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STABILIZATION OF A SEMILINEAR WAVE EQUATION WITH VARIABLE COEFFICIENTS AND A DELAY TERM IN THE BOUNDARY FEEDBACK

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Abstract. We study the uniform stabilization of a semilinear wave equation with variable coefficients and a delay term in the boundary feedback. The Riemannian geometry method is applied to prove the exponential stability of the system by introducing an equivalent energy function.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega = \Gamma_0 \bigcup \Gamma_1$. Assume that Γ_0 is nonempty and relatively open in $\partial\Omega$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Define

$$\mathcal{A}u = -\operatorname{div}(A(x)\nabla u) \quad \text{for } u \in H^1(\Omega), \tag{1.1}$$

where $\operatorname{div}(X)$ denote the divergence of the vector field X in the Euclidean metric, $A(x) = (a_{ij}(x))$ is a matrix function with $a_{ij} = a_{ji}$ of class C^1, satisfying

$$\begin{gathered} \lambda \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda \sum_{i=1}^{n} \xi_i^2 \quad \forall x \in \Omega, \\ 0 \neq \xi = (\xi_1, \xi_2 \cdots \xi_n)^T \in \mathbb{R}^n, \end{gathered} \tag{1.2}$$

for some positive constants λ, Λ.

We consider the initial boundary value problem

$$\begin{gathered} u_{tt}(x,t) + \mathcal{A}u(x,t) + h(\nabla u) + f(u) = 0 \quad \text{in } \Omega \times (0,+\infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu_A} = -\mu_1 u_t(x,t) - \mu_2 u_t(x,t-\tau) \quad \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\ u_t(x,t-\tau) = g_0(x,t-\tau) \quad \text{on } \Gamma_1 \times [0,\tau], \end{gathered} \tag{1.3}$$

where

$$\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \nu_i,$$

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and $\nu(x) = (\nu_1, \nu_2, \cdots \nu_n)^T$ denotes the outside unit normal vector of the boundary, $\nu_A = A\nu$. $f : R \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuous nonlinear functions satisfying some assumptions (see (A1), (A2)). Here, $\tau > 0$ is a time delay, μ_1, μ_2 are positive real numbers, and the initial values (u_0, u_1, g_0) belong to suitable spaces.

The problem of uniform stabilization for the solution to the wave equation has been widely investigated. We refer the reader to [3, 6, 8, 10, 11]. The system (1.3) was claimed to be a nondissipative wave system in the literature. The stability of a nondissipative system is a important mathematical problem and has attracted much attention in recent years. On the other hand, delay effects arise in many applications and practical problems and it is well-known that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in absence of the delay, see [4, 12, 13, 16]. Consequently, we consider the stabilization for a nondissipative wave system with a delay term in the boundary feedback.

When $A(x) \equiv I$, we say that the system (1.3) is of constant coefficients. In this case, many results on such problems are available in the literature, see [4, 6, 10, 12, 13, 16]. The coefficients matrix $A(x)$ is related to the material in applications. Our main goal is to dispense with the restriction $A(x) \equiv I$, and we consider the variable coefficients case. The main tool is the Riemannian geometric method which was first introduced in [17] to obtain the observability inequality. This method was then applied to established the controllability and stabilization in [1, 2, 9, 15, 18] for second-order hyperbolic equations with the variable coefficients principal part. For a survey on the Riemannian geometric method, we refer the reader to [7].

We will show that the nondissipative system (1.3) is essentially a dissipative system by introducing an equivalent energy function of the system. A similar nondissipative system with variable coefficients has been studied in [8]. However, the delay term was not considered. The appearance of the delay term often brings great difficulty. We will select a new equivalent energy function, which is different from the equivalent energy function in [8], to obtain the exponential stability of the solution to (1.3).

Our paper is organized as follows. In Section 2, some necessary notation is introduced and the main results are presented. In Section 3, some preliminary results and the main theorem are proved. The proof of the existence theorem of the solution is presented in the Appendix.

2. Notation and statement of results

All definitions and notation are standard and classical in the literature, see [14]. Set

$$G(x) = (g_{ij}(x)) = A^{-1}(x). \tag{2.1}$$

For each $x \in \mathbb{R}^n$ in the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$, we denote the inner product and the norm as

$$g(X,Y) = \langle X, Y\rangle_g = \sum_{i,j=1}^n g_{ij}(x)\alpha_i\beta_j, \quad |X|_g = \langle X, X\rangle_g^{1/2} \tag{2.2}$$

for all

$$X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x, \quad x \in \mathbb{R}^n.$$

From [17, Lemma 2.1], it holds that

$$\langle X(x), A(x)Y(x)\rangle_g = X(x)\cdot Y(x) \quad x\in\mathbb{R}^n, \tag{2.3}$$

where the central dot denotes the Euclidean product of $\mathbb{R}^n$.

It is easy to check that $(\mathbb{R}^n, g)$ is a Riemannian manifold with the metric g.

Denote as D the Levi-Civita connection in the Riemannian metric g. Let H be a vector field on $(\mathbb{R}^n, g)$. Then the covariant differential DH of H determines a bilinear form on $\mathbb{R}^n_x\times\mathbb{R}^n_x$ for each $x\in\mathbb{R}^n$, by

$$DH(X,Y) = \langle D_Y H, X\rangle_g \quad \forall X,Y\in\mathbb{R}^n_x, \tag{2.4}$$

where $D_Y H$ stands for the covariant derivative of vector field H with respect to Y.

Denote as $\nabla_g u$ the the gradient of u in the Riemannian metric g. It follows from [17, Lemma 2.1] that

$$\nabla_g u = \sum_{i=1}^n \Big(\sum_{j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)\frac{\partial}{\partial x_i}, \quad |\nabla_g u|_g^2 = \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{2.5}$$

We refer the reader to [17] for further relationships.

The following assumptions are needed for proving our results.

(A1) $f:\mathbb{R}\to\mathbb{R}$ is a C^1-function deriving from a potential

$$F(s) = \int_0^s f(\tau)d\tau \ge 0 \quad \forall s\in\mathbb{R}, \tag{2.6}$$

and satisfies

$$|f(s)|\le b_1|s|^\rho + b_2, \quad |f'(s)|\le b_1|s|^{\rho-1}+b_2, \tag{2.7}$$

where b_1, b_2 are positive constants and the parameter ρ satisfies

$$1\le\rho\le\begin{cases} 2, & n\le 3,\\ \frac{n}{n-2}, & n\ge 4.\end{cases} \tag{2.8}$$

(A2) $h:\mathbb{R}^n\to\mathbb{R}$ is a C^1-function and there exist two constants $\beta>0$ and $L>0$ such that

$$|h(\xi)|\le\beta\sqrt{\lambda}|\xi|, \quad |\nabla h(\xi)|\le L \quad \forall\xi\in\mathbb{R}^n. \tag{2.9}$$

(A3) There exists a vector field H on the Riemannian manifold $(\mathbb{R}^n, g)$ such that

$$DH(X,X) = c(x)|X|_g^2 \quad \forall x\in\overline{\Omega},\ X\in\mathbb{R}^n_x. \tag{2.10}$$

Let $b=\min_{\overline{\Omega}} c(x) > 0$ and $B=\max_{\overline{\Omega}} c(x)$ such that

$$B<\min\big\{b+\frac{2b-3\varepsilon_0}{n}, r\big(b-\frac{\varepsilon_0}{n}\big)\big\} \quad \text{for some } \varepsilon_0\in(0,b) \text{ and } r>1. \tag{2.11}$$

Moreover,

$$H\cdot\nu\le 0 \text{ on } \Gamma_0 \text{ and } H\cdot\nu\ge\delta>0 \text{ on } \Gamma_1 \text{ for some constant } \delta. \tag{2.12}$$

Note that (A3) implies that

$$nb\le\operatorname{div}(H)\le nB. \tag{2.13}$$

A number of examples of such a vector field H on $(\mathbb{R}^n, g)$ for which the condition (2.10) is satisfied without any constraints on B are presented in [17].

When $A(x)\equiv I$, condition (2.10) is automatically satisfied by choosing $H = x-x_0$.

If $\mu_2 = 0$, that is, in absence of the delay term, the energy of the system (1.3) is exponentially decaying to zero, see [8]. On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the boundary condition on Γ_1, the system (1.3) becomes unstable. See, for instance [5]. So it is interesting to seek a stabilization result when both μ_1 and μ_2 are nonzero. In this case, the boundary feedback is composed of two parts and only one of them has a delay.

The stability of a linear wave equation with constant coefficients and a delay in the boundary feedback has been studied in [12]. There it is shown that if $\mu_1 = \mu_2$, then there exists a sequence of arbitrary small (and large) delays such that instabilities occur, if $\mu_2 > \mu_1$; delays which destabilize the system were also obtained.

In this article, in agreement with [12], we assume that

$$\mu_2 < \mu_1. \tag{2.14}$$

Set

$$V = \{v \in H^1(\Omega) | v = 0 \quad \text{on } \Gamma_0\}, \quad W = H^2(\Omega) \cap V.$$

Theorem 2.1. *Under assumptions* (A1), (A2) *and* (2.14), *for any given initial values* $(u_0, u_1) \in W \times W$, $g_0 \in C^1([-\tau, 0]; L^2(\Gamma_1))$, *satisfying*

$$\frac{\partial u_0}{\partial \nu_A} = -\mu_1 u_1 - \mu_2 g_0(x, -\tau) \quad \text{on } \Gamma_1 \tag{2.15}$$

and $T > 0$, *system* (1.3) *admits a unique strong solution* u *on* $(0, T)$ *such that*

$$u \in L^\infty(0, T; V), \quad u_t \in L^\infty(0, T; V), \quad u_{tt} \in L^\infty(0, T; L^2(\Omega)). \tag{2.16}$$

Moreover, if $(u_0, u_1) \in V \times L^2(\Omega)$, $g_0 \in L^2(-\tau, 0; L^2(\Gamma_1))$, *then* (1.3) *possess at least a weak solution in the space* $C([0, T]; V) \cap C^1([0, T]; L^2(\Omega))$.

The Galerkin's approximation will be used for proving Theorem 2.1. Under assumption (2.14), define the energy of (1.3) as

$$E(t) = \frac{1}{2} \int_\Omega [|u_t|^2 + |\nabla_g u|_g^2 + 2F(u)]dx + \frac{\xi}{2} \int_0^1 \int_{\Gamma_1} u_t^2(x, t - \tau\rho) d\Gamma d\rho, \tag{2.17}$$

where ξ is a strictly positive constant satisfying

$$\tau\mu_2 \le \xi \le \tau(2\mu_1 - \mu_2). \tag{2.18}$$

Denote $E_s(t)$ as

$$E_s(t) = \frac{1}{2} \int_\Omega [|u_t|^2 + |\nabla_g u|_g^2 + 2F(u)]dx. \tag{2.19}$$

Our main result is the following theorem.

Theorem 2.2. *Let* u *be a (strong or weak) solution of* (1.3). *Suppose that* (A1)–(A3) *and* (2.14) *hold. In addition assume that* f *satisfies*

$$2rF(s) \le sf(s) \quad \text{for some constant } r > 1, \text{ and all } s \in \mathbb{R}. \tag{2.20}$$

If β *in* (2.9) *is sufficiently small, then there exist positive constants* C *and* ω *independent of initial values such that*

$$E(t) \le CE(0) \exp\{-\omega t\} \quad \forall t \ge 0. \tag{2.21}$$

3. Proof of Theorem 2.2

Proposition 3.1. *Let u be a (strong or weak) solution to the system* (1.3)*, the following estimate holds:*

$$\frac{dE(t)}{dt} \le -C_1 \int_{\Gamma_1} [u_t^2(x,t) + u_t^2(x,t-\tau)] d\Gamma + \beta E_s(t), \tag{3.1}$$

with C_1 is a positive constant to be specified later.

Proof. Differentiating (2.17), we obtain

$$\begin{aligned}\frac{dE(t)}{dt} &= \int_\Omega [u_t u_{tt} + \langle \nabla_g u, \nabla_g u_t\rangle_g + f(u)u_t]\,dx \\ &\quad + \xi \int_0^1 \int_{\Gamma_1} u_t(x,t-\tau\rho) u_{tt}(x,t-\tau\rho) d\Gamma d\rho \\ &= \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} u_t d\Gamma - \int_\Omega u_t h(\nabla u) dx \\ &\quad + \xi \int_0^1 \int_{\Gamma_1} u_t(x,t-\tau\rho) u_{tt}(x,t-\tau\rho) d\Gamma d\rho.\end{aligned} \tag{3.2}$$

Now, let $y(x,\rho) = u(x,t-\tau\rho)$. So we have

$$u_t = -\frac{1}{\tau} y_\rho, \quad u_{tt} = \frac{1}{\tau^2} y_{\rho\rho}. \tag{3.3}$$

Therefore,

$$\int_0^1 \int_{\Gamma_1} u_t(x,t-\tau\rho) u_{tt}(x,t-\tau\rho) d\Gamma d\rho = -\frac{1}{\tau^3} \int_0^1 \int_{\Gamma_1} y_\rho(x,\rho) y_{\rho\rho}(x,\rho) d\Gamma d\rho. \tag{3.4}$$

Integrating by parts in ρ, we obtain

$$\begin{aligned}&\int_0^1 \int_{\Gamma_1} y_\rho(x,\rho) y_{\rho\rho}(x,\rho) d\Gamma d\rho \\ &= \Big(\int_{\Gamma_1} y_\rho(x,\rho) y_\rho(x,\rho) d\Gamma \Big)\Big|_0^1 - \int_0^1 \int_{\Gamma_1} y_{\rho\rho}(x,\rho) y_\rho(x,\rho) d\Gamma d\rho \\ &= \int_{\Gamma_1} \big[y_\rho^2(x,1) - y_\rho^2(x,0) \big]\, d\Gamma - \int_0^1 \int_{\Gamma_1} y_{\rho\rho}(x,\rho) y_\rho(x,\rho) d\Gamma d\rho.\end{aligned} \tag{3.5}$$

That is

$$\int_0^1 \int_{\Gamma_1} y_\rho(x,\rho) y_{\rho\rho}(x,\rho) d\Gamma d\rho = \frac{1}{2} \int_{\Gamma_1} \big[y_\rho^2(x,1) - y_\rho^2(x,0) \big]\, d\Gamma.$$

Therefore,

$$\begin{aligned}&\int_0^1 \int_{\Gamma_1} u_t(x,t-\tau\rho) u_{tt}(x,t-\tau\rho) d\Gamma d\rho \\ &= -\frac{1}{2\tau^3} \int_{\Gamma_1} \big[y_\rho^2(x,1) - y_\rho^2(x,0) \big]\, d\Gamma \\ &= \frac{1}{2\tau} \int_{\Gamma_1} \big[u_t^2(x,t) - u_t^2(x,t-\tau) \big]\, d\Gamma\end{aligned} \tag{3.6}$$

which, together with the boundary condition of (1.3) on Γ_1 and (3.2), leads to

$$\begin{aligned}\frac{dE(t)}{dt} &= -\mu_1 \int_{\Gamma_1} u_t^2(x,t)d\Gamma - \mu_2 \int_{\Gamma_1} u_t(x,t)u_t(x,t-\tau)d\Gamma - \int_\Omega u_t(x,t)h(\nabla u)dx \\ &\quad + \frac{\xi}{2\tau}\int_{\Gamma_1}\big[u_t^2(x,t) - u_t^2(x,t-\tau)\big]\,d\Gamma.\end{aligned} \tag{3.7}$$

Applying the Cauchy-Schwarz inequality to (3.7), from (2.9) and the fact $F(s) \geq 0$, we have

$$\begin{aligned}\frac{dE(t)}{dt} &\leq \Big(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\Big)\int_{\Gamma_1} u_t^2(x,t)d\Gamma + \Big(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\Big)\int_{\Gamma_1} u_t^2(x,t-\tau)d\Gamma \\ &\quad + \frac{\beta}{2}\int_\Omega [|u_t|^2 + |\nabla_g u|_g^2]dx \\ &\leq \Big(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\Big)\int_{\Gamma_1} u_t^2(x,t)d\Gamma + \Big(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\Big)\int_{\Gamma_1} u_t^2(x,t-\tau)d\Gamma \\ &\quad + \beta E_s(t),\end{aligned} \tag{3.8}$$

which implies

$$\frac{dE(t)}{dt} \leq -C_1 \int_{\Gamma_1}\big[u_t^2(x,t) + u_t^2(x,t-\tau)\big]\,d\Gamma + \beta E_s(t), \tag{3.9}$$

with

$$C_1 = \min\{\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{\mu_2}{2}\}.$$

Due to (2.18), we have $C_1 > 0$. The proof is complete. □

Remark 3.2. From inequality (3.9), it seems that the system (1.3) is not dissipative. However, this is a wrong impression. Actually, by introducing an equivalent energy function, we will find that the system (1.3) is essentially dissipative under some suitable conditions.

Lemma 3.3. *Let H be a vector field on $\overline{\Omega}$. For any (strong or weak) solution to* (1.3) *we have*

$$\frac{\partial u}{\partial \nu_A} H(u) = |\nabla_g u|_g^2 (H \cdot \nu) \quad \text{on } \Gamma_0. \tag{3.10}$$

Proof. Let $x \in \Gamma_0$. We decompose $\nabla_g u$ into a direct sum in $(\mathbb{R}^n_x, g)$

$$\nabla_g u(x) = \big\langle \nabla_g u(x), \frac{\nu_A(x)}{|\nu_A|_g}\big\rangle_g \frac{\nu_A(x)}{|\nu_A|_g} + Y(x), \tag{3.11}$$

where $Y(x) \in \mathbb{R}^n_x$ with $\langle Y(x), \nu_A(x)\rangle_g = 0$. Taking (2.3) into account, we obtain

$$Y(x)\cdot\nu(x) = \langle Y(x), A(x)\nu(x)\rangle_g = \langle Y(x), \nu_A(x)\rangle_g = 0, \tag{3.12}$$

which imply $Y(x) \in \Gamma_{0x}$, the tangent space of Γ_0 at x.

Since $u = 0$ on Γ_0, it follows from (3.11) and (3.12) that

$$\begin{aligned}|\nabla_g u|_g^2 = \nabla_g u(u) &= \frac{1}{|\nu_A|_g^2}\langle \nabla_g u(x), \nu_A(x)\rangle_g^2 + Y(u) \\ &= \frac{1}{|\nu_A|_g^2}\Big|\frac{\partial u}{\partial \nu_A}\Big|^2.\end{aligned} \tag{3.13}$$

Similarly, H can be decomposed into a direct sum

$$H = \langle H(x), \frac{\nu_A(x)}{|\nu_A|_g}\rangle_g \frac{\nu_A(x)}{|\nu_A|_g} + Z(x), \tag{3.14}$$

where $Z(x) \in \Gamma_{0x}$.

Recalling that $u = 0$ on Γ_0, from (2.3) and (3.14), we obtain

$$H(u) = \frac{\langle H(x), \nu_A(x)\rangle_g}{|\nu_A|_g^2}\big(\frac{\partial u}{\partial \nu_A}\big) = \frac{H(x)\cdot\nu(x)}{|\nu_A|_g^2}\big(\frac{\partial u}{\partial \nu_A}\big) \tag{3.15}$$

which, together with (3.13), leads to (3.10). The proof is complete. □

Let

$$P(t) = \int_\Omega [2H(u) + (nb - \varepsilon_0)u]\, u_t dx \quad \text{for some } \varepsilon_0 \in (0, b). \tag{3.16}$$

Proposition 3.4. *Let u be a (strong or weak) solution of* (1.3)*, under the assumptions of Theorem 2.2, there exist two positive constants θ and N such that*

$$\frac{dP(t)}{dt} \le -2\theta E_s(t) + N\int_{\Gamma_1} [u_t^2(x,t) + u_t^2(x, t-\tau)] d\Gamma. \tag{3.17}$$

Proof. Differentiating (3.16) with respect to t we obtain

$$\begin{aligned} \frac{dP(t)}{dt} &= \int_\Omega u_t[2H(u_t) + (nb - \varepsilon_0)u_t]dx - \int_\Omega \mathcal{A}u[2H(u) + (nb-\varepsilon_0)u]dx \\ &\quad - \int_\Omega h(\nabla u)[2H(u) + (nb-\varepsilon_0)u]dx - \int_\Omega f(u)[2H(u) + (nb-\varepsilon_0)u]dx \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t), \end{aligned} \tag{3.18}$$

where

$$\begin{gathered} I_1(t) = \int_\Omega u_t[2H(u_t) + (nb-\varepsilon_0)u_t]dx, \\ I_2(t) = -\int_\Omega \mathcal{A}u[2H(u) + (nb-\varepsilon_0)u]dx, \\ I_3(t) = -\int_\Omega h(\nabla u)[2H(u) + (nb-\varepsilon_0)u]dx, \\ I_4(t) = -\int_\Omega f(u)[2H(u) + (nb-\varepsilon_0)u]dx. \end{gathered}$$

Now we estimate $I_i(t)$, $(i = 1, 2, 3, 4)$. Noting that $u = 0$ on Γ_0, we have

$$\begin{aligned} I_1(t) &= \int_\Omega H(u_t^2)dx + \int_\Omega (nb-\varepsilon_0)u_t^2 dx \\ &= \int_{\Gamma_1} u_t^2(H\cdot\nu)d\Gamma - \int_\Omega [\operatorname{div}(H) - nb]u_t^2 dx - \varepsilon_0\int_\Omega u_t^2 dx, \end{aligned}$$

where $\operatorname{div}(H)$ denote the divergence of the vector field H in the Euclidean metric. Denoting $M = \max_{\overline{\Omega}} |H|_g$, from (2.13) we obtain

$$I_1(t) \le M\int_{\Gamma_1} u_t^2 d\Gamma - \varepsilon_0 \int_\Omega u_t^2 dx. \tag{3.19}$$

Next, we estimate $I_2(t)$.

$$\begin{aligned}
I_2(t) &= 2\int_{\partial\Omega}\frac{\partial u}{\partial\nu_A}H(u)d\Gamma - 2\int_\Omega \langle\nabla_g u, \nabla_g(H(u))\rangle_g dx + \int_{\partial\Omega}(nb-\varepsilon_0)u\frac{\partial u}{\partial\nu_A}d\Gamma \\
&\quad - \int_\Omega (nb-\varepsilon_0)|\nabla_g u|_g^2 dx \\
&= -2\int_\Omega DH(\nabla_g u, \nabla_g u)dx + \int_\Omega [\operatorname{div}(H) - nb + \varepsilon_0]\,|\nabla_g u|_g^2 dx \\
&\quad + \int_{\Gamma_1}\Big[2\frac{\partial u}{\partial\nu_A}H(u) - |\nabla_g u|_g^2(H\cdot\nu) + (nb-\varepsilon_0)u\frac{\partial u}{\partial\nu_A}\Big]d\Gamma \\
&\quad + \int_{\Gamma_0}|\nabla_g u|_g^2(H\cdot\nu)d\Gamma,
\end{aligned} \tag{3.20}$$

where the validity of the last step comes from the fact $u = 0$ on Γ_0 and (3.10). Since

$$\int_{\Gamma_1} 2\frac{\partial u}{\partial\nu_A}H(u)d\Gamma \le \int_{\Gamma_1}\Big[\delta|\nabla_g u|_g^2 + \frac{M^2}{\delta}|\frac{\partial u}{\partial\nu_A}|^2\Big]d\Gamma, \tag{3.21}$$

from (2.10), (2.12), (2.13), (3.20), (3.21) we obtain

$$\begin{aligned}
I_2(t) &\le \int_\Omega [\operatorname{div}(H) - (n+2)b + \varepsilon_0]\,|\nabla_g u|_g^2 dx \\
&\quad + \int_{\Gamma_1}\Big[\delta|\nabla_g u|_g^2 + \frac{M^2}{\delta}|\frac{\partial u}{\partial\nu_A}|^2 - \delta|\nabla_g u|_g^2 + (nb-\varepsilon_0)u\frac{\partial u}{\partial\nu_A}\Big]d\Gamma \\
&\le [nB - (n+2)b + \varepsilon_0]\int_\Omega |\nabla_g u|_g^2 dx \\
&\quad + \int_{\Gamma_1}\Big[\frac{M^2}{\delta}|\frac{\partial u}{\partial\nu_A}|^2 + (nb-\varepsilon_0)u\frac{\partial u}{\partial\nu_A}\Big]d\Gamma.
\end{aligned} \tag{3.22}$$

Using the trace theorem,

$$\int_{\Gamma_1}|v|^2 d\Gamma \le \widetilde{C}\int_\Omega |\nabla_g v|_g^2 dx$$

for some constant $\widetilde{C} > 0$, for all $v \in V$, and the boundary condition of (1.3) on Γ_1, we estimate the last term on the right-hand side of (3.22) as

$$\begin{aligned}
&\int_{\Gamma_1}\Big[\frac{M^2}{\delta}|\frac{\partial u}{\partial\nu_A}|^2 + (nb-\varepsilon_0)u\frac{\partial u}{\partial\nu_A}\Big]d\Gamma \\
&\le \int_{\Gamma_1}\frac{M^2}{\delta}|\frac{\partial u}{\partial\nu_A}|^2 d\Gamma + (nb-\varepsilon_0)\int_{\Gamma_1}\Big[\eta|u|^2 + \frac{1}{4\eta}|\frac{\partial u}{\partial\nu_A}|^2\Big]d\Gamma \\
&\le \widetilde{C}(nb-\varepsilon_0)\eta\int_\Omega |\nabla_g u|_g^2 dx + \Big(\frac{nb-\varepsilon_0}{4\eta} + \frac{M^2}{\delta}\Big)\int_{\Gamma_1}|\frac{\partial u}{\partial\nu_A}|^2 d\Gamma \\
&\le \widetilde{C}(nb-\varepsilon_0)\eta\int_\Omega |\nabla_g u|_g^2 dx \\
&\quad + C_2\Big(\frac{nb-\varepsilon_0}{4\eta} + \frac{M^2}{\delta}\Big)\int_{\Gamma_1}\big[u_t^2(x,t) + u_t^2(x,t-\tau)\big]\,d\Gamma \\
&= \varepsilon_0\int_\Omega |\nabla_g u|_g^2 dx + M_1\int_{\Gamma_1}\big[u_t^2(x,t) + u_t^2(x,t-\tau)\big]\,d\Gamma,
\end{aligned} \tag{3.23}$$

where $\eta = \frac{\varepsilon_0}{\widetilde{C}(nb-\varepsilon_0)}$, $M_1 = C_2\big(\frac{nb-\varepsilon_0}{4\eta} + \frac{M^2}{\delta}\big)$ were used in the last step and C_2 is a positive constant. Substitute (3.23) into (3.22) to obtain

$$I_2(t) \le [nB - (n+2)b + 2\varepsilon_0]\int_\Omega |\nabla_g u|_g^2 dx + M_1 \int_{\Gamma_1} [u_t^2(x,t) + u_t^2(x,t-\tau)] d\Gamma. \tag{3.24}$$

Applying the Cauchy inequality and recalling (2.9), we can obtain the estimation of $I_3(t)$ as follows:

$$\begin{aligned} I_3(t) &\le 2\beta M \int_\Omega |\nabla_g u|_g^2 dx + \beta(nb-\varepsilon_0)\int_\Omega |\nabla_g u|_g |u| dx \\ &\le \beta\Big[2M + \frac{nb-\varepsilon_0}{2}(1+\overline{C})\Big]\int_\Omega |\nabla_g u|_g^2 dx, \end{aligned} \tag{3.25}$$

where $\overline{C}$ is a positive constant satisfying $\int_\Omega |u|^2 \le \overline{C}\int_\Omega |\nabla_g u|_g^2 dx$ for all $u \in V$.

Finally, we estimate $I_4(t)$. By (2.12), (2.13), (2.20), the nonnegativity of $F, F(0) = 0$, $u = 0$ on Γ_0, we have

$$\begin{aligned} I_4(t) &\le -(nb-\varepsilon_0) r \int_\Omega 2F(u)dx - 2\int_\Omega H(F(u))dx \\ &= -\int_\Omega [(nb-\varepsilon_0)r - \operatorname{div}(H)] 2F(u) dx - \int_{\Gamma_1} 2F(u)(H\cdot\nu) d\Gamma \\ &\le [nB - (nb-\varepsilon_0)r]\int_\Omega 2F(u)dx. \end{aligned} \tag{3.26}$$

Let

$$0 < \beta < \frac{\varepsilon_0}{2M + \frac{(nb-\varepsilon_0)}{2}(1+\overline{C})}.$$

Combine (3.18), (3.19), (3.24), (3.25) and (3.26) to obtain (3.17), where

$$\begin{gathered} \theta := \min\{(n+2)b - nB - 3\varepsilon_0, (nb-\varepsilon_0)r - nB, \varepsilon_0\}, \\ N := M_1 + M. \end{gathered} \tag{3.27}$$

By (2.11) and the values of M and M_1, we have $\theta > 0, N > 0$. The proof is complete. □

Proof of Theorem 2.2. Define

$$S(t) := \int_{t-\tau}^t \int_{\Gamma_1} e^{s-t} u_t^2(x,s) d\Gamma ds. \tag{3.28}$$

We can easily estimate

$$\begin{aligned} \frac{dS(t)}{dt} &= \int_{\Gamma_1} u_t^2(x,t)d\Gamma - \int_{\Gamma_1} e^{-\tau} u_t^2(x,t-\tau)d\Gamma - \int_{t-\tau}^t\int_{\Gamma_1} e^{s-t}u_t^2(x,s)d\Gamma ds \\ &\le \int_{\Gamma_1} u_t^2(x,t)d\Gamma - e^{-\tau}\int_{\Gamma_1} u_t^2(x,t-\tau)d\Gamma - e^{-\tau}\int_{t-\tau}^t\int_{\Gamma_1} u_t^2(x,s)d\Gamma ds. \end{aligned} \tag{3.29}$$

Let us define a new energy function for (1.3) as

$$L(t) := E(t) + \gamma_1 P(t) + \gamma_2 S(t), \tag{3.30}$$

where γ_1, γ_2 are suitable positive small constants that will be specified later on.

Note that $L(t)$ is equivalent to the energy $E(t)$ if γ_1, γ_2 are small enough. In particular, there exist a positive constant C_3 and suitable positive constants α_1, α_2 such that

$$\alpha_1 E(t) \le L(t) \le \alpha_2 E(t) \quad \forall 0 \le \gamma_1, \gamma_2 \le C_3. \tag{3.31}$$

Therefore, $L(t)$ is an equivalent energy function of(1.3) for small γ_1, γ_2.

Differentiating the function $L(t)$ and recalling (3.1), (3.17), (3.29) we deduce

$$\begin{aligned} \frac{dL(t)}{dt} &= \frac{dE(t)}{dt} + \gamma_1 \frac{dP(t)}{dt} + \gamma_2 \frac{dS(t)}{dt} \\ &\le (-2\gamma_1\theta + \beta)E_s(t) + (-C_1 + \gamma_1 N + \gamma_2) \int_{\Gamma_1} u_t^2(x,t) d\Gamma \\ &\quad + (-C_1 + \gamma_1 N - \gamma_2 e^{-\tau}) \int_{\Gamma_1} u_t^2(x, t-\tau) d\Gamma \\ &\quad - \gamma_2 e^{-\tau} \int_{t-\tau}^{t} \int_{\Gamma_1} u_t^2(x,s) d\Gamma ds. \end{aligned} \tag{3.32}$$

Note that

$$\begin{aligned} E(t) &= E_s(t) + \frac{\xi}{2} \int_0^1 \int_{\Gamma_1} u_t^2(x, t-\tau\rho) d\Gamma d\rho \\ &= E_s(t) + \frac{\xi}{2\tau} \int_{t-\tau}^{t} \int_{\Gamma_1} u_t^2(x,s) d\Gamma ds. \end{aligned} \tag{3.33}$$

Choosing γ_1, γ_2 sufficiently small such that $-C_1 + \gamma_1 N + \gamma_2 < 0$, $-C_1 + \gamma_1 N - \gamma_2 e^{-\tau} < 0$ and choosing $\beta > 0$ small enough such that $-2\gamma_1\theta + \beta < 0$, from (3.32) and (3.33), we have

$$\frac{dL(t)}{dt} \le -\widehat{C} E(t), \tag{3.34}$$

with $\widehat{C}$ is a positive constant. Applying the second inequality of (3.31), from (3.34), we have

$$\frac{dL(t)}{dt} \le -\frac{\widehat{C}}{\alpha_2} L(t). \tag{3.35}$$

Then, we easily obtain

$$L(t) \le L(0) \exp(-\omega t) \quad \forall t \ge 0, \tag{3.36}$$

with ω is a positive constant. Using (3.31) again, we deduce the estimate (2.21). The proof is complete. □

4. Appendix: Proof of Theorem 2.1

As in [8], we use Galerkin approximations to prove the well-posedness of (1.3). The change of variable

$$v(x,t) = u(x,t) - \phi(x,t), \tag{4.1}$$

where

$$\phi(x,t) = u_0(x) + t u_1(x) \quad (x,t) \in Q := \Omega \times (0,T), \tag{4.2}$$

gives the following problem, which is equivalent to (1.3),

$$
\begin{gathered}
v_{tt} - \operatorname{div}(A(x)\nabla v) + h(\nabla v + \nabla \phi) + f(v + \phi) = \mathcal{F} \quad \text{in } \Omega \times (0,T), \\
v = 0 \quad \text{on } \Gamma_0 \times (0,T), \\
\frac{\partial v}{\partial \nu_A} = -\mu_1[v_t(x,t) + u_1] - \mu_2[v_t(x,t-\tau) + u_1] + \mathcal{B} \quad \text{on } \Gamma_1 \times (0,T), \\
v(x,0) = v_t(x,0) = 0 \quad \text{in } \Omega, \\
v_t(x,t-\tau) = g_0(x,t-\tau) - u_1 \quad \text{on } \Gamma_1 \times [0,\tau],
\end{gathered}
\tag{4.3}
$$

where $\mathcal{F} = \operatorname{div}(A(x)\nabla\phi)$, $\mathcal{B} = -\frac{\partial \phi}{\partial \nu_A}$ and $\operatorname{div}(X)$ denote the divergence of the vector field X in the Euclidean metric.

Let $\{w_i\}_{i\in N}$ be a basis for W that is orthonormal in $L^2(\Omega)$, and let V_m be the space spanned by $w_1 \cdots w_m$.

When $g_0 \in C^1([-\tau,0];L^2(\Gamma_1))$, we choose a sequence $g_{0m} \to g_0$ strongly in $C^1([-\tau,0];L^2(\Gamma_1))$. Now we define the approximation

$$
v_m(t) = \sum_{j=1}^{m} \gamma_j(t) w_j,
$$

where $v_m(t)$ are solutions to the Cauchy problem

$$
\begin{gathered}
\int_\Omega v_{mtt}(t) w dx + \int_\Omega \langle \nabla_g v_m(t), \nabla_g w\rangle_g dx + \int_\Omega h(\nabla v_m(t) + \nabla\phi(t)) w dx \\
+ \int_\Omega f(v_m(t)+\phi(t)) w dx + \int_{\Gamma_1} [\mu_1(v_{mt}(t) + u_1) + \mu_2(v_{mt}(t-\tau) + u_1)] w d\Gamma \\
= \int_\Omega \mathcal{F}(t) w dx + \int_{\Gamma_1} \mathcal{B} w d\Gamma, \\
v_m(0) = v_{mt}(0) = 0, \\
v_{mt}(x,t) = g_{0m}(x,t) - u_1 \quad \text{on } \Gamma_1 \times [-\tau, 0],
\end{gathered}
\tag{4.4}
$$

for all $w \in V_m$.

According to the standard theory of ordinary differential equations, the finite dimensional problem (4.4) has solutions $v_m(t)$ defined on some interval $[0,T_m)$. The a priori estimates that follow imply that $T_m = T$.

Step 1: The first-order estimate of v_m. Replacing w by $v_{mt}(t)$ in (4.4) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega [|v_{mt}(t)|^2 + |\nabla_g v_m(t)|_g^2 + 2F(v_m(t)+\phi(t))] dx\Big) \\
&+ \int_\Omega h(\nabla v_m(t) + \nabla\phi(t)) v_{mt}(t) dx - \int_\Omega f(v_m(t)+\phi(t)) u_1 dx \\
&= \int_\Omega \mathcal{F}(t) v_{mt}(t) dx + \frac{d}{dt}\Big(\int_{\Gamma_1} \mathcal{B}(t) v_m(t) d\Gamma\Big) - \int_{\Gamma_1} \mathcal{B}_t(t) v_m(t) d\Gamma \\
&\quad - \int_{\Gamma_1} [\mu_1(v_{mt}(t)+u_1) + \mu_2(v_{mt}(t-\tau)+u_1)][v_{mt}(t)+u_1] d\Gamma \\
&\quad + \int_{\Gamma_1} [\mu_1(v_{mt}(t)+u_1) + \mu_2(v_{mt}(t-\tau)+u_1)] u_1 d\Gamma.
\end{aligned}
\tag{4.5}
$$

Using the Sobolev imbedding theorem, Hölder's inequality, (A1) and the regularities of the initial values, we infer that

$$
\begin{aligned}
\int_\Omega f(v_m(t)+\phi(t))u_1 dx &\le C\int_\Omega |v_m(t)+\phi(t)|^\rho |u_1| dx + C\int_\Omega |u_1| dx \\
&\le C\Big(\int_\Omega |v_m(t)|^\rho |u_1| dx + \int_\Omega |\phi(t)|^\rho |u_1| dx\Big) + C \\
&\le C\Big(\int_\Omega |v_m(t)|^{2\rho} dx\Big)^{1/2}\Big(\int_\Omega |u_1|^2 dx\Big)^{1/2} \\
&\quad + C\Big(\int_\Omega |u_0|^\rho |u_1| + t^\rho |u_1|^{\rho+1}\Big) dx + C \\
&\le C\Big(\int_\Omega |\nabla_g v_m(t)|_g^2 dx\Big)^{\rho/2} + Ct^\rho + C.
\end{aligned} \tag{4.6}
$$

Here and in what follows, we use the constant $C > 0$ to denote some constants independent of functions involved although it may have different values in different contexts.

By (A2), it holds

$$
\begin{aligned}
&\int_\Omega h(\nabla v_m(t) + \nabla\phi(t)) v_{mt}(t) dx \\
&\le \frac{\beta^2}{2}\int_\Omega |\nabla_g v_m(t) + \nabla_g \phi(t)|_g^2 dx + \frac{1}{2}\int_\Omega |v_{mt}(t)|^2 dx.
\end{aligned} \tag{4.7}
$$

Combining (4.5)–(4.7), recalling the trace theorem,

$$
\int_{\Gamma_1} |v|^2 d\Gamma \le \widetilde{C}\int_\Omega |\nabla_g v|_g^2 dx
$$

for some constant $\widetilde{C} > 0$ and all $v \in V$, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega [|v_{mt}(t)|^2 + |\nabla_g v_m(t)|_g^2 + 2F(v_m(t)+\phi(t))] dx\Big) \\
&\le C\Big(\int_\Omega |\nabla_g v_m(t)|_g^2 dx\Big)^{\rho/2} + Ct^\rho + \frac{\beta^2}{2}\int_\Omega |\nabla_g v_m(t) + \nabla_g\phi(t)|_g^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |v_{mt}(t)|^2 dx + \frac{1}{2}\int_\Omega |\mathcal{F}(t)|^2 dx + \frac{1}{2}\int_\Omega |v_{mt}(t)|^2 dx \\
&\quad + \frac{d}{dt}\Big(\int_{\Gamma_1} \mathcal{B}(t) v_m(t) d\Gamma\Big) + \frac{\widetilde{C}}{2}\int_{\Gamma_1} |\mathcal{B}_t(t)|^2 d\Gamma + \frac{1}{2}\int_\Omega |\nabla_g v_m(t)|_g^2 dx + C \\
&\quad - \int_{\Gamma_1} [\mu_1(v_{mt}(t)+u_1) + \mu_2(v_{mt}(t-\tau)+u_1)][v_{mt}(t)+u_1] d\Gamma \\
&\quad + \int_{\Gamma_1} [\mu_1(v_{mt}(t)+u_1) + \mu_2(v_{mt}(t-\tau)+u_1)] u_1 d\Gamma\,.
\end{aligned}
$$

Integrating the obtained result over the interval $(0,t)$, noticing $v_m(0)=v_{mt}(0)=0$, $\frac{\rho}{2}\le 1$ and applying the trace theorem, we obtain

$$
\begin{aligned}
&\int_\Omega \big[|v_{mt}(t)|^2+|\nabla_g v_m(t)|_g^2+2F(v_m(t)+\phi(t))\big]\,dx\\
&\le (C+2\beta^2+1)\int_0^t\int_\Omega |\nabla_g v_m(s)|_g^2dxds+Ct^{\rho+1}+2\int_0^t\int_\Omega |v_{ms}(s)|^2dxds\\
&\quad+2\beta^2\int_0^t\int_\Omega |\nabla_g\phi(s)|_g^2dxds+\int_0^t\int_\Omega|\mathcal{F}(s)|^2dxds+2\int_{\Gamma_1}\mathcal{B}(t)v_m(t)d\Gamma\\
&\quad+\widetilde{C}t\int_{\Gamma_1}\Big|\frac{\partial u_1}{\partial\nu_A}\Big|^2d\Gamma+Ct+C\\
&\quad-\int_0^t\int_{\Gamma_1}[\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)][v_{ms}(s)+u_1]d\Gamma ds\\
&\quad+\int_0^t\int_{\Gamma_1}[\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)]u_1d\Gamma ds\\
&\le (C+2\beta^2+1)\int_0^t\int_\Omega |\nabla_g v_m(s)|_g^2dxds+2\int_0^t\int_\Omega |v_{ms}(s)|^2dxds\\
&\quad+\zeta\int_\Omega|\nabla_g v_m(t)|_g^2dx+C(t^{\rho+1}+t+t^3)+C\\
&\quad-\int_0^t\int_{\Gamma_1}[\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)][v_{ms}(s)+u_1]d\Gamma ds\\
&\quad+\int_0^t\int_{\Gamma_1}[\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)]u_1d\Gamma ds,
\end{aligned}
\tag{4.8}
$$

where $\zeta>0$ is a sufficiently small constant that will be specified later on. Using the Cauchy-Schwartz inequality, we deduce

$$
\begin{aligned}
&-\int_0^t\int_{\Gamma_1}[\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)][v_{ms}(s)+u_1]d\Gamma ds\\
&\le\int_0^t\int_{\Gamma_1}\Big[\big(\frac{\mu_2}{2}-\mu_1\big)|v_{ms}(s)+u_1|^2+\frac{\mu_2}{2}|v_{ms}(s-\tau)+u_1|^2\Big]\,d\Gamma ds.
\end{aligned}
\tag{4.9}
$$

Now, using the history values about $v_{mt}(t)$ $t\in[-\tau,0]$, the second term in the right-hand side of (4.9) can be rewritten as

$$
\begin{aligned}
&\int_0^t\int_{\Gamma_1}|v_{ms}(s-\tau)+u_1|^2d\Gamma ds\\
&=\int_{-\tau}^{t-\tau}\int_{\Gamma_1}|v_{m\rho}(\rho)+u_1|^2d\Gamma d\rho\\
&=\int_{-\tau}^{0}\int_{\Gamma_1}|v_{m\rho}(\rho)+u_1|^2d\Gamma d\rho+\int_0^{t-\tau}\int_{\Gamma_1}|v_{m\rho}(\rho)+u_1|^2d\Gamma d\rho\\
&=\int_{-\tau}^{0}\int_{\Gamma_1}|g_{0m}(\rho)|^2d\Gamma d\rho+\int_0^{t-\tau}\int_{\Gamma_1}|v_{m\rho}(\rho)+u_1|^2d\Gamma d\rho\\
&\le C_0+\int_0^{t}\int_{\Gamma_1}|v_{m\rho}(\rho)+u_1|^2d\Gamma d\rho,
\end{aligned}
\tag{4.10}
$$

where C_0 is a positive constant. From (4.9) and (4.10), we deduce

$$\begin{aligned} &-\int_0^t\int_{\Gamma_1}[\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)][v_{ms}(s)+u_1]d\Gamma ds\\ &\leq \int_0^t\int_{\Gamma_1}(\mu_2-\mu_1)|v_{ms}(s)+u_1|^2 d\Gamma ds + C. \end{aligned} \tag{4.11}$$

On the other hand, taking the Cauchy-Schwartz inequality, the inequality (4.10) and the regularities of the initial values, we deduce

$$\begin{aligned} &\int_0^t\int_{\Gamma_1}[\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)]u_1 d\Gamma ds\\ &\leq \eta\int_0^t\int_{\Gamma_1}|\mu_1(v_{ms}(s)+u_1)+\mu_2(v_{ms}(s-\tau)+u_1)|^2 d\Gamma ds\\ &\quad + C(\eta)\int_0^t\int_{\Gamma_1}|u_1|^2 d\Gamma ds\\ &\leq 2\eta\int_0^t\int_{\Gamma_1}|\mu_1(v_{ms}(s)+u_1)|^2 d\Gamma ds\\ &\quad + 2\eta\int_0^t\int_{\Gamma_1}|\mu_2(v_{ms}(s-\tau)+u_1)|^2 d\Gamma ds + C'(\eta)\\ &\leq 2(\mu_1^2+\mu_2^2)\eta\int_0^t\int_{\Gamma_1}|v_{ms}(s)+u_1|^2 d\Gamma ds + C + C'(\eta) \quad t\in[0,T], \end{aligned} \tag{4.12}$$

where $\eta > 0$ is a sufficiently small constant that will be specified later on and $C(\eta), C'(\eta)$ are positive constants.

Substituting (4.11), (4.12) into (4.8) and choosing $\zeta > 0$ small enough, we obtain

$$\begin{aligned} &\int_\Omega \left[|v_{mt}(t)|^2 + |\nabla_g v_m(t)|_g^2 + 2F(v_m(t)+\phi(t))\right]dx\\ &+ [\mu_1-\mu_2-2(\mu_1^2+\mu_2^2)\eta]\int_0^t\int_{\Gamma_1}|v_{ms}(s)+u_1|^2 d\Gamma ds\\ &\leq (C+2\beta^2+1)\int_0^t\int_\Omega|\nabla_g v_m(s)|_g^2 dxds + 2\int_0^t\int_\Omega|v_{ms}(s)|^2 dxds\\ &\quad + C(t^{\rho+1}+t+t^3)+C. \end{aligned} \tag{4.13}$$

Finally, noting the fact $\mu_2 < \mu_1$, $F(s) \geq 0$ for all $s \in \mathbb{R}$, choosing $\eta > 0$ sufficiently small, by Gronwall's lemma, we obtain the first-order estimate of v_m

$$\begin{aligned} &\int_\Omega \left[|v_{mt}(t)|^2 + |\nabla_g v_m(t)|_g^2 + 2F(v_m(t)+\phi(t))\right]dx\\ &\quad + \int_0^t\int_{\Gamma_1}|v_{ms}(s)+u_1|^2 d\Gamma ds \leq C_4, \end{aligned} \tag{4.14}$$

where $C_4 > 0$ is a constant independent of $m \in N$ and $t \in [0,T]$.

Step 2: The second-order estimate of v_m. We estimate the term $\|v_{mtt}(0)\|_{L^2(\Omega)}$. Take $t = 0$ in (4.4) and notice the fact $v_m(0) = v_{mt}(0) = 0$, to obtain

$$\int_\Omega v_{mtt}(0)wdx + \int_\Omega h(\nabla u_0)wdx + \int_\Omega f(u_0)wdx + \int_{\Gamma_1}[\mu_1 u_1 + \mu_2 g_{0m}(-\tau)]wd\Gamma$$

$$= \int_\Omega \operatorname{div}(A(x)\nabla u_0) w dx + \int_{\Gamma_1} \Big(-\frac{\partial u_0}{\partial \nu_A}\Big) w d\Gamma \quad \forall w \in V_m$$

which, together with (2.15), leads to

$$\begin{aligned}
&\int_\Omega v_{mtt}(0) w dx + \int_\Omega h(\nabla u_0) w dx + \int_\Omega f(u_0) w dx + \int_{\Gamma_1} [\mu_2 g_{0m}(-\tau)] w d\Gamma \\
&= \int_\Omega \operatorname{div}(A(x)\nabla u_0) w dx + \int_{\Gamma_1} \mu_2 g_0(-\tau) w d\Gamma \quad \forall w \in V_m,
\end{aligned}$$

which, together with (A1), (A2) and the regularities of the initial values, lead to

$$\|v_{mtt}(0)\|_{L^2(\Omega)} \le C_5,$$

where $C_5 > 0$ is a constant independent of $m \in N$.

Next, differentiate (4.4) with respect to t and replace w by v_{mtt}, to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[\int_\Omega \big(|v_{mtt}(t)|^2 + |\nabla_g v_{mt}(t)|_g^2\big)\, dx \Big] \\
&+ \int_\Omega \nabla h(\nabla v_m(t) + \nabla\phi(t))(\nabla v_{mt}(t) + \nabla u_1) v_{mtt}(t) dx \\
&+ \int_\Omega f'(v_m(t) + \phi(t))(v_{mt}(t) + u_1) v_{mtt}(t) dx \\
&+ \int_{\Gamma_1} [\mu_1 v_{mtt}(t) + \mu_2 v_{mtt}(t-\tau)] v_{mtt}(t) d\Gamma \\
&= \int_\Omega \mathcal{F}_t(t) v_{mtt}(t) dx + \frac{d}{dt}\Big(\int_{\Gamma_1} \mathcal{B}_t(t) v_{mt}(t) d\Gamma \Big).
\end{aligned} \tag{4.15}$$

Taking (A2) into account, we infer that

$$\begin{aligned}
&\int_\Omega \nabla h(\nabla v_m(t) + \nabla\phi(t))(\nabla v_{mt}(t) + \nabla u_1) v_{mtt}(t) dx \\
&\le C\Big(1 + \int_\Omega |\nabla_g v_{mt}(t)|_g^2 dx + \int_\Omega |v_{mtt}(t)|^2 dx\Big).
\end{aligned} \tag{4.16}$$

We use Hölder's inequality, the Sobolev imbedding theorem, and the trace theorem, by noticing (A1), (4.14) and the regularities of the initial values, to obtain

$$\begin{aligned}
&\int_\Omega f'(v_m(t) + \phi(t))(v_{mt}(t) + u_1) v_{mtt}(t) dx \\
&\le C \int_\Omega (|v_m(t)|^{\rho-1} + |\phi(t)|^{\rho-1} + C)(|v_{mt}(t)| + |u_1|)|v_{mtt}(t)| dx \\
&\le C \int_\Omega (|v_m(t)|^{2(\rho-1)} |v_{mt}(t)|^2 dx + C \int_\Omega |\phi(t)|^{2(\rho-1)} |v_{mt}(t)|^2 dx \\
&\quad + C \int_\Omega |v_{mtt}(t)|^2 dx + C \\
&\le C\Big(\int_\Omega (|v_m(t)|^{2(\rho-1)\cdot\frac{n}{2}} dx\Big)^{2/n} \Big(\int_\Omega (|v_{mt}(t)|^{2\cdot\frac{n}{n-2}} dx\Big)^{(n-2)/n} \\
&\quad + C \int_\Omega |\phi(t)|^{2(\rho-1)} |v_{mt}(t)|^2 dx + C \int_\Omega |v_{mtt}(t)|^2 dx + C \\
&\le C \int_\Omega \big(|\nabla_g v_{mt}(t)|_g^2 + |v_{mtt}(t)|^2\big)\, dx + C
\end{aligned} \tag{4.17}$$

and

$$\int_\Omega \mathcal{F}_t(t) v_{mtt}(t)dx \le C\int_\Omega |v_{mtt}(t)|^2 dx + C, \tag{4.18}$$

$$\int_{\Gamma_1} \mathcal{B}_t(t) v_{mt}(t) d\Gamma \le C\frac{\widetilde{C}}{4\xi} + \xi\int_\Omega |\nabla_g v_{mt}(t)|_g^2 dx, \tag{4.19}$$

where $\xi > 0$ is a sufficiently small constant that will be specified later on.

Finally, combining (4.16)–(4.19), integrating (4.15) over $(0,t)$, choosing $\xi > 0$ sufficiently small and recalling $\|v_{mtt}(0)\|_{L^2(\Omega)} \le C_5$, we obtain

$$\begin{aligned} &\int_\Omega \big(|v_{mtt}(t)|^2 + |\nabla_g v_m(t)|_g^2\big)\,dx + \int_0^t\int_{\Gamma_1} [\mu_1 v_{\rm mss}(s) + \mu_2 v_{\rm mss}(s-\tau)] v_{\rm mss}(s) d\Gamma ds \\ &\le C\int_0^t\int_\Omega \big(|v_{\rm mss}(s)|^2 + |\nabla_g v_{ms}(s)|_g^2\big)\,dx\,ds + Ct + C. \end{aligned} \tag{4.20}$$

Note that

$$\begin{aligned} &\int_0^t\int_{\Gamma_1} |v_{\rm mss}(s-\tau)|^2 d\Gamma ds \\ &= \int_{-\tau}^{t-\tau}\int_{\Gamma_1} |v_{m\rho\rho}(\rho)|^2 d\Gamma d\rho \\ &= \int_{-\tau}^{0}\int_{\Gamma_1} |g_{0m\rho}(\rho)|^2 d\Gamma d\rho + \int_0^{t-\tau}\int_{\Gamma_1} |v_{m\rho\rho}(\rho)|^2 d\Gamma d\rho \\ &\le C_0' + \int_0^t\int_{\Gamma_1} |v_{m\rho\rho}(\rho)|^2 d\Gamma d\rho, \end{aligned} \tag{4.21}$$

where C_0' is a positive constant. From (4.21), we infer

$$\begin{aligned} &\int_0^t\int_{\Gamma_1} [\mu_1 v_{\rm mss}(s) + \mu_2 v_{\rm mss}(s-\tau)] v_{\rm mss}(s)\, d\Gamma ds \\ &\ge \int_0^t\int_{\Gamma_1} \Big[\big(\mu_1 - \frac{\mu_2}{2}\big)|v_{\rm mss}(s)|^2 - \frac{\mu_2}{2}|v_{\rm mss}(s-\tau)|^2\Big]\, d\Gamma ds \\ &\ge \int_0^t\int_{\Gamma_1} (\mu_1-\mu_2)|v_{\rm mss}(s)|^2 d\Gamma ds - C \end{aligned} \tag{4.22}$$

which, together with (4.20), leads to

$$\begin{aligned} &\int_\Omega \big(|v_{mtt}(t)|^2 + |\nabla_g v_m(t)|_g^2\big)\,dx + \int_0^t\int_{\Gamma_1} (\mu_1-\mu_2)|v_{\rm mss}(s)|^2 d\Gamma ds \\ &\le \int_0^t\int_\Omega \big(|v_{\rm mss}(s)|^2 + |\nabla_g v_m(s)|_g^2\big)\,dx\,ds + Ct + C. \end{aligned} \tag{4.23}$$

Recalling the fact $\mu_2 < \mu_1$, by Gronwall's lemma, we obtain the second-order estimate of v_m,

$$\int_\Omega \big(|v_{mtt}(t)|^2 + |\nabla_g v_m(t)|_g^2\big)\,dx + \int_0^t\int_{\Gamma_1} (\mu_1-\mu_2)|v_{\rm mss}(s)|^2 d\Gamma ds \le C_6,$$

where C_6 is a positive constant independent of $m \in N$ and $t \in [0,T]$.

For the delay term, using the same method as the one in (4.9)-(4.11), the proof can be completed arguing as in [8, Theorem 3.1].

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