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EXISTENCE OF EXPONENTIAL ATTRACTORS FOR THE PLATE EQUATIONS WITH STRONG DAMPING

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Abstract. We show the existence of $(H_0^2(\Omega)\times L^2(\Omega), H_0^2(\Omega)\times H_0^2(\Omega))$-global attractors for plate equations with critical nonlinearity when $g \in H^{-2}(\Omega)$. Furthermore we prove that for each fixed $T > 0$, there is an $(H_0^2(\Omega) \times L^2(\Omega), H_0^2(\Omega) \times H_0^2(\Omega))_T$-exponential attractor for all $g \in L^2(\Omega)$, which attracts any $H_0^2(\Omega)\times L^2(\Omega)$-bounded set under the stronger $H^2(\Omega)\times H^2(\Omega)$-norm for all $t \geq T$.

1. Introduction

We consider the long-time behavior of the solutions for the following equation on a bounded domain $\Omega \subset \mathbb{R}^5$ with smooth boundary $\partial\Omega$:

$$\begin{gathered} u_{tt} + \Delta^2 u_t + \Delta^2 u + f(u) = g(x), \quad x \in \Omega, \\ u\big|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $g \in H^{-2}(\Omega)$, $f \in \mathcal{C}^1(\mathbb{R})$, $f(0) = 0$ and satisfies the following conditions:

$$|f'(s)| \leq C(1 + |s|^8), \quad \forall s \in \mathbb{R}, \tag{1.2}$$

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1^2, \tag{1.3}$$

where λ_1 is the first eigenvalue of Δ^2 on $H_0^2(\Omega)$.

Problem (1.1) stems from the elastic equation established by Woinowsky-Krieger [10]. The asymptotic behavior and the existence of global solutions of the linear plate equations were studied by Ball [1, 2] in 1973. The asymptotic behavior of the plate equations with linear damping and nonlinear damping have been extensively studied, see for example [3, 4, 11, 12, 13]. The existence of the global attractors of the autonomous plate equations with critical exponent on the unbounded domain was investigated by several authors in [4, 5, 11]. In [12, 13], the authors discussed the existence of compact attractors for the autonomous and non-autonomous plate equations in a bounded domain, respectively. For the best of our knowledge, the existence of bi-space global attractor and exponential attractor of (1.1) has not been

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published. Therefore, it is necessary to continue researching. As we know, existence and regularity of global attractors of the wave equations with strong damping have been studied in [6, 7, 14, 15, 16]. The authors in [14] proved the existence of global attractors for the wave equation when the nonlinearity is critical and $g \in L^2(\Omega)$. Then in [16], they showed the existence of a global attractor when nonlinearity is critical and $g \in H^{-1}(\Omega)$; moreover, they showed the existence of exponential attractor for $g \in L^2(\Omega)$. In this article, we borrow the ideas and methods in [14, 16] to prove existence of bi-space global attractor for $g \in H^{-2}(\Omega)$ and bi-space T-exponential attractor for $g \in L^2(\Omega)$. For other results of attractors about the dynamical systems, please refer the reader to [8, 9, 15] and the references therein.

2. Preliminaries

Let $A = \Delta^2$ with domain $D(A) = H_0^2(\Omega) \cap H^4(\Omega)$. Consider the family of Hilbert spaces $D(A^{s/2})$, $s \in \mathbb{R}$ with inner products and norms

$$(\cdot,\cdot)_{D(A^{s/2})} = (A^{s/2}\cdot, A^{s/2}\cdot), \quad \|\cdot\|_{D(A^{s/2})} = \|A^{s/2}\cdot\|,$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ mean the $L^2(\Omega)$ inner product and norm respectively. For convenience, we denote $\mathcal{H}_s = D(A^{(1+s)/2}) \times D(A^{s/2})$, $\forall s \in \mathbb{R}$, whose norm is $\|\cdot\|_s$. In particular, $\mathcal{H}_0 = H_0^2(\Omega) \times L^2(\Omega)$ and $\mathcal{V} = H_0^2(\Omega) \times H_0^2(\Omega)$. Note that

$$\begin{gathered} D(A^{s/2}) \hookrightarrow D(A^{r/2}), \quad \text{for } s > r; \\ D(A^{s/2}) \hookrightarrow L^{10/(5-4s)}(\Omega), \quad \text{for } s \in [0, \frac{5}{4}). \end{gathered} \tag{2.1}$$

Given $s > r > q$, for any $\epsilon > 0$, there exists $C_\epsilon = C_\epsilon(s, r, q)$ such that

$$\|A^{r/2}u\| \le \epsilon\|A^{s/2}u\| + C_\epsilon\|A^{\frac{q}{2}}u\|, \quad \text{for any } u \in D(A^{s/2}). \tag{2.2}$$

For the nonlinear function f, we know that f allows the decomposition

$$f = f_0 + f_1, \tag{2.3}$$

where $f_0, f_1 \in \mathcal{C}(\mathbb{R})$ and satisfy

$$|f_0(u)| \le C(|u| + |u|^9) \quad \text{for all } u \in \mathbb{R}, \tag{2.4}$$

$$f_0(u)u \ge 0 \quad \text{for all } u \in \mathbb{R}, \tag{2.5}$$

$$|f_1(u)| \le C(1 + |u|^\gamma) \quad \text{for all } u \in \mathbb{R},\ \gamma < 9, \tag{2.6}$$

$$\liminf_{|u|\to\infty} \frac{f_1(u)}{u} > -\lambda_1^2, \tag{2.7}$$

where C is a positive constant. Denote

$$\sigma = \min\{\frac{1}{8}, \frac{9-\gamma}{4}\}. \tag{2.8}$$

Under the above assumptions, equation (1.1) has an unique weak solution satisfying

$$u \in C([0,T], H_0^2(\Omega)), \quad u_t \in C([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^2(\Omega)).$$

We also need the following properties.

Lemma 2.1 ([16]). *Let $\mathscr{T}$ be a Hölder mapping from $(\mathscr{X}, \|\cdot\|_1)$ to $(\mathscr{X}, \|\cdot\|_2)$ with constant $\mathscr{L}$ and Hölder exponent $\gamma \in (0,1]$; that is,*

$$\|\mathscr{T}x_1 - \mathscr{T}x_2\|_2 \le \mathscr{L}\|x_1 - x_2\|_1^\gamma, \quad \forall x_1, x_2 \in \mathscr{X},$$

Then for any $\mathcal{E} \subset \mathscr{X}$, the following estimates hold:

(i) $\dim_F(\mathscr{T}\mathcal{E}, \|\cdot\|_2) \le \frac{1}{\gamma}\dim_F(\mathcal{E}, \|\cdot\|_1)$;

(ii) *if, further,* $\{S(t)\}_{t\ge 0}$ *is a semigroup on* $\mathscr{X}$, *satisfies* $S(t)\mathscr{X} \subset \mathscr{X}$ *for all* $t \ge 0$, *then*

$$\operatorname{dist}_{\|\cdot\|_2}(\mathscr{T}S(t)\mathscr{X}, \mathscr{T}\mathcal{E}) \le 2\mathscr{L}\operatorname{dist}^{\gamma}_{\|\cdot\|_1}(S(t)\mathscr{X}, \mathcal{E}), \quad \forall t \ge 0, \tag{2.9}$$

where $\operatorname{dist}_{\|\cdot\|_i}(\cdot,\cdot)$ *is the Hausdorff semidistance of two sets with respect to* $\|\cdot\|_i$, $i = 1, 2$.

3. Global attractors and regularity for g in $H^{-2}(\Omega)$

Since the injection $i : L^2(\Omega) \hookrightarrow H^{-2}(\Omega)$ is dense, we know that for every $g \in H^{-2}(\Omega)$ and any $\eta > 0$, there is a $g_\eta \in L^2(\Omega)$ which depends on g and η such that

$$\|g - g_\eta\|_{H^{-2}} < \eta. \tag{3.1}$$

We decompose the solution $u(t)$ of (1.1) corresponding to initial data (u_0, u_1) as $u(t) = v^\eta(t) + w^\eta(t)$, where $v^\eta(t) and w^\eta(t)$ satisfy the following two equations

$$\begin{gathered} v^\eta_{tt} + \Delta^2 v^\eta_t + \Delta^2 v^\eta + f_0(v^\eta) = g - g_\eta, \\ (v^\eta(0), v^\eta_t(0)) = (u_0, u_1), \quad v^\eta|_{\partial\Omega} = 0 \end{gathered} \tag{3.2}$$

and

$$\begin{gathered} w^\eta_{tt} + \Delta^2 w^\eta_t + \Delta^2 w^\eta + f(u) - f_0(v^\eta) = g_\eta, \\ (w^\eta(0), w^\eta_t(0)) = (0, 0), \quad w^\eta|_{\partial\Omega} = 0. \end{gathered} \tag{3.3}$$

We first recall some results for the bounded dissipative case.

Lemma 3.1. *Let* f *satisfy* (1.2) *and* (1.3), $g \in H^{-2}(\Omega)$ *and* $\{S(t)\}_{t\ge 0}$ *be the semigroup generated by the weak solution of* (1.1) *in the natural energy space* $\mathcal{H}_0$. *Then* $\{S(t)\}_{t\ge 0}$ *has a bounded absorbing set* B_0 *in* $\mathcal{H}_0$; *that is, for any bounded subset* $B \subset \mathcal{H}_0$, *there exists* $T = T(B_0)$ *such that*

$$S(t)B \subset B_0, \quad \forall t \ge T. \tag{3.4}$$

The proof of the above lemma and the following corollary are similar to those in [14, 16], so we omit them.

Corollary 3.2. *Under the assumptions of Lemma 3.1, for a given* $R > 0$, *there exists* $K_0 = K_0(R)$ *and* $\Lambda_0 = \Lambda_0(R)$, *for* $\|z_0\|_0 \le R$, *the corresponding solution* $S(t)z_0 = (u(t), u_t(t))$ *satisfy*

$$\begin{gathered} \|S(t)z_0\|_0 \le K_0, \quad \forall t \in \mathbb{R}^+; \\ \int_0^{+\infty} \|\Delta u_t(y)\|^2 dy \le \Lambda_0. \end{gathered}$$

Next, we obtain the existence of the global attractors, so we need the following asymptotic compactness result.

Lemma 3.3. *For any* $\epsilon > 0$, *there is a* $\eta = \eta(\epsilon, g)$ *such that the solutions of* (3.2) *satisfy*

$$\|v^\eta_t\|^2 + \|\Delta v^\eta\|^2 \le Q_0(\|z_0\|_0)e^{-Ct} + \epsilon, \quad \forall t \ge 0, \tag{3.5}$$

where the constant C *only depends on* $\|z_0\|_0$ *and* $\|g - g_\eta\|_{H^{-2}}$, $Q_0(\cdot)$ *is a nondecreasing function on* $[0, \infty)$.

Proof. Multiplying (3.2) by $(v_t^\eta+\delta v^\eta)$ and integrating over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|v_t^\eta+\delta v^\eta\|^2+(1+\delta)\|\Delta v^\eta\|^2+2\int_\Omega F(v^\eta)\Big)+\frac{\delta}{2}\|\Delta v^\eta\|^2\\ &+\frac{1}{2}\|\Delta v_t^\eta\|^2+\Big(\frac{\lambda_1}{2}-\delta-\frac{\delta^2}{2}\Big)\|v_t^\eta\|^2+\frac{\delta(\lambda_1-\delta)}{2}\|v^\eta\|^2\\ &\le 4\|g-g_\eta\|_{H^{-2}}^2+\frac{1}{4}\|\Delta v_t^\eta\|^2+\frac{\delta^2}{4}\|\Delta v^\eta\|^2, \end{aligned}\tag{3.6}$$

where $F(v^\eta)=\int_0^{v^\eta}f_0(s)ds$.

Let δ be small enough, then from (3.6) we have the estimate

$$\begin{aligned} &\frac{d}{dt}\Big(\|v_t^\eta+\delta v^\eta\|^2+(1+\delta)\|\Delta v^\eta|^2+2\int_\Omega F(v^\eta)\Big)\\ &+C_\delta(\|\Delta v_t^\eta\|^2+\|\Delta v^\eta\|^2)\le 4\|g-g_\eta\|_{H^{-2}}^2. \end{aligned}\tag{3.7}$$

Multiplying (3.2) by v_t^η we can deduce that (similar to Lemma 3.1)

$$\|v_t^\eta\|^2+\|\Delta v^\eta\|^2\le Q'(\|z_0\|_0,\|g-g_\eta\|_{H^{-2}}):=M_0,\quad \forall t\ge 0. \tag{3.8}$$

On the other hand, this inequality and (2.4) yield

$$\int_\Omega F(v^\eta)dx\le C\int_\Omega(|v^\eta(t)|^2+|v^\eta(t)|^{10})dx \tag{3.9}$$

which combining with (3.8) imply

$$\int_\Omega F(v^\eta)dx\le C_{M_0}\int_\Omega|\Delta v^\eta|^2dx. \tag{3.10}$$

Hence, from (3.7) and (3.10), taking C_{δ,M_0} small enough, we have

$$\begin{aligned} &\frac{d}{dt}\Big(\|v_t^\eta+\delta v^\eta\|^2+(1+\delta)\|\Delta v^\eta\|^2+2\int_\Omega F(v^\eta)dx\Big)\\ &+C_{\delta,M}(\|v_t^\eta+\delta v^\eta\|^2+(1+\delta)\|\Delta v^\eta\|^2+2\int_\Omega F(v^\eta)dx)\\ &\le 4\|g-g_\eta\|_{H^{-2}}^2. \end{aligned}\tag{3.11}$$

Applying Gronwall lemma, we obtain

$$\|v_t^\eta+\delta v^\eta\|^2+(1+\delta)\|\Delta v^\eta\|^2+2\int_\Omega F(v^\eta)dx\le Q_0(\|z_0\|_0)e^{-C_{\delta,M}t}+\frac{\|g-g_\eta\|_{H^{-2}}^2}{4C_{\delta,M_0}}.$$

Therefore, we can complete our proof by taking $\eta^2\le 4C_{\delta,M_0}\epsilon$ in (3.1). □

Lemma 3.4. *For any $T>0$ and $\eta>0$, there is a positive constant $M_1=M_1(T,\eta)$ which depends on (T,η), such that the solutions of (3.3) satisfy*

$$\|w^\eta(T)\|_{1+\sigma}^2+\|w_t^\eta(T)\|_\sigma^2\le M_1, \tag{3.12}$$

where $\sigma=\min\{\frac{1}{8},\frac{9-\gamma}{4}\}$.

Proof. According to Corollary 3.2 and Lemma 3.3,

$$\|\Delta u\|+\|\Delta v^\eta\|\le M_2,\quad t\ge 0. \tag{3.13}$$

Multiplying (3.3) by $A^\sigma w_t^\eta$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|A^{\frac{\sigma}{2}}w_t^\eta\|^2+\|A^{\frac{\sigma+1}{2}}w^\eta\|^2)+\|A^{\frac{\sigma+1}{2}}w_t^\eta\|^2\\ &=-(f(u)-f_0(v^\eta),A^\sigma w_t^\eta)+(g_\eta,A^\sigma w_t^\eta). \end{aligned}\tag{3.14}$$

Recall that the nonlinear term $f(u)$ satisfies

$$|(f(u)-f_0(v^\eta),A^\sigma w_t^\eta)|\le |(f(u)-f(v^\eta),A^\sigma w_t^\eta)|+|(f_1(v^\eta),A^\sigma w_t^\eta)|.$$

From (1.2), (3.13) and using the Hölder inequality, we have

$$\begin{aligned}|(f(u)-f(v^\eta),A^\sigma w_t^\eta)| &\le C\int_\Omega (1+|u|^8+|v^\eta|^8)|w^\eta||A^\sigma w_t^\eta|\\ &\le C(1+\|u\|_{L^{10}}^8+\|v^\eta\|_{L^{10}}^8)\|w^\eta\|_{L^{\frac{10}{1-4\sigma}}}\|A^\sigma w_t^\eta\|_{L^{\frac{10}{1+4\sigma}}}\\ &\le C(1+\|\Delta u\|^8+\|\Delta v^\eta\|^8)\|A^{\frac{\sigma+1}{2}}w^\eta\|\|A^{\frac{\sigma+1}{2}}w_t^\eta\|\\ &\le C_{M_2}\|A^{\frac{\sigma+1}{2}}w^\eta\|^2+\frac13\|A^{\frac{\sigma+1}{2}}w_t^\eta\|^2;\end{aligned}$$

In addition, noticing that $\frac{\gamma}{9-4\sigma}\le 1$, we obtain

$$\begin{aligned}|(f_1(v^\eta),A^\sigma w_t^\eta| &\le C(1+\|v^\eta\|^\gamma_{L^{\frac{10\gamma}{9-4\sigma}}})\|A^\sigma w_t^\eta\|_{L^{\frac{10}{1+4\sigma}}}\\ &\le C(1+\|\Delta v^\eta\|^\gamma)\|A^{\frac{\sigma+1}{2}}w_t^\eta\|\\ &\le C_{M_2}+\frac13\|A^{\frac{\sigma+1}{2}}w_t^\eta\|^2;\end{aligned}$$

Finally, for $\sigma<1$, we obtain

$$|(g_\eta,A^\sigma w_t^\eta)|\le C\|g_\eta\|^2+\frac13\|A^{\frac{\sigma+1}{2}}w_t^\eta\|^2. \tag{3.15}$$

Combining (3.14) and (3.15), it follows that

$$\frac{d}{dt}(\|A^{\frac{\sigma}{2}}w_t^\eta\|^2+\|A^{\frac{\sigma+1}{2}}w^\eta\|^2)\le C_{M_2}(\|A^{\frac{\sigma}{2}}w_t^\eta\|^2+\|A^{\frac{\sigma+1}{2}}w^\eta\|^2)+C'_{M_2}.$$

Thus, we can complete our proof by applying Gronwall lemma. □

Using Lemmas 3.3 and 3.4, we have the following lemma.

Lemma 3.5. *Let f satisfy* (1.2) *and* (1.3)*, $g\in H^{-2}(\Omega)$ and $\{S(t)\}_{t\ge0}$ be the semigroup generated by the weak solution of* (1.1) *in the natural energy space $\mathcal{H}_0$. Then $\{S(t)\}_{t\ge0}$ is asymptotically smooth in $\mathcal{H}_0$.*

To prove that the global attractors $\mathscr{A}_{\mathcal{H}_0}$ in $\mathcal{H}_0$ are bounded in $\mathcal{V}$, we need the following lemma.

Lemma 3.6. *Under conditions of Lemma 3.5, and* (1.2), (1.3)*, for every $t>0$, the following estimate holds:*

$$\min\{1,t\}\|\Delta u_t\|^2+\min\{1,t^2\}\|u_{tt}\|^2\le Q_1(\|z_0\|_0+\|g\|_{H^{-2}}),$$

where $Q_1(\cdot)$ is a nondecreasing function on $[0,\infty)$, and $(u(t),u_t(t))$ is the solution corresponding to the initial data $z_0\in\mathcal{H}_0$.

The results in the above lemma, are obtained suing the same derivation process as in [14, 16]. Combining Lemmas 3.1, 3.5 and 3.6, according to the abstract conclusion in [9, 14, 16], we have the following theorem.

Theorem 3.7. *Under the assumptions of Lemma 3.5, $\{S(t)\}_{t\ge0}$ has a global attractor $\mathscr{A}_{\mathcal{H}_0}$ in $\mathcal{H}_0$, and $\mathscr{A}_{\mathcal{H}_0}$ is bounded in $\mathcal{V}$.*

Next, we prove that $\mathscr{A}_{\mathcal{H}_0}$ is a $(\mathcal{H}_0,\mathcal{V})$-global attractor. First, By Theorem 3.7 and Lemma 3.6, we have the following statement.

Lemma 3.8. *Let f satisfy (1.2) and (1.3), $g \in H^{-2}(\Omega)$, then the semigroup $\{S(t)\}_{t\geq 0}$ possesses $(\mathcal{H}_0, \mathcal{V})$-bounded absorbing set, that is, there exists $B_{\mathcal{V}} \subset \mathcal{V}$ such that, for any bounded set $B \subset \mathcal{H}_0$, there exists $T_1 = T_1(B)$, there holds*

$$S(t)B \subset B_{\mathcal{V}}, \quad \forall t \geq T_1.$$

Therefore, to obtain the existence of $(\mathcal{H}_0, \mathcal{V})$-global attractor, we only need prove $\{S(t)\}_{t\geq 0}$ is $(\mathcal{H}_0, \mathcal{V})$-asymptotic compactness and continuity.

Let $\bar{B}_1 = \cup_{t\geq T_{B_{\mathcal{V}}}} S(t)B_{\mathcal{V}}$, where $T_{B_{\mathcal{V}}} = \max\{T_1, 1\}$, T_1 is from Lemma 3.8. Then $\bar{B}_1$ is bounded absorbing set, and positive invariant. At the same time, due to Lemma 3.6 and uniqueness of the solution, for any initial value $(u_0, u_1) \in \bar{B}_1$, we have the estimate

$$\|u_{tt}\|^2 \leq C_{\|B_{\mathcal{V}}\|, \|g\|_{H^{-2}}}, \quad \forall t \geq 0.$$

Lemma 3.9. *Suppose that $z_0^n = (u_0^n, u_1^n) \in \bar{B}_1, n = 1, 2, \dots$ is convergent sequence about $\mathcal{H}$-norm, then for any $t \geq 0$, $S(t)z_0^n$ is convergent sequence about $\mathcal{V}$-norm in $\bar{B}_1$.*

Proof. Suppose that $(u^i(t), u_t^i(t))(i = 1, 2)$ is the solution for the initial value $(u_0^i, u_1^i) \in \bar{B}_1$, let $z(t) = u^1(t) - u^2(t)$. Then z satisfy

$$z_{tt} + \Delta^2 z_t + \Delta^2 z + f(u^1) - f(u^2) = 0, \tag{3.16}$$

the corresponding initial condition $(z(0), z_t(0)) = (u_0^1, u_1^1) - (u_0^2, u_1^2)$boundary value conditions $z|_{\partial\Omega} = 0$.

Multiplying (3.16) by z_t, we have

$$\|\Delta z_t\|^2 = -(z_{tt}, z_t) - (\Delta^2 z, z_t) - (f(u^1) - f(u^2), z_t).$$

Due to

$$| - (z_{tt}, z_t) - (\Delta^2 z, z_t)| \leq \|z_{tt}\|\|z_t\| + \|\Delta z\|^2 + \frac{1}{4}\|\Delta z_t\|^2,$$

and

$$| - (f(u^1) - f(u^2), z_t)| \leq C \int_\Omega |f'(u^1 + \theta(u^1 - u^2))||z||z_t| \leq C_M\|\Delta z\|^2 + \frac{1}{4}\|\Delta z_t\|^2,$$

we get

$$\|\Delta z_t\|^2 \leq C_M(\|z_t\| + \|\Delta z\|^2),$$

where C_M only depends on $\|\bar{B}_1\|_0$. By means of the continuity of semigroup $S(t)$ about $\mathcal{H}_0$-norm and the arbitrariness of (u_0^i, u_1^i), we can easily obtain the results of Lemma 3.9 hold. □

So, according to Theorem 3.7 and Lemma 3.9, we have $(\mathcal{H}_0, \mathcal{V})$-asymptotic compactness.

Lemma 3.10. *Under the assumptions of Lemma 3.5, $\{S(t)\}_{t\geq 0}$ is $(\mathcal{H}_0, \mathcal{V})$-asymptotic compact.*

Now we have the existence of $(\mathcal{H}_0, \mathcal{V})$-Global Attractors:

Theorem 3.11. *Let f satisfy (1.2), (1.3), $g \in H^{-2}(\Omega)$ and $\{S(t)\}_{t\geq 0}$ be the semigroup generated by the weak solution of (1.1) in the natural energy space $\mathcal{H}_0$. Then $\{S(t)\}_{t\geq 0}$ has a $(\mathcal{H}_0, \mathcal{V})$-global attractor $\mathscr{A}$; that is, $\mathscr{A}$ is compact, invariant in $\mathcal{V}$, and attracts every bounded (in $\mathcal{H}_0$) subset of $\mathcal{H}_0$ under the $\mathcal{V}$-norm.*

4. Exponential attractor for g in $L^2(\Omega)$

In this section, we consider a slightly stronger $(\mathcal{H}_0, \mathcal{V})$-exponential attraction for $\{S(t)\}_{t\geq 0}$. Borrowing the main idea and methods in [14, 16] we prove the following main results.

Theorem 4.1. *Let $g \in L^2(\Omega)$ and f satisfy (1.2), (1.3). Then there exists a set $\mathcal{E}$ which is compact in $\mathcal{V}$ and bounded in $D(A) \times H_0^2(\Omega)$, satisfying the following conditions:*

- (i) *$\mathcal{E}$ is positive invariant; i.e., $S(t)\mathcal{E} \subset \mathcal{E}$, for all $t \geq 0$;*
- (ii) *$\dim_F(\mathcal{E}, \mathcal{V}) < \infty$; i.e., $\mathcal{E}$ has finite fractal dimension in $\mathcal{V}$;*
- (iii) *there exists an increasing function $\tilde{Q} : \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha > 0$ such that for any subset $B \subset \mathcal{H}_0$ with $\sup_{z_0 \in B} \|z_0\|_{\mathcal{H}_0} \leq R$ there holds*
$$\operatorname{dist}_{\mathcal{V}}(S(t)B, \mathcal{E}) \leq \tilde{Q}(R)\frac{1}{\sqrt{t}}e^{-\alpha t}, \quad \text{for all } t > 0.$$

Remark 4.2. From the proof of Theorem 4.1 given below, we can require in Theorem 4.1 that $\mathcal{E}$ be bounded in $D(A) \times D(A)$.

We first state a crucial result about the asymptotic regularity of the solutions of (1.1) with $g \in L^2(\Omega)$, which can be found in [16].

Theorem 4.3 ([14, 16])**.** *Let f satisfy (1.2) and (1.3), $g \in L^2(\Omega)$, B_0 be a bounded absorbing set of $\{S(t)\}_{t\geq 0}$ in the natural energy space $H_0^2(\Omega) \times L^2(\Omega)$. Then the global attractor $\mathcal{A}_{\mathcal{H}_0}$ is bounded in $D(A) \times D(A)$. Moreover, there exists positive constants M (which depends only on the $H_0^2 \times L^2$-bounds of B_0) and v (which is independent of B_0 but may depend on the coefficients in (1.1)), and a set $\mathcal{B}_1$, closed and bounded in $D(A) \times D(A)$, such that*
$$\operatorname{dist}_{\mathcal{H}}(S(t)B_0, \mathcal{B}_1) \leq Me^{-\nu t}, \quad \forall t \geq 0, \tag{4.1}$$
where $\operatorname{dist}_{\mathcal{H}}$ denotes the usual Hausdorff semidistance in $\mathcal{H}_0$.

As a results, based on the regularity and exponential attraction results, Theorem 4.3, we can repeat the process in [6, 16] to prove the existence of the exponential attractor in $\mathcal{H}_0$ for the critical case. That is,

Proposition 4.4. *Let $g \in L^2(\Omega)$ and f satisfy (1.2) and (1.3). Then the semigroup $\{S(t)\}_{t\geq 0}$ has an exponential attractor $\mathcal{E}_0$ in $\mathcal{H}_0$; that is,*

- (i) *$\mathcal{E}_0$ is positive invariant; i.e., $S(t)\mathcal{E}_0 \subset \mathcal{E}_0$, for all $t \geq 0$;*
- (ii) *$\dim_F(\mathcal{E}_0, \mathcal{H}_0) < \infty$; i.e., $\mathcal{E}_0$ has finite fractal dimension in $\mathcal{H}_0$;*
- (iii) *There exists an increasing function $\mathscr{J} : \mathbb{R}^+ \to \mathbb{R}^+$ and μ_0 such that for any subset $B \subset \mathcal{H}_0$ with $\sup_{z_0 \in B} \|z_0\|_{\mathcal{H}_0} \leq R$ there holds*
$$\operatorname{dist}_{\mathcal{H}_0}(S(t)B, \mathcal{E}_0) \leq \mathscr{J}(R)e^{-\mu_0 t}, \quad \forall t > 0.$$

As in [6, 16], we have the following Lipschitz continuity in $\mathcal{H}_0$.

Lemma 4.5. *For any bounded subset $B \subset \mathcal{H}_0$ and each fixed $T > 0$, there exists a positive constant $M_{T,B}$ which depends only on T and $\|B\|_{\mathcal{H}_0}$ such tat*
$$\|S(T)z_0 - S(T)z_1\|_{\mathcal{H}_0} \leq M_{T,B}\|z_0 - z_1\|_{\mathcal{H}_0}, \quad \forall z_0, z_1 \in B. \tag{4.2}$$
and, $S(t)$ maps the bounded set of $\mathcal{H}_0$ into a bounded set of $\mathcal{H}_0$, that is, there exists an increasing function $Q_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any subset $B \subset \mathcal{H}_0$,
$$\|S(t)B\|_{\mathcal{H}_0} \leq Q_1(\|B\|_{\mathcal{H}_0}), \quad \forall t \geq 0. \tag{4.3}$$

Thanks to Lemma 3.6, we can deduce the following Hölder continuity.

Lemma 4.6. *For any bounded subset $B \subset \mathcal{H}_0$ and each fixed $T > 0$, the mapping $S(T) : (\cup_{t\geq 0} S(t)B, \|\cdot\|_{\mathcal{H}_0}) \to (\cup_{t\geq T} S(t)B, \|\cdot\|_{\mathcal{V}})$ is $\frac{1}{2}$-Hölder continuous; that is, there exists an increasing function $Q_T(\cdot) : [0,\infty) \to [0,\infty)$, which depends only on T, such that*

$$\|S(T)z_0 - S(T)z_1\|_{\mathcal{V}} \leq Q_T(\|B\|_{\mathcal{H}_0})\|z_0 - z_1\|_{\mathcal{H}_0}^{1/2}, \quad \text{for all } z_0, z_1 \in \cup_{t\geq 0} S(t)B. \tag{4.4}$$

Proof. From Lemma 3.6 we know that $\cup_{t\geq T} S(t)B$ is bounded in $\mathcal{V}$ for every $T > 0$. For any $z^i = (u_0^i, u_1^i) \in \mathcal{H}_0 (i = 1, 2)$, let $(u_i(t), u_{i_t}(u)) = S(t)z^i$ be the corresponding solution of (1.1), and denote $z(t) = u_1(t) - u_2(t)$, then z satisfies

$$\begin{gathered} z_{tt} + \Delta^2 z_t + \Delta^2 z + f(u^1) - f(u^2) = 0, \\ (z(0), z_t(0)) = z_1 - z_2, \quad z|_{\partial\Omega} = 0. \end{gathered} \tag{4.5}$$

Multiplying (4.5) by z_t and integrating over Ω, we have

$$\|\Delta z_t\|^2 \leq \|z_{tt}\|\|z_t\| + \|\Delta z_t\|\|\Delta z\| + \int_\Omega |f(u_1) - f(u_2)||z_t|.$$

From (1.2) and using the Hölder inequality, we have

$$\begin{aligned} \int_\Omega |f(u_1) - f(u_2)||z_t| &\leq C \int_\Omega (1 + |u_1|^8 + |u_2|^8)|z||z_t| \\ &\leq C_M \|z\|_{L^{10}} \|z_t\|_{L^{10}} \\ &\leq C_M \|\Delta z\|\|\Delta z_t\|, \end{aligned}$$

where the constant C_M depends only on the $\mathcal{H}_0$-bounds of B. The above inequality with Lemma 4.5 and Lemma 3.6 imply

$$\|\Delta z_t\|^2 \leq \bar{M}_1(\|z_0 - z_1\|_{\mathcal{H}_0} + \|z_0 - z_1\|_{\mathcal{H}_0}^2) \leq \bar{M}_2 \|z_0 - z_1\|_{\mathcal{H}_0},$$

where $\bar{M}_1, \bar{M}_2$ depend only on T and $\|B\|_{\mathcal{H}_0}$; Which, noticing (4.2) again, implies (4.4). □

For convenience, we first iterate the following so-called T-exponential attractor.

Definition 4.7 ([16])**.** Let X, Y be two Banach spaces, $Y \hookrightarrow X$ and $\{S(t)\}_{t\geq 0}$ be a semigroup on X. A set $\mathcal{E}_T \subset Y$ is called a $(X,Y)_T$-exponential attractor for $\{S(t)\}_{t\geq 0}$ if the following conditions hold:

- (i) $\mathcal{E}_T$ is compact in Y and positive invariant; that is, $S(t)\mathcal{E}_T \subset \mathcal{E}_T$, for every $t \geq 0$;
- (ii) $\dim_F(\mathcal{E}_T, Y) < \infty$; that is $\mathcal{E}_T$ has finite fractal dimension in Y;
- (iii) There exists an increasing function $J_T : \mathbb{R}^+ \to \mathbb{R}^+$ and $k > 0$ such that, for any set $B \subset X with \sup_{z_0 \in B} \|z_0\|_X \leq R$ there holds

$$\operatorname{dist}_Y(S(t)B, \mathcal{E}_T) \leq J_T(R)e^{-kt}, \quad \text{for all } t \geq T.$$

Then, we have the existence of an $(\mathcal{H}_0, \mathcal{V})_T$-exponential attractor.

Lemma 4.8. *Let f satisfy* (1.2) *and* (1.3)*, $g \in L^2(\Omega)$. Then for each fixed $T > 0$, $\{S(t)\}_{t\geq 0}$ has an $(\mathcal{H}_0, \mathcal{V})_T$-exponential attractor.*

Proof. For each fixed $T > 0$, we will verify $S(T)\mathcal{E}_0$ is an $(\mathcal{H}_0, \mathcal{V})_T$-exponential attractor, where $\mathcal{E}_0$ is the exponential attractor given in Proposition 4.4.

We verify that $S(T)\mathcal{E}_0$ satisfies all the conditions of Definition 4.7 corresponding to spaces $\mathcal{H}_0$ and $\mathcal{V}$ as follows

(1) The positive invariance of $S(T)\mathcal{E}_0$ is obvious since $\mathcal{E}_0$ is positive invariant; The compactness of $S(T)\mathcal{E}_0$ in $\mathcal{V}$ follows from the compactness of $\mathcal{E}_0$ in $\mathcal{H}_0$ and continuity (Lemma 4.6) of $S(T)$.

(2) Applying property (i) of Lemma 2.1, the finiteness of $\dim_F(S(T)\mathcal{E}_0, \mathcal{V})$ follows from Lemma 4.6 and the finiteness of $\dim_F(\mathcal{E}_0, \mathcal{H}_0)$.

(3) For any bounded subset $B \subset \mathcal{H}_0$, denote $\hat{B} = B \cup \mathcal{E}_0$. Then from Lemma 4.6 we have $S(T) : (\cup_{t\geq 0} S(t)B, \|\cdot\|_{\mathcal{H}_0}) \to (\cup_{t\geq T} S(t)B, \|\cdot\|_{\mathcal{V}})$ is $\frac{1}{2}-$Hölder continuous. Hence, applying property (ii) of Lemma 2.1, the exponential attraction of $S(T)\mathcal{E}_0$ with respect to $\mathcal{V}$-norm follows from the exponential attraction of $\mathcal{E}_0$ with respect to $\mathcal{H}_0$-norm immediately. □

Proof of Theorem 4.1. For any fixed $T_0 \geq 1$, let $\mathcal{E}_{T_0}$ be the $(\mathcal{H}_0, \mathcal{V})_{T_0}$-exponential attractor obtained in Lemma 4.8. Then we claim that $\mathcal{E}_{T_0}$ satisfies conditions (i)-(iii) of Definition 4.7.

We need to verify only (iii). Let $J_{T_0}(\cdot)$ and k_0 be the mapping and exponent given in Definition 4.7 and Lemma 4.8 corresponding to T_0. Note that there is a $t_0 > 0$ such that

$$e^{-\frac{k_0}{2}t} \leq \frac{1}{\sqrt{t}}, \quad \text{for all } t \geq t_0.$$

Then, to complete the proof, we can set $\alpha = \frac{k_0}{2}$ and

$$\tilde{Q}(\cdot) = (J_{T_0}(\cdot) + Q_0(\cdot + \|\mathcal{E}_{T_0}\|_{\mathcal{H}_0}) + Q_1(\cdot + \|\mathcal{E}_{T_0}\|_{\mathcal{H}_0} + \|g\|_{H^{-2}}))e^{(t_0+T_0)\alpha},$$

where $Q(\cdot)$ is given in Lemma 3.6 and $Q_1(\cdot)$ is given in (4.3). □

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