogonal system in Y, B and $D(A_1)$, with their natural inner products; we assume that it is normalized in Y.

We will also use the following orthogonal projections: for each n, define

$$\begin{gathered} \overline{P}_n : Y \to M_n = \operatorname{span}\{\overline{\phi}^1, \overline{\phi}^2, \dots, \overline{\phi}^n\}, \\ f \to \overline{P}_n(f) = \sum_{\ell=1}^n (f, \overline{\phi}^\ell)\overline{\phi}^\ell, \end{gathered} \tag{2.4}$$

Standard computations with the fractional powers of A_1, using the previous results, give us the following:

$$|\nabla\overline{P}_n\varphi| \le |A_1^{1/2}\varphi| \le (\theta + 2UC_P)^{1/2}|\nabla\varphi|, \quad \forall\varphi \in B, \tag{2.5}$$

$$\begin{aligned} (\theta - 2UC_P)^{1/2}|\nabla(\varphi - \overline{P}_n\varphi)| \le |A_1^{1/2}(\varphi - \overline{P}_n\varphi)| &\le \frac{1}{\beta_{n+1}^{1/2}}|A_1\varphi| \\ &\le \frac{\theta}{\beta_{n+1}^{1/2}}|\Delta\varphi|, \quad \forall\varphi \in D(A_1). \end{aligned} \tag{2.6}$$

We will need an inequality similar to the last one, but holding for a larger set of functions. For this, we consider the following extension of the operator A_1: let $\tilde{A}_1$ be the Friedrichs extension of the symmetric operator $\overline{P}(-\theta\Delta)$, but now with domain $D(A_1) = \{\varphi \in H^2(\Omega); \theta\frac{\partial\varphi}{\partial\mathbf{n}} - Un_3\varphi = 0 \text{ on } \partial\Omega\}$. By observing that the subspace span$\{1\}$ is the orthogonal complement of Y in $L^2(\Omega)$, and in particular, $L^2(\Omega) = \operatorname{span}\{1\} \oplus Y$, we have that $\tilde{A}_1 = 0 \oplus A_1$; we conclude that $\tilde{A}_1$ is a semipositive selfadjoint operator with eigenvalues $\{\beta_i\}_{i=0}^{\infty}$, where $\beta_0 = 0$ and its associated eigenfunction is the constant function 1; the other eigenvalues are exactly the same ones of A_1, with the same eigenfunctions. As before, we have that $|\tilde{A}_1^{1/2}(\varphi - \tilde{P}_n\varphi)| \le \frac{1}{\beta_{n+1}^{1/2}}|\tilde{A}_1\varphi|$ for all $\varphi \in D(\tilde{A}_1)$, where $\tilde{P}_n$ is now the $L^2(\Omega)$-orthogonal projection on $\operatorname{span}\{1\} \oplus M_n$. For $n \ge 1$, by using the definitions of fractional powers in terms of the eigenvalues and eigenfunctions, we see that $\tilde{A}_1^{1/2}(\varphi - \tilde{P}_n\varphi) = A_1^{1/2}(\varphi - P_n\varphi)$. Finally, from the previous results, by proceeding as before, we finally also obtain

$$(\theta - 2UC_P)^{1/2}|\nabla(\varphi - \overline{P}_n\varphi)| \le \frac{\theta}{\beta_{n+1}^{1/2}}|\Delta\varphi|, \quad \forall \varphi \in D(\tilde{A}_1) \tag{2.7}$$

3. Existence of weak solutions

To analyze our problem, it is convenient to introduce the following change of variables

$$\overline{m} = m - E,$$

where

$$E(x) = C_\alpha \exp\Big(\frac{U}{\theta}x_3\Big),$$

and the constant C_α is chosen such that $\int_\Omega E(x)dx = \alpha$.

The idea behind this change of variable is the following: since $-\theta\Delta E + U\frac{\partial E}{\partial x_3} = 0$ and $\theta\frac{\partial E}{\partial \mathbf{n}} - Un_3E = 0$, we have that $E(\cdot)$ is a particular solution of equation (1.1) (iii) in the special case of no fluid motion, that is, when $\mathbf{u} \equiv 0$; thus, our intension to look for solutions in a neighborhood of this special microorganism distribution.

By writing Problem (1.1) - (1.2) in terms of variables $\mathbf{u}$ and $\overline{m}$, we obtain

$$\begin{gathered}
-2\operatorname{div}(\nu(\overline{m}+E)D(\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla(q + \frac{\theta}{U}E) = \overline{m}\cdot\chi + \mathbf{f},\\
\operatorname{div}\mathbf{u} = 0,\\
-\theta\Delta\overline{m} + \mathbf{u}\cdot\nabla(\overline{m}+E) + U\frac{\partial\overline{m}}{\partial x_3} = 0 \quad \text{in } \Omega.\\
\mathbf{u} = 0 \quad \text{on } S,\\
\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma,\\
\nu(\overline{m}+E)[D(\mathbf{u})\mathbf{n} - \mathbf{u}\cdot(D(\mathbf{u})\mathbf{n})\mathbf{n}] = \mathbf{b}_1 \quad \text{on } \Gamma,\\
\theta\frac{\partial\overline{m}}{\partial\mathbf{n}} - Un_3\overline{m} = 0 \quad \text{on } \partial\Omega,\\
\int_\Omega \overline{m}dx = 0.
\end{gathered} \tag{3.1}$$

Next, we give the definition of a weak solution of our problem.

Definition. Let $\mathbf{f} \in X(\Omega)$; a pair of functions $(\mathbf{u},\overline{m}) \in J_0(\Omega) \times B$ is called a *weak solution* of (3.1) when the following two equalities are satisfied for all $(\mathbf{v},\phi) \in J_0(\Omega) \times B$:

$$2(\nu(\overline{m}+E)D(\mathbf{u}), D(\mathbf{v})) + B_0(\mathbf{u},\mathbf{u},\mathbf{v}) + (\overline{m}\cdot\chi,\mathbf{v}) - 2\int_\Gamma \mathbf{b}_1\mathbf{v}d\sigma = (\mathbf{f},\mathbf{v}), \tag{3.2}$$

$$\theta(\nabla\overline{m},\nabla\phi)+B_1(\mathbf{u},\overline{m}+E,\phi)-U(\overline{m},\frac{\partial\phi}{\partial x_3})=0. \tag{3.3}$$

This variational formulation is obtained, as usual, by working in a formal way. To give an idea on how to do that, here we remark that it is obtained by using the following computations done, by simplicity, on the original form of the first equation. Assume that $\mathbf{u},\mathbf{v}\in\dot{\mathbf{H}}(\Omega)$ and $q\in C^1$; then we have

$$\begin{aligned}&\int_\Omega[-2\operatorname{div}(\nu(m)D(\mathbf{u}))+\nabla q]\mathbf{v}dx\\&=2\int_\Omega\nu(m)D(\mathbf{u}):\nabla\mathbf{v}dx-2\int_{\partial\Omega}\nu(m)D(\mathbf{u})\mathbf{n}\cdot\mathbf{v}dS+\int_{\partial\Omega}q\mathbf{n}\cdot\mathbf{v}dS.\end{aligned}$$

The third term is zero since $\mathbf{v}\in\dot{\mathbf{H}}(\Omega)$ and thus $\mathbf{v}|_S=0$ and $\mathbf{v}\cdot\mathbf{n}|_\Gamma=0$. Next, we can split $\nabla\mathbf{v}$ in its symmetric and antisymmetric parts, and observe that the symmetric part is exactly $D(\mathbf{v})$; since $D(\mathbf{u})$ is also a symmetric matrix, a standard computation then gives that we can rewrite the first term as

$$\int_\Omega\nu(m)D(\mathbf{u}):\nabla\mathbf{v}dx=\int_\Omega\nu(m)D(\mathbf{u}):D(\mathbf{v})dx.$$

On the other hand, by using the tangential and normal components at each point of the boundary, the second term becomes

$$\begin{aligned}&-2\int_{\partial\Omega}\nu(m)D(\mathbf{u})\cdot\mathbf{n}\ \mathbf{v}dS\\&=-2\int_{\partial\Omega}(\nu(m)D(\mathbf{u})\mathbf{n})\cdot\mathbf{n}\ (\mathbf{v}\cdot\mathbf{n})dS\\&\quad-2\int_{\partial\Omega}[\nu(m)D(\mathbf{u})\mathbf{n}-(\nu(m)D(\mathbf{u})\mathbf{n})\cdot\mathbf{n})\mathbf{n}]\cdot(\mathbf{v}-(\mathbf{v}\cdot\mathbf{n})\mathbf{n})dS\\&\quad-2\int_\Gamma\nu(m)[D(\mathbf{u})\mathbf{n}-\mathbf{n}\cdot(D(\mathbf{u})\mathbf{n})\mathbf{n}]\cdot\mathbf{v}dS\\&=-2\int_\Gamma\mathbf{b}_1\cdot\mathbf{v}dS\end{aligned}$$

were we have used the fact that $\mathbf{v}\in\dot{\mathbf{H}}(\Omega)$ and the boundary condition on Γ. Thus,

$$\int_\Omega[-2\operatorname{div}(\nu(m)D(\mathbf{u}))+\nabla q]\mathbf{v}dx=2(\nu(m)D(\mathbf{u}),D(\mathbf{v}))-2\int_\Gamma\mathbf{b}_1\cdot\mathbf{v}dS.$$

Analogously, the second equation in the model (already with the previous change of variable) can be formally treated as

$$\begin{aligned}&\int_\Omega-\theta\Delta\overline{m}\phi\,dx+\int_\Omega U\frac{\partial\overline{m}}{\partial x_3}\phi\,dx\\&=\theta\int_\Omega\nabla\overline{m}\nabla\phi\,dx-\int_{\partial\Omega}\theta\frac{\partial\overline{m}}{\partial n}\phi\,dS-U\int_\Omega\overline{m}\frac{\partial\phi}{\partial x_3}dx+\int_{\partial\Omega}Un_3\overline{m}\phi\,dS.\\&=\theta\int_\Omega\nabla\overline{m}\nabla\phi dx-U\int_\Omega\overline{m}\frac{\partial\phi}{\partial x_3}dx\,.\end{aligned}$$

Then we have the following result.

Theorem 3.1. *Let ν be a continuous function satisfying*

$$\nu_0=\inf\{\nu(m),m\in\mathbb{R}\}>0,\quad \nu_1=\sup\{\nu(m),m\in\mathbb{R}\}<+\infty; \tag{3.4}$$

$\mathbf{f} \in X(\Omega)$ *and* $\frac{U}{\theta} < (C_P)^{-1}$*, where* C_P *is the Sobolev constant appearing in the Poincarè-Friedrichs inequality. Then there exists a weak solution of Problem* (3.1) *satisfying*

$$|D(\mathbf{u})|^2 + |\nabla \overline{m}|^2 \leq C(|\mathbf{f}|^2 + \|\mathbf{b}_1\|^2_{H^{1/2}(\Gamma)} + |\nabla E|^2),$$

with a constant C *independent of* $\mathbf{f}$*,* $\mathbf{b}_1$ *and* E*.*

Proof. We consider the following Schauder bases formed by the eigenfunctions described in the previous section: $(\overline{\mathbf{w}}^j)_1^\infty$ for $\mathbf{J}_0(\Omega)$ and $(\overline{\phi}^j)_1^\infty$ for B. For each $n \in \mathbb{N}$, we define $\mathbf{W}_n = \text{span}\{\overline{\mathbf{w}}^j,\ 1 \leq j \leq n\}$ and $M_n = \text{span}\{\phi^\ell,\ 1 \leq \ell \leq n\}$ and consider the Galerkin approximations

$$\mathbf{u}^n = \sum_{j=1}^n c_{n,j} \overline{\mathbf{w}}^j \in \mathbf{W}_n. \quad \overline{m}^n = \sum_{\ell=1}^n d_{n,\ell} \overline{\phi}^\ell \in M_n,$$

satisfying the following approximate problem

$$\begin{aligned} &2(\nu(\overline{m}^n + E)D(u^n), D(\overline{v})) + B_0(\mathbf{u}^n, \mathbf{u}^n, \overline{v}) \\ &+ (\overline{m}^n \cdot \chi, \overline{v}) - 2\int_\Gamma \mathbf{b}_1 \overline{v} d\sigma = (\mathbf{f}, \overline{v}), \end{aligned} \tag{3.5}$$

$$\theta(\nabla \overline{m}^n, \nabla \overline{\phi}) + B_1(\mathbf{u}^n, \overline{m}^n + E, \overline{\phi}) - U(\overline{m}^n, \frac{\partial \overline{\phi}}{\partial x_3}) = 0, \tag{3.6}$$

for all $\overline{v} \in \mathbf{W}_n$ and all $\overline{\phi} \in M_n$.

Firstly, by assuming the existence of $(\mathbf{u}^n, \overline{m}^n)$ for all $n \in \mathbb{N}$ (such existence will be proved later on), we will prove that they indeed converge, along subsequences, to a solution of our problem. To do this, it will be necessary to obtain estimates for the gradients of the unknowns.

We set $\overline{v} = u^n$ in (3.5) to obtain

$$\begin{aligned} &2(\nu(\overline{m}^n + E)D(\mathbf{u}^n), D(\mathbf{u}^n)) + B_0(\mathbf{u}^n, \mathbf{u}^n, \mathbf{u}^n) \\ &+ (\overline{m}^n \cdot \chi, \mathbf{u}^n) - 2\int_\Omega \mathbf{b}_1 \mathbf{u}^n d\sigma = (\mathbf{f}, \mathbf{u}^n). \end{aligned} \tag{3.7}$$

By observing that $B_0(\mathbf{u}^n, \mathbf{u}^n, \mathbf{u}^n) = 0$ (see (2.3)) and using Hölder inequality together with (3.4), its follows that

$$\begin{aligned} 2\nu_0 |D(\mathbf{u}^n)|^2 &\leq |(\overline{m}^n \cdot \chi, \mathbf{u}^n) + (\mathbf{f}, \mathbf{u}^n) + 2\int_\Omega \mathbf{b}_1 \mathbf{u}^n d\sigma| \\ &\leq |\overline{m}^n|\ |\mathbf{u}^n| + |\mathbf{f}|\ |\mathbf{u}^n| + 2c(\Gamma)\Big(\int_\Gamma \mathbf{b}_1^2 d\sigma\Big)^{1/2} |D(\mathbf{u}^n)|. \end{aligned}$$

By the Sobolev embeddings, Korn, Young and Poincarè-Friedrichs inequalities, we then obtain

$$\nu_0 |D(\mathbf{u}^n)|^2 \leq \frac{3}{4\nu_0}(C_p^2 \gamma |\nabla \overline{m}^n|^2 + \gamma |\mathbf{f}|^2 + 4c^2(\Gamma)\|\mathbf{b}_1\|^2_{H^{1/2}(\Gamma)}). \tag{3.8}$$

Next, for each n we consider the orthogonal projection $\overline{P}_n$ defined in (2.4), denote by $|\Omega|$ the Lebesgue measure of Ω, and take $\overline{\phi} = \overline{m}^n + \overline{P}_n(E - \alpha/|\Omega|)$ to get

$$\begin{aligned} &\theta |\nabla \overline{m}^n|^2 + \theta(\nabla \overline{m}^n, \nabla \overline{P}_n(E - \alpha/|\Omega|)) + B_1(\mathbf{u}^n, \overline{m}^n + E, \overline{m}^n \\ &+ \overline{P}_n(E - \alpha/|\Omega|)) - U(\overline{m}^n, \frac{\partial}{\partial x_3}(\overline{m}^n + \overline{P}_n(E - \alpha/|\Omega|))) = 0. \end{aligned} \tag{3.9}$$

Now, we observe that

$$\begin{aligned} &B_1(\mathbf{u}^n, \overline{m}^n + E, \overline{m}^n + \overline{P}_n(E - \alpha/|\Omega|)) \\ &= B_1(\mathbf{u}^n, \overline{m}^n + E - \alpha/|\Omega|, \overline{m}^n + \overline{P}_n(E - \alpha/|\Omega|)) \\ &= \ B_1(\mathbf{u}^n, \overline{m}^n + \overline{P}_n(E - \alpha|/\Omega|), \overline{m}^n + \overline{P}_n(E - \alpha/|\Omega|)) \\ &\quad + B_1(\mathbf{u}^n, E - \alpha|/\Omega| - \overline{P}_n(E - \alpha|/\Omega|), \overline{m}^n) \\ &\quad + B_1(\mathbf{u}^n, E - \alpha|\Omega| - \overline{P}_n(E - \alpha|/\Omega|), \overline{P}_n(E - \alpha/|\Omega|)), \end{aligned} \tag{3.10}$$

We also remark that, due to the fact that $(1, \overline{\phi}_j) = 0$ for all $j \geq 1$, that the gradients of a constant is zero, and that $E \in D(\tilde{A}_1)$, by using (2.7), we obtain

$$|\nabla(E - \alpha/|\Omega| - \overline{P}_n(E - \alpha/|\Omega|))| = |\nabla(E - \overline{P}_n E)| \leq \frac{\theta}{\beta_{n+1}^{1/2}(\theta - 2UC_P)^{1/2}} |\Delta E| \tag{3.11}$$

Using (3.10) in (3.9), recalling that $B_1(\mathbf{u}^n, \overline{m}^n + \overline{P}_n(E - \alpha|/\Omega|), \overline{m}^n + \overline{P}_n(E - \alpha/|\Omega|)) = 0$ (see (2.3)), suitably using Hölder inequality and properties (2.5) and (3.11) and the fact that $z \leq 1 + z^2$ for all real z, together with the hypothesis that $\theta - UC_P > 0$, we obtain that

$$|\nabla \overline{m}^n|^2 \leq K_1\Big(\frac{1}{\beta_{n+1}} + \frac{1}{\beta_{n+1}^{1/2}}\Big)|\Delta E||D(u^n)| + K_2|\nabla E|^2, \tag{3.12}$$

with positive constants K_1 and K_2 that do not depend on n.

By multiplying (3.12) by $\frac{3}{2\nu_0}C_p^2$, adding the result to (3.8), for $n \geq N_0$ such that

$$\frac{3}{2\nu_0}C_p^2 K_1\Big(\frac{1}{\beta_{N_0+1}} + \frac{1}{\beta_{N_0+1}^{1/2}}\Big)|\Delta E| < \frac{\nu_0}{2},$$

we have

$$|D(\mathbf{u}^n)|^2 + |\nabla \overline{m}^n|^2 \leq C(|\mathbf{f}|^2 + \|\mathbf{b}_1\|^2_{H^{1/2}\Gamma)} + |\nabla E|^2). \tag{3.13}$$

Therefore, the sequence $\{(\mathbf{u}^n, \overline{m}^n)\}$ is bounded in $\mathbf{J}_0(\Omega) \times B$.

Next, since $\mathbf{J}_0(\Omega)$ is compactly immersed in $\mathbf{X}(\Omega)$, and B is compactly immersed in Y, there are elements $\mathbf{u} \in \mathbf{J}_0(\Omega), \overline{m} \in B$ and a subsequence, which for simplicity we still denote by $\{(\mathbf{u}^n, \overline{m}^n)\}$), such that

$$\begin{gathered} \mathbf{u}^n \to \mathbf{u} \quad \text{weakly in } \mathbf{J}_0(\Omega) \text{ and strongly in } \mathbf{X}(\Omega), \\ \overline{m}^n \to m \quad \text{weakly in } B \text{ and strongly in } Y, \\ D(\mathbf{u}^n) \to D(\mathbf{u}) \quad \text{weakly in } (L^2(\Omega))^9, \\ \nabla(\overline{m}^n) \to \nabla(\overline{m}) \quad \text{weakly in } (L^2(\Omega))^3. \end{gathered}$$

These convergences are enough to allow us to take the limit as $n \to +\infty$ in (3.5) and (3.6) to obtain

$$\begin{gathered} 2(\nu(\overline{m} + E)D(\mathbf{u}), D(\overline{\mathbf{w}}^j)) + B_0(\mathbf{u}, \mathbf{u}, \overline{\mathbf{w}}^j) \\ +(\overline{m} \cdot \chi, \overline{\mathbf{w}}^j) - 2\int_\Gamma \mathbf{b}_1 \cdot \overline{\mathbf{w}}^j d\sigma = (\mathbf{f}, \overline{\mathbf{w}}^j), \\ \theta(\nabla \overline{m}, \nabla \overline{\phi}^\ell) + B_1(\mathbf{u}, \overline{m} + E, \overline{\phi}^\ell) - U\Big(\overline{m}, \frac{\partial \overline{\phi}^\ell}{\partial x_3}\Big) = 0, \end{gathered} \tag{3.14}$$

for all $j, \ell \in \mathbb{N}$.

In fact, it is well known [11] that with the previous convergences we obtain

$$B_0(\mathbf{u}^n,\mathbf{u}^n,\mathbf{w}) \to B_0(\mathbf{u},\mathbf{u},\mathbf{w}), \quad \forall \mathbf{w}\in \mathbf{J}_0(\Omega),$$
$$B_1(\mathbf{u}^n,\overline{m}^n,\phi) \to B_1(\mathbf{u},\overline{m},\phi), \quad \forall \phi\in B.$$

We also note that

$$(\nu(\overline{m}^n+E)D(\mathbf{u}^n),D(\mathbf{w}^j)) \to (\nu(\overline{m}+E)D(\mathbf{u}),D(\mathbf{w}^j))$$

since

$$(\nu(\overline{m}^n+E)D(\mathbf{u}^n),D(\mathbf{w}^j)) = (D(\mathbf{u}^n),\nu(\overline{m}^n+E)D(\mathbf{w}^j)),$$
$$(D(\mathbf{u}),\nu(\overline{m}+E)D(\mathbf{w}^j)) = (\nu(\overline{m}+E)D(\mathbf{u}),D(\mathbf{w}^j)),$$
$$\nu(\overline{m}^n+E)D(\mathbf{w}^j) \to \nu(\overline{m}+E)D(\mathbf{w}^j)$$

strongly in $(L^2(\Omega))^9$ due to the Lebesgue dominated convergence theorem.

Finally, as the $\{\overline{\mathbf{w}}^j\}$ and $\{\overline{\phi}^\ell\}$ are Schauder bases, respectively in $\mathbf{J}_0(\Omega)$ and B, by using (3.14), we conclude that $(\mathbf{u},\overline{m})$ satisfies (3.2), (3.3), and thus $(\mathbf{u},\overline{m})$ is the required weak solution.

3.1. **Existence of approximate solutions.** It remains to prove that, for each $n\in\mathbb{N}$, equations (3.5)-(3.6) have solutions. For this, we proceed similarly as in Lorca and Boldrini [6].

Let $\mathbf{W}_n$ and M_n be as in the last section. Given any $(\mathbf{z},\xi)\in \mathbf{W}_n\times M_n$, we consider the unique solution $(\mathbf{v},\Psi)\in \mathbf{W}_n\times M_n$ of the linearized equations

$$\begin{aligned}&2(\nu(\xi+E)D(\mathbf{v}),D(\overline{\mathbf{w}}^j)) + B_0(\mathbf{z},\mathbf{v},\overline{\mathbf{w}}^j)\\ &+ (\Psi\cdot\chi,\overline{\mathbf{w}}^j) - 2\int_\Gamma \mathbf{b}_1\cdot\overline{\mathbf{w}}^j d\sigma - (\mathbf{f},\overline{\mathbf{w}}^j) = 0,\end{aligned} \tag{3.15}$$

$$\theta(\nabla\Psi,\nabla\overline{\phi}^\ell) + B_1(\mathbf{z},\Psi+E,\overline{\phi}^\ell) - U\Big(\Psi,\frac{\partial\overline{\phi}^\ell}{\partial x_3}\Big) = 0, \tag{3.16}$$

for $1\le j,\ell\le n$.

To prove that there is only one such solution $(\mathbf{v},\Psi)\in \mathbf{W}_n\times M_n$, we observe that (3.15), (3.16) constitute in fact a linear system with $2n$ equations for the $2n$ coefficients of the expansions

$$\mathbf{v} = \sum_{j=1}^n c_j\overline{\mathbf{w}}^j, \quad \Psi = \sum_{\ell=1}^n d_\ell\overline{\phi}^\ell.$$

Thus, to show the existence and uniqueness of solutions of system (3.15), (3.16), it is enough to prove that the only solution of its associated homogeneous linear system, that is, the corresponding equations with $\mathbf{b}_1 = 0$ and $\mathbf{f}=0$, is the trivial null solution. For this, let $(\mathbf{v},\Psi)$ be any solution of the such homogeneous system, and c_j and d_ℓ its corresponding coefficients as in the previous expansions. Then, by multiplying (3.15) by c_j and (3.16) by d_ℓ, and adding in j and ℓ from 1 to n, we obtain

$$2(\nu(\xi+E)D(\mathbf{v}),D(\mathbf{v})) + B_0(\mathbf{z},\mathbf{v},\mathbf{v}) + (\Psi\cdot\chi,\mathbf{v}) = 0, \tag{3.17}$$

$$\theta|\nabla\Psi|^2 + B_1(\mathbf{z},\Psi,\Psi) - U\Big(\Psi,\frac{\partial\Psi}{\partial x_3}\Big) = 0. \tag{3.18}$$

As $B_1(\mathbf{z},\Psi,\Psi)=0$, by using Hölder and Poincarè-Friedrichs inequalities, (3.18) becomes $(\theta-UC_P)|\nabla\Psi|^2\le 0$. Thus, $(\theta-UC_P)>0$ implies $|\nabla W|^2=0$; i.e., Ψ

is constant; since $\int_\Omega \Psi dx = 0$, we conclude that $\Psi = 0$. Since $B_0(\mathbf{z}, \mathbf{v}, \mathbf{v}) = 0$ and $\Psi = 0$, (3.17) gives $2\nu_0|D(\mathbf{u})|^2 \le 0$, which in turn implies that $\mathbf{v} = 0$.

Thus, for each n we have a well defined operator

$$T_n : \mathbf{W}_n \times M_n \to \mathbf{W}_n \times M_n, \tag{3.19}$$

such that to each $(\mathbf{z}, \xi) \in \mathbf{W}_n \times M_n$ associates $T(\mathbf{z}, \xi) = (\mathbf{v}, \Psi)$, where $(\mathbf{v}, \Psi) \in \mathbf{W}_n \times M_n$ is the unique solution of (3.15). Moreover, since $\mathbf{W}_n$ and M_n are finite dimensional vector spaces, it is rather standard to prove that T_n is continuous for any chosen norms.

Next, by proceeding exactly as before in the derivation of the estimates for $(\mathbf{u}^n, \overline{m}^n)$, we obtain for system (3.15), (3.16) the same kind of estimate as the one in (3.13); i. e.,

$$|D(\mathbf{v})|^2 + |\nabla\Psi|^2 \le C(|\mathbf{f}|^2 + \|\mathbf{b}_1\|^2_{H^{1/2}(\Gamma)} + |\nabla E|^2) = R_1^2,$$

with R_1 independent of n and $(\mathbf{z}, \xi) \in \mathbf{W}_n \times M_n$. By denoting $F_n = \{(\mathbf{z}, \xi) \in \mathbf{W}_n \times M_n; (|D(\mathbf{z})|^2 + |\nabla\xi|^2) \le R_1^2\}$, and restricting the operator T_n to such F_n. we have a continuous operator $T_n : F_n \to F_n$ acting from a finite dimensional closed convex set convex F_n into itself. Brower fixed point theorem then gives us the existence of at least one fixed point, $(\mathbf{u}, \overline{m})$, which is a solution of (3.5)-(3.6). This completes the proof. □

4. Existence of strong solutions

Here we prove the existence of solutions that are more regular than the ones obtained in the previous section. The main difficulty for this will be to obtain the necessary higher order estimates for the nonlinear terms present in the equations.

Theorem 4.1. *Assume that* $\mathbf{b}_1 = 0$, $\mathbf{f} \in \mathbf{X}(\Omega)$ *and* ν *is a* C^1*-function satisfying* (3.4)*. Then, for a small enough* U*, there exists a strong solution of* (3.1)*; that is, there exists a pair of functions* $(\mathbf{u}, \overline{m}) \in (J_0(\Omega) \cap H^2(\Omega)) \times (Y \cap H^2(\Omega))$ *satisfying*

$$P[-2\operatorname{div}(\nu(\overline{m} + E)D(\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + \overline{m}\cdot\chi - \mathbf{f}] = 0 \quad \text{in } (L^2(\Omega))^3,$$
$$\overline{P}\big[-\theta\Delta\overline{m} + \mathbf{u}\cdot\nabla(\overline{m} + E) + U\frac{\partial\overline{m}}{\partial x_3}\big] = 0 \quad \text{in } L^2(\Omega).$$

Proof. We start by recalling that the $L^2(\Omega)$-norm of the Stokes operator A (respectively the operator A_1) applied to an element and the norm of $\mathbf{J}_0(\Omega) \cap (H^2(\Omega))^3$ (respectively the norm of $Y \cap H^2(\Omega)$) of the same element are equivalent.

Next, we repeat the construction used in the proof of Theorem 3.1 to show the existence of approximations $\mathbf{u}^n$ and $\overline{m}^n$. In the present situation, however, under the conditions of Theorem 4.1, we will show that each of the previous operators T_n admits fixed points satisfying an estimate independent of n in a more regular space.

Since all the estimates obtained in the previous section hold true with the same proofs, we proceed with the derivation of further $H^2(\Omega)$-estimates for $(\mathbf{v}, \Psi)$ solution of (3.15)- (3.16).

For this, we multiply (3.15) by $\alpha_j c_j$, (3.16) by $\beta_\ell d_\ell$ and add in j and ℓ from 1 to n to obtain

$$-2(\operatorname{div}(\nu(\xi + E)D(\mathbf{v})), A\mathbf{v}) + B_0(\mathbf{z}, \mathbf{v}, A\mathbf{v}) + (\overline{m}\cdot\chi, A\mathbf{v}) - (\mathbf{f}, A\mathbf{v}) = 0, \tag{4.1}$$

$$\theta(A_1\Psi, A_1\Psi) + B_1(\mathbf{z}, \Psi + E, A_1\Psi) + U\left(\frac{\partial}{\partial x_3}\Psi, A_1\Psi\right) = 0, \tag{4.2}$$

again with B_0 and B_1 given by (2.2).

Using the identity $2\operatorname{div}(\nu(\xi+E)D(\mathbf{v})) = \nu(\xi+E)\Delta\mathbf{v} + 2\nu'(\xi+E)\nabla(\xi+E)D(\mathbf{v})$ in (4.1), we have

$$\begin{aligned} -(\nu(\xi+E)\Delta\mathbf{v}, A\mathbf{v}) &= -B_0(z, \mathbf{v}, A\mathbf{v}) - (\Psi\cdot\chi, A\mathbf{v}) + (\mathbf{f}, A\mathbf{v}) \\ &\quad + (2\nu'(\xi+E)\nabla(\xi+E)D(\mathbf{v}), A\mathbf{v}). \end{aligned} \tag{4.3}$$

Next, from Helmholtz decomposition, we know that there exists $\overline{q} \in H^1(\Omega)$ such that $-\Delta\mathbf{v} = A\mathbf{v} + \nabla\overline{q}$, and we have the estimate

$$\|\overline{q}\|_1 \le c|\Delta\mathbf{v}|. \tag{4.4}$$

Therefore, (4.3) becomes

$$\begin{aligned} (\nu(\xi+E)A\mathbf{v}, A\mathbf{v}) &= -B_0(\mathbf{z}, \mathbf{v}, A\mathbf{v}) - (\Psi\cdot\chi, A\mathbf{v}) + (\mathbf{f}, A\mathbf{v}) \\ &\quad + 2(\nu'(\xi+E)\nabla(\xi+E)D(\mathbf{v}), A\mathbf{v}) - (\nu(\xi+E)\nabla\overline{q}, A\mathbf{v}). \end{aligned}$$

Using Korn, Hölder and Poincarè-Friedrichs inequalities and Sobolev embeddings, together with (3.4), we obtain

$$\begin{aligned} \nu_0|A\mathbf{v}|^2 &\le C_0|D(\mathbf{z})||A\mathbf{v}|^2 + \overline{C}_P|A_1\Psi||A\mathbf{v}| + |\mathbf{f}||A\mathbf{v}| \\ &\quad + 2\nu_1'(|A_1\xi| + |\nabla E|_4)|A\mathbf{v}|^2) + |(\nu(\xi+E)\nabla\overline{q}, A\mathbf{v})| \end{aligned}$$

where $\nu_1' = \sup\{|\nu'(r)|, r \in \mathbb{R}\}$. Since $A\mathbf{u} \in \mathbf{W}_n$, we observe that

$$(\nu(\xi+E)\nabla\overline{q}, A\mathbf{v}) = -(\overline{q},\ \operatorname{div}(\nu(\xi+E)A\mathbf{v})) = -(\overline{q}, (\nu'(\xi+E)\nabla(\xi+E)A\mathbf{v})),$$

and thus

$$|(\nu(\xi+E)\nabla\overline{q}, A\mathbf{v})| \le \nu_1'|\overline{q}|_4|\nabla(\xi+E)|_4|A\mathbf{v}| \le c\nu_1'(|A_1\xi| + |\nabla E|_4)\|\overline{q}\|_1 A\mathbf{v}|. \tag{4.5}$$

Combining (4.4)-(4.5), using Young and Poincarè-Friedrichs inequalities, its follows that

$$\frac{\nu_0}{2}|A\mathbf{v}|^2 \le C_1(|A\mathbf{z}|^2 + |A_1\xi|^2 + |\nabla E|_4^2)\ |A\mathbf{v}|^2 + \frac{2\overline{C}_P^2}{\nu_0}|A_1\Psi|^2 + \frac{2}{\nu_0}|\mathbf{f}|^2. \tag{4.6}$$

Similarly, by using Hölder, Korn and Friedrichs inequalities and Sobolev embeddings, from (4.2) we obtain

$$\theta|A_1\Psi|^2 \le C_2|A\mathbf{z}|^2(|A_1\Psi|^2 + |\nabla E|_4^2) + U\overline{C}_P|A_1\Psi|^2. \tag{4.7}$$

Multiplying (4.6) by $\delta = \frac{\nu_0}{2\overline{C}_P^2}(\theta - U\overline{C}_P)$ and adding the result to (4.7), we obtain

$$\begin{aligned} &\frac{\nu_0^2}{4\overline{C}_P^2}(\theta - U\overline{C}_P)|A\mathbf{v}|^2 + \frac{1}{2}(\theta - U\overline{C}_P)|A_1\Psi|^2 \\ &\le \delta(C_1(|A\mathbf{z}|^2 + |A_1\xi|^2 + |\nabla E|_4^2)|A\mathbf{v}|^2 + \frac{2}{\nu_0}|\mathbf{f}|^2) + C_2|A\mathbf{z}|^2(|A_1\Psi|^2 + |\nabla E|_4^2). \end{aligned}$$

Since $\frac{U}{\theta} < (\overline{C}_P)^{-1}$, there exists a positive constant C_4 such that

$$|A\mathbf{v}|^2 + |A_1\Psi|^2 \le C_4([|A\mathbf{z}|^2 + |A_1\xi|^2 + |\nabla E|_4^2]|A\mathbf{v}|^2 + |\mathbf{f}|^2 + |A\mathbf{z}|^2(|A_1\Psi|^2 + |\nabla E|_4^2)),$$

which implies

$$|A\mathbf{v}|^2 + |A_1\Psi|^2 \le C_4(|A\mathbf{z}|^2 + |A_1\xi|^2 + |\nabla E|_4^2)(|A\mathbf{v}|^2 + |A_1\Psi|^2 + |\nabla E|_4^2) + C_4|\mathbf{f}|^2.$$

This expression can be rewritten for a positive constant C as

$$[1-C(|A\mathbf{z}|^2+|A_1\xi|^2](|A\mathbf{v}|^2+|A_1\Psi|^2) \le C(|A\mathbf{z}|^2+|A_1\xi|^2)|\nabla E|_4^2+C|\nabla E|_4^4+C|\mathbf{f}|^2, \tag{4.8}$$

Next, we will use (4.8) to find $R_2 > 0$ and suitable conditions that will guarantee that when $|A\mathbf{z}|^2 + |A_1\xi|^2 \le R_2^2$ we also have $|A\mathbf{v}|^2 + |A_1\Psi|^2 \le R_2^2$. For this, let $R_2^2 = 1/(4C)$ and then take U small enough such that $|\nabla E|_4^2 \le 1/(4C)$. Under these choices, when $|A\mathbf{z}|^2 + |A_1\xi|^2 \le R_2^2$, inequality (4.8) implies

$$(1/2)(|A\mathbf{v}|^2+|A_1\Psi|^2) \le (1/4)R_2^2+(1/64C)+C|\mathbf{f}|^2 = (1/4)R_2^2+(1/16)R_2^2+C|\mathbf{f}|^2,$$

which is rewritten as $(|A\mathbf{v}|^2 + |A_1\Psi|^2) \le (5/8)R_2^2 + C|\mathbf{f}|^2$. Hence, when $|\mathbf{f}| \le (3/8)R_2^2 = 3/(32C)$, we obtain that $(|A\mathbf{v}|^2 + |A_1\Psi|^2 \le R_2^2$, as required, and such R_2 is independent of n and $(\mathbf{z}, \xi)$.

Under these conditions, we can consider again the operators T_n, but now as continuous operators $T_n : G_n \to G_n$ acting from a finite dimensional closed convex set $G_n = \{(\mathbf{z}, \xi) \in W_n \times M_n; |A\mathbf{z}|^2 + |A_1\xi|^2 \le R_2^2\}$ in itself.

Now, proceeding again similarly as before, we can apply Brower fixed point theorem for each of such operator T_n; this gives a sequence of approximate solutions $(\mathbf{u}^n, \overline{m}^n)$ satisfying (3.5), (3.6), which are now uniformly bounded in $H^2(\Omega)$. By taking the limit as $n \to +\infty$ and proceeding exactly as before, we obtain a solution of our original problem. This proves the theorem. □

Remarks. To each given solution of Problem (3.1), it is trivially associated a solution of (1.1)-(1.2). Moreover, the regularities obtained for $(\mathbf{u}, \overline{m})$ also hold for $(\mathbf{u}, m)$.

As in Temam [11], there exists a unique function q (the hydrostatic pressure) in $H^1(\Omega) \cap L_0^2(\Omega)$, where $L_0^2(\Omega) = \{h \in L^2(\Omega);\ (h, 1) = 0\}$, which satisfies

$$-2\operatorname{div}(\nu(m)D(\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} + m\cdot\chi - \mathbf{f} = -\nabla q.$$

Finally, the positivity of the concentration established by Kan-On, Narukawa and Teramoto in [4] also holds true for our problem, with exactly the same proof. That is, we have

Lemma 4.2 (Positivity of the concentration)**.** *Let* $(\mathbf{u}, q, m)$ *be a solution of* (1.1)-(1.2) *given by Theorem 4.1. Then* $m(x) > 0$ *for all* $x \in \Omega$.

5. Uniqueness of the solution

Theorem 5.1. *Assume that* $\nu(\cdot)$ *is a Lipschitz continuous function and that* $(\mathbf{u}, \overline{m})$ *is a weak solution of the Problem* (3.1) *such that* $(|D(\mathbf{u})|+|A_1\overline{m}|+|\nabla E|_4))$ *is small enough; then such solution is unique.*

Note that as before, this last requirement on $|\nabla E|_4$ can be attained for small enough U.

Proof. Let $(\mathbf{u}_1, \overline{m}_1), (\mathbf{u}_2, \overline{m}_2)$ be two weak solutions of (3.1), with $\overline{m}_1$ and $\overline{m}_2$ in $(H^2(\Omega) \cap Y)$ and both satisfying the stated smallness condition. By denoting $\mathbf{z} = \mathbf{u}_1 - \mathbf{u}_2$ and $\varphi = \overline{m}_1 - \overline{m}_2$, then we have that $\mathbf{z} \in \mathbf{J}_0$, $\varphi \in (H^2(\Omega) \cap Y)$ and the pair $(\mathbf{z},\varphi)$ satisfies, for all $\mathbf{w}\in \mathbf{J}_0(\Omega)$ and $\phi\in L^2(\Omega)$, the following equations:

$$\begin{gathered} 2(\nu(\overline{m}_1)D(\mathbf{z}),D(\mathbf{w}))+2((\nu(\overline{m}_1)-\nu(\overline{m}_2))D(\mathbf{u}_2),D(\mathbf{w}))\\ +B_0(\mathbf{z},\mathbf{u}_1,\mathbf{w})+B_0(\mathbf{u}_2,\mathbf{z},\mathbf{w})+(\varphi\cdot\chi,\mathbf{w})=0,\\ \theta(\nabla\varphi,\nabla\phi)+B_1(\mathbf{z},\overline{m}_1+E,\phi)+B_1(\mathbf{u}_2,\varphi,\phi)-U\Big(\varphi,\frac{\partial\phi}{\partial x_3}\Big)=0. \end{gathered} \tag{5.1}$$

By setting $\mathbf{w}=\mathbf{z}$ and $\phi=-A_1\varphi$ in (5.1), we obtain

$$\begin{aligned} 2\nu_0|D(\mathbf{z})|^2 &\le |2((\nu(\overline{m}_1)-\nu(\overline{m}_2))D(\mathbf{u}_2),D(\mathbf{z}))+B_0(\mathbf{z},\mathbf{u}_1,\mathbf{z})+(\varphi\cdot\chi,\mathbf{z})|\\ &\le 2C_1|\varphi|_\infty|D(\mathbf{u}_2)|\ |D(\mathbf{z})|+C_0|D(\mathbf{z})|\ |D(\mathbf{u}_1)|\ |D(\mathbf{z})|\\ &\quad +\overline{C}_PC_P|A_1\varphi|\ |D(\mathbf{z})|, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \theta|A_1\varphi|^2 &\le \Big|B_1(\mathbf{z},\overline{m}_1+E,A_1\varphi)+B_1(\mathbf{u}_2,\varphi,A_1\varphi)+U\Big(\frac{\partial\varphi}{\partial x_3},A_1\varphi\Big)\Big|\\ &\le C_0|D(\mathbf{z})|(|A_1\overline{m}_1|+|\nabla E|_4)|A_1\varphi|+C_0|D(\mathbf{u}_2)|\ |A_1\varphi|^2+U\overline{C}_P|A_1\varphi|^2, \end{aligned}$$

where C_1 is the Lipschitz constant of $\nu(\cdot)$; that is, $|\nu(r)-\nu(s)|\le C_1|r-s|$, for all $r,s\in\mathbb{R}$. Thus, by taking U small enough so that $\theta-U\overline{C}_P>0$, we obtain

$$|A_1\varphi|\le [C_0/(\theta-U\overline{C}_P)](|D(\mathbf{z})|(|A_1\overline{m}_1|+|\nabla E|_4)+|D(\mathbf{u}_2)|\ |A_1\varphi|),$$

which, under the condition that $(C_0/(\theta-U\overline{C}_P))|D(\mathbf{u}_2)|<1/2$, gives

$$|A_1\varphi|\le [C_0/(\theta-U\overline{C}_P)](|A_1\overline{m}_1|+|\nabla E|_4)|D(\mathbf{z})|. \tag{5.3}$$

Since $|\varphi|_\infty\le C|A_1\varphi|$, by combining (5.2) and (5.3), we obtain $|D(\mathbf{z})|\le D_3|D(\mathbf{z})|$, where

$$\begin{aligned} D_3=&\frac{C_0}{\nu_0(\theta-U\overline{C}_P)}\Big(C_1C(|A_1\overline{m}_1|+|\nabla E|_4)|D(\mathbf{u}_2)|+\frac{\overline{C}_PC_P}{2}(|A_1\overline{m}_1|+|\nabla E|_4)\Big)\\ &+\frac{C_0}{2\nu_0}|D(\mathbf{u}_1)|. \end{aligned}$$

When $D_3<1$, we conclude that $|D(\mathbf{z})|=0$ and, from (5.3), that $|A_1\varphi|=0$. Since $\mathbf{z}\in\mathbf{J}_0(\Omega)$ and $\varphi\in H^2(\Omega)\cap Y$, its follows that $\mathbf{z}=0$, $\varphi=0$ in Ω; consequently, $\mathbf{u}_1=\mathbf{u}_2$ and $\overline{m}_1=\overline{m}_2$. □

Acknowledgments. J. L. Boldrini was partially supported by grant 305467/2011-5 from CNPq (Brazil), grants 2007/06638-6 and 2009/15098-0 from FAPESP (Brazil), and grant 1120260 from Fondecyt (Chile).

M. A. Rojas-Medar was partially supported by grant MTM2012-32325 from DGI-MEC (Spain), and grants 1120260, GI/C-UBB 121909 from Fondecyt-Chile.

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