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DECAY OF SOLUTIONS FOR A SYSTEM OF NONLINEAR WAVE EQUATIONS

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Abstract. This article concerns the decay of solutions of the semi-linear wave equation
$$u_{tt}+\delta u_t-\phi(x)\triangle u=\lambda u|u|^{\beta-1}\quad x\in\mathbb{R}^N\ t\geq 0$$
Introducing an appropriate Lyaponuv function, we find exponential decay for certain initial data.

1. Introduction

In this article, we consider the initial boundary value problem
$$\begin{gathered}u_{tt}+\delta u_t-\phi(x)\triangle u=\lambda u|u|^{\beta-1}\quad x\in\mathbb{R}^N,\ t\geq 0\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x)\quad x\in\mathbb{R}^N\end{gathered}\tag{1.1}$$
with initial conditions $u_0(x)$, $u_1(x)$ in appropriate function spaces and $\delta>0$. Models of this type are of interest in applications in various areas of mathematical physics [3, 13, 22, 23], as well as in geophysics and ocean acoustics, where, for example, the coefficient $\phi(x)$ represents the speed of sound at the point $x\in\mathbb{R}^N$ (see [14]). Throughout this article, we assume that the functions $\phi(x)$ and $g:\mathbb{R}^N\to R$ satisfy the following conditions:

(G0) $\phi(x)>0$, for all $x\in\mathbb{R}^N$, $(\phi(x))^{-1}=:g(x)$ is $\mathcal{C}^{0,\gamma}(\mathbb{R}^N)$-smooth, for some $\gamma\in(0,1)$ and $g\in L^{N/2}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$,

Examples of functions ϕ of this type can be found in [23, p. 632].

The questions of global existence, nonexistence and blow-up of solutions of the Cauchy problem for nonlinear wave equations have been studied by many authors; see for example [10, 15, 20]. In general, global existence happens, when the damping terms dominate the source terms, while blow-up appears in the opposite situation and under the assumption that the initial data is sufficiently large (for example when the initial energy is assumed to be sufficiently negative). In [12] it is shown that for sufficiently small initial data global existence can be obtained, even when the influence of the source term is stronger than that of the damping term. In the works [1, 2, 5, 6, 8, 9, 12, 16, 21] the spatial domain is assumed to be bounded. On the other hand, in [19] the problem is considered in the whole of $\mathbb{R}^N$ and the method

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of modified potential well is used to construct the global solutions. In [9, 12, 19] the coefficient $\phi(x) = 1$, which makes possible the treatment of the equations in the classical Sobolev space setting. In [17, 18, 24] decay properties of solutions of wave equations, involving weighted dissipative terms, are discussed. Cavalcanti [7] considers the nonlinear evolution equation with source and damping terms on a compact manifold.

The purpose of this article is to obtain decay estimate of solutions to the problem (1.1). More precisely we show that we can always find initial data in the stable set for which the solution of (1.1) decays exponentially. The key tool in the proof is an idea of Zuazua [11, 24], which is based on the construction of a suitable Lyapunov function.

Notation: For simplicity we use the symbols L^p and $D^{1,2}$, for the spaces $L^p(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N)$, respectively, with $1 \le p \le \infty$. We use $\|\cdot\|_p$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$. Also differentiation with respect to time is denoted by a dot over the function. The constants C and c are considered in a generic sense.

2. Asymptotic stability

In this section we introduce and prove our main result. For this purpose we use the definition of the solution of problem (1.1) given by Karachalios and Stavrakakis in [13]. For later use, we briefly mention here some facts, notation and results from paper [13].

The space setting for the initial conditions and the solutions of the problem (1.1) is the product space $\mathcal{X}_0 = D^{1,2}(\mathbb{R}^N) \times L^2_g(\mathbb{R}^N)$. The space $D^{1,2}(\mathbb{R}^N)$ is defined as the closure of $C_0^\infty(\mathbb{R}^N)$ functions with respect to the energy norm $\|u\|_{D^{1,2}} =: \int_{\mathbb{R}^N} |\nabla u|^2\,dx$. It is well known that

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \nabla u \in (L^2(\mathbb{R}^N))^N\}$$

and that $D^{1,2}$ is embedded continuously in $L^{\frac{2N}{N-2}}$.i.e., there exists $k > 0$ such that

$$\|u\|_{\frac{2N}{N-2}} \le k\|u\|_{D^{1,2}} \tag{2.1}$$

We shall frequently use the following generalized version of Poincaré inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \ge \alpha \int_{\mathbb{R}^N} gu^2\,dx \tag{2.2}$$

for all $u \in C_0^\infty$ and $g \in L^{N/2}$, where $\alpha =: k^{-2}\|g\|^{-1}_{N/2}$ (see [6, Lemma 2.1]). It has been shown that $D^{1,2}(\mathbb{R}^N)$ is a separable Hilbert space. The space $L^2_g(\mathbb{R}^N)$ is defined to be the closure of $u \in C_0^\infty(\mathbb{R}^N)$ functions with respect to the inner product

$$(u,v)_{L^2_g(\mathbb{R}^N)} =: \int_{\mathbb{R}^N} guv\,dx \tag{2.3}$$

Clearly, $L^2_g(\mathbb{R}^N)$ is a separable Hilbert space.

We consider the potential well

$$\mathcal{W} =: \{u \in D^{1,2}(\mathbb{R}^N) : K(u) =: \|u\|^2_{D^{1,2}} - \lambda\|u\|^{\beta+1}_{L_g^{\beta+1}} > 0\}$$

Also consider the functional

$$\mathcal{J}(u) =: \frac{1}{2}\|u\|^2_{D^{1,2}} - \frac{\lambda}{\beta+1}\int_{\mathbb{R}^N} g(x)|u(t)|^{\beta+1}\,dx. \tag{2.4}$$

The energy of the problem is defined as

$$\mathcal{E}^*(u(t),u_t(t)) = \mathcal{E}^*(t) := \frac{1}{2}\|u_t(t)\|^2_{L^2_g} + \mathcal{J}(u). \tag{2.5}$$

From equation (1.1), we have

$$\mathcal{E}^*(t) = \mathcal{E}^*(0) - \delta\int_0^t \|u_t(\tau)\|^2_{L^2_g}\,d\tau \tag{2.6}$$

Under certain assumptions on the initial data, solutions exist globally in the energy space $\mathcal{X}_0$. In addition to the principal condition (G0) in the introduction, we shall use the following additional hypotheses for the function g and the nonlinearity exponent β.

- (G1) $g \in L^1(\mathbb{R}^N)$ and $1 < \beta \le \frac{N}{N-2}$, for all $N \ge 3$.
- (G2) $N \ge 3$ and $\frac{N+2}{N} \le \beta \le \frac{N}{N-2}$.
- (G3) $N = 3, 4$ and $\frac{N+4}{N} \le \beta \le \frac{N}{N-2}$.

Let us note that since $g \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ by hypothesis (G0), then any g satisfying hypothesis (G1) belongs to all spaces $L^p(\mathbb{R}^N)$, for $p \in [1,+\infty)$.

A weak solution of (1.1) is a function $u(x,t)$ such that

- (i) $u \in L^2[0,T;D^{1,2}(\mathbb{R}^N)]$, $u_t \in L^2[0,T;L^2_g(\mathbb{R}^N)]$, $u_{tt} \in L^2[0,T;D^{-1,2}(\mathbb{R}^N)]$,
- (ii) for all $v \in C_0^\infty([0,T]\times\mathbb{R}^N)$, u satisfies the generalized formula

$$\begin{aligned}
&\int_0^T (u_{tt}(\tau),v(\tau))_{L^2_g}\,d\tau + \delta\int_0^T (u_t(\tau),v(\tau))_{L^2_g}\,d\tau \\
&+ \int_0^T\int_{\mathbb{R}^N} \nabla u(\tau)\nabla v(\tau)\,dx\,d\tau - \lambda\int_0^T (f(u(\tau)),v(\tau))_{L^2_g}\,d\tau = 0,
\end{aligned} \tag{2.7}$$

where $f(s) = |s|^{\beta-1}s$, and

(iii) u satisfies the initial conditions

$$u(x,0) = u_0(x) \in D^{1,2}(\mathbb{R}^N), \quad u_t(x,0) = u_1(x) \in L^2_g(\mathbb{R}^N).$$

The following two lemmas come from [13].

Lemma 2.1 ([13, Proposition 3.1])**.** *Let g, β, N satisfy conditions* (G0) or (G2)*. Suppose that the constants $\delta > 0$, $\lambda < \infty$ and the initial conditions*

$$u_0 \in D^{1,2}(\mathbb{R}^N) \quad\text{and}\quad u_1 \in L^2_g(\mathbb{R}^N). \tag{2.8}$$

are given. Then for sufficiently small $T > 0$ the problem (1.1) *admits a unique (weak) solution such that*

$$u \in C[0,T;D^{1,2}(\mathbb{R}^N)], \quad u_t \in C[0,T;L^2_g(\mathbb{R}^N)]. \tag{2.9}$$

Lemma 2.2 ([13, Theorem 3.2]])**.** *Let condition* (G3) *be satisfied and $u_0 \in \mathcal{W}$. Assume that the initial data satisfy* (2.8) *and they are sufficiently small in the sense*

$$\mathcal{E}^*(0) < \Big(\frac{1}{C_0\lambda\mu_0^{p_1}}\Big)^{1/p_2} \tag{2.10}$$

where

$$p_1 = \frac{2(\beta+1) - N(\beta-1)}{2}, \quad p_2 = \frac{N\beta - N - 4}{4}.$$

Then the (weak) solution of (1.1) *is such that*

$$u \in C[0,\infty;D^{1,2}(\mathbb{R}^N)], \quad u_t \in C[0,\infty;L^2_g(\mathbb{R}^N)]. \tag{2.11}$$

Theorem 2.3. *If the assumption of Lemma 2.2 is satisfied, then there exist two positive constants $\widehat{C}$ and ξ, independent of t such that*

$$0 < \mathcal{E}^*(t) \le \widehat{C} e^{-\xi t} \quad \text{for all } t \ge 0 \tag{2.12}$$

Proof. From (2.4), we have

$$\mathcal{E}^*(t) = \frac{1}{2}\|u_t(t)\|_{L_g^2}^2 + \frac{\beta-1}{2(\beta+1)}\|u\|_{D^{1,2}}^2 + K(u).$$

Theorem 3.2 in [13] shows that for all $t \ge 0$, $u(t) \in \mathcal{W}$, so we have

$$0 < \mathcal{E}^*(t) \quad \text{for} \quad \text{all} \quad t \ge 0. \tag{2.13}$$

The proof of the other inequality relies on the construction of a Lyapunov function by performing a suitable modification of the energy. To this end, for $\varepsilon > 0$, to be chosen later, we define

$$L(t) = \mathcal{E}^*(t) + \varepsilon \int g u u_t \, dx \tag{2.14}$$

It is straightforward to see that $L(t)$ and $\mathcal{E}^*(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 depending on ε such that for $t \ge 0$,

$$\beta_1 \mathcal{E}^*(t) \le L(t) \le \beta_2 \mathcal{E}^*(t) \tag{2.15}$$

By taking the time derivative of the function L defined above in equation (2.14), using problem (1.1), and performing several integration by parts, we obtain

$$\begin{aligned}
&\frac{dL(t)}{dt} \\
&= -\delta\|u_t\|_{L_g^2}^2 + \varepsilon \int g u_t^2 \, dx + \varepsilon \int g u u_{tt} \, dx \\
&= -\delta\|u_t\|_{L_g^2}^2 + \varepsilon\|u_t\|_{L_g^2}^2 + \varepsilon \int g u \lambda u |u|^{\beta-1} \, dx - \varepsilon \int g u \delta u_t \, dx + \varepsilon \int g u \phi \Delta u \, dx \\
&= -\delta\|u_t\|_{L_g^2}^2 + \varepsilon\|u_t\|_{L_g^2}^2 + \varepsilon\lambda\|u\|_{L_g^{\beta+1}}^{\beta+1} - \varepsilon\|\nabla u\|_2^2 - \varepsilon\delta \int g u u_t \, dx
\end{aligned} \tag{2.16}$$

Using Young inequality and Sobolev inequality, for any $\gamma > 0$, we obtain

$$\int g u u_t \, dx \le \frac{1}{4\gamma} \int g u_t^2 \, dx + \gamma \int g u^2 \, dx \le \frac{1}{4\gamma} \int g u_t^2 \, dx + \frac{\gamma}{\alpha}\|\nabla u\|_2^2 \tag{2.17}$$

where α is the Sobolev constant.

Using the result [13, (3.20) in Theorem 3.2], we have

$$\|u(t)\|_{L_g^{\beta+1}}^{\beta+1} \le C_0 \mu_0^{p_1} \mathcal{E}^*(0)^{p_2} \|\nabla u\|_2^2$$

Consequently, inserting (2.17) into (2.16), we have

$$\frac{dL(t)}{dt} \le (\varepsilon + \frac{\varepsilon\delta}{4\gamma} - \delta)\|u_t\|_{L_g^2}^2 + (\varepsilon\lambda C_0 \mu_0^{p_1} \mathcal{E}^*(0)^{p_2} - \varepsilon + \frac{\varepsilon\delta\gamma}{\alpha})\|\nabla u\|_2^2 \tag{2.18}$$

By the condition $\mathcal{E}^*(0)^{p_2} C_0 \lambda \mu_0^{p_1} < 1$, let us choose γ small enough such that

$$\varepsilon(\lambda C_0 \mu_0^{p_1} \mathcal{E}^*(0)^{p_2} - 1 + \frac{\delta\gamma}{\alpha}) < 0 \tag{2.19}$$

From this inequality we may find $\eta > 0$, which depends only on γ such that

$$\frac{dL(t)}{dt} \le \Big(\varepsilon(\frac{\delta}{4\gamma} + 1) - \delta\Big)\|u_t\|_{L_g^2}^2 - \varepsilon\eta\|\nabla u\|_2^2 \tag{2.20}$$

Consequently, using the definition of the energy (2.5), for any positive constant M (we will choose the suitable M), we always obtain

$$\frac{dL(t)}{dt} \leq -M\varepsilon\mathcal{E}^*(t) + \Big(\varepsilon(\frac{\delta}{4\gamma} + 1 + \frac{M}{2}) - \delta\Big)\|u_t\|^2_{L^2_g} + \varepsilon(\frac{M}{2} - \eta)\|\nabla u\|_2^2 \tag{2.21}$$

Choose $M < 2\eta$, and ε small enough such that

$$\varepsilon(\frac{\delta}{4\gamma} + 1 + \frac{M}{2}) - \delta < 0 \tag{2.22}$$

inequality (2.21) becomes

$$\frac{dL(t)}{dt} \leq -M\varepsilon\mathcal{E}^*(t) \quad \text{for all} \quad t \geq 0 \tag{2.23}$$

On the other hand, by (2.15), setting $\xi = \frac{M\varepsilon}{\beta_2}$, the last inequality becomes

$$\frac{dL(t)}{dt} \leq -\xi L(t) \quad \text{for all} \quad t \geq 0 \tag{2.24}$$

Integrating this differential inequality between 0 and t gives the following estimate for the function L

$$L(t) \leq Ce^{-\xi t} \quad \text{for all } t \geq 0 \tag{2.25}$$

Consequently, by using (2.15) once again, we conclude

$$E(t) \leq \widehat{C}e^{-\xi t} \quad \text{all } dt \geq 0 \tag{2.26}$$

This completes the proof. □

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