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DIRECT AND INVERSE BIFURCATION PROBLEMS FOR NON-AUTONOMOUS LOGISTIC EQUATIONS

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Abstract. We consider the semilinear eigenvalue problem
$$-u''(t) + k(t)u(t)^p = \lambda u(t), \quad u(t) > 0, \quad t \in I := (-1/2, 1/2),$$
$$u(-1/2) = u(1/2) = 0,$$
where $p > 1$ is a constant, and $\lambda > 0$ is a parameter. We propose a new inverse bifurcation problem. Assume that $k(t)$ is an unknown function. Then can we determine $k(t)$ from the asymptotic behavior of the bifurcation curve? The purpose of this paper is to answer this question affirmatively. The key ingredient is the precise asymptotic formula for the L^q-bifurcation curve $\lambda = \lambda(q, \alpha)$ as $\alpha \to \infty$ $(1 \le q < \infty)$, where $\alpha := \|k^{1/(p-1)}u_\lambda\|_q$.

1. Introduction

We consider the semilinear non-autonomous logistic equation of population dynamics

$$-u''(t) + k(t)u(t)^p = \lambda u(t), \quad t \in I := (-1/2, 1/2), \tag{1.1}$$
$$u(t) > 0 \quad t \in I, \tag{1.2}$$
$$u(-1/2) = u(1/2) = 0, \tag{1.3}$$

where $p > 1$ is a given constant, and $\lambda > 0$ is a parameter. We assume that $k(t) \in C^2(\bar{I})$ satisfies the following conditions.

$$k(t) > 0, \quad k(t) = k(-t), \quad t \in \bar{I}, \tag{1.4}$$
$$k'(t) \ge 0, \quad 0 \le t \le 1/2. \tag{1.5}$$

The local and global structure of the bifurcation diagrams of (1.1)–(1.3) have been investigated by many authors in L^∞-framework. We refer to [1, 6, 7, 10, 11, 12]. In particular, the following basic properties are well known from [1, 9].

(a) For each $\lambda > \pi^2$, there exists a unique solution $u_\lambda \in C^2(\bar{I})$ such that (λ, u_λ) satisfies (1.1)–(1.3).
(b) The set $\{(\lambda, u_\lambda) : \lambda > \pi^2\}$ gives all the solutions of (1.1)–(1.3) and is a continuous unbounded curve in $\mathbb{R}_+ \times C(\bar{I})$ emanating from $(\pi^2, 0)$.
(c) $\pi^2 < \mu < \lambda$ holds if and only if $u_\mu < u_\lambda$ in I.

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We also emphasize that (1.1)–(1.3) is the model equation of population density for some species when $p = 2$. Here, λ and k are regarded as the reciprocal number of its diffusion rate and the effect of crowd for the species, respectively. Furthermore, the L^1-norm of the solution represents the total number of the species. From this biological background, it is significant to study the global structure of the bifurcation diagram in L^q-framework ($1 \le q < \infty$); we refer to [2, 3, 4, 5, 9, 14, 15, 16, 17]. In this article, we parameterize the solution set as follows. For a given $\alpha > 0$, we denote by $(\lambda(q,\alpha), u_\alpha) \in \{\lambda > \pi^2\} \times C^2(\bar{I})$ the solution pair of (1.1)–(1.3) with $\|k^{1/(p-1)}u_\alpha\|_q = \alpha$, which uniquely exists by (c) above. We call the graph $\lambda = \lambda(q,\alpha) \quad (\alpha > 0)$ the L^q-bifurcation diagram of (1.1)–(1.3). From [1, 9] we see that

(d) $\lambda(q,\alpha)$ is increasing for $\alpha > 0$ and $\lambda(q,\alpha) \to \infty$ as $\alpha \to \infty$.

From this asymptotic property, we propose a new inverse bifurcation problem (NIBP) for (1.1)–(1.3), under the following condition on $k(t)$: hypothesis

(H1) Assume that $k(t)$ satisfies (1.4) and (1.5). Furthermore, $K'(t)/K(t)$ and $K''(t)/K(t)$ are non-increasing for $0 \le t \le 1/2$, where $K(t) := k(t)^{-1/(p-1)}$.

Typical examples of $k(t)$ satisfying (H1) are as follows.

$$k(t) = (1-t^2)^{1-p},$$
$$k(t) = k_b(t) = \cos^{1-p}(bt) \quad (0 \le b < \pi, \text{ with } b \text{ constant}).$$

Now, the new inverse bifurcation problem is stated as follows.

(NIBP) Assume that the unknown function $k(t)$ satisfies (H1). Let $\lambda_0(q,\alpha)$ be the L^q-bifurcation diagram of (1.1)–(1.3) with $k(t) \equiv 1$. Suppose that as $\alpha \to \infty$,

$$\lambda(q,\alpha) - \lambda_0(q,\alpha) = o(1). \tag{1.6}$$

Then can we determine $k(t)$?

Inverse problems such as the one above seem to be new for nonlinear problems and it corresponds to the linear inverse eigenvalue problems, which determine unknown potential from the information about eigenvalues. Therefore, it seems worth considering.

The main purpose here is to answer the inverse bifurcation problem (NIBP) affirmatively. To do this, we first consider the direct problem of (1.1)–(1.3), namely, we establish the precise asymptotic formula for $\lambda(q,\alpha)$ as $\alpha \to \infty$. Comparing to the autonomous case, however, there are no works which obtain precise asymptotic formula in non-autonomous case. We refer to [14, 15, 17]. By the terms which come from k, k', k'' and u', the tools for autonomous case are not useful any more in non-autonomous problems.

To overcome this difficulty, we adopt a new parameter $\|k^{1/(p-1)}u_\alpha\|_q = \alpha$ to parameterize the bifurcation curve $\lambda(q,\alpha)$. Indeed, in [14, 15], $\lambda(q,\alpha)$ was parameterized by $\|u_\alpha\|_2 = \alpha$ ($q = 2$) and the calculation there became too complicated, and the optimal estimate for the third term of $\lambda(q,\alpha)$ as $\alpha \to \infty$ has not been obtained there. By the new idea above, the tools for autonomous problems can be available to our non-autonomous case.

Before stating the results for (NIBP), we first state the result for the direct problem.

Theorem 1.1. *Let $p > 1$ and $q \ge 1$ be fixed constants.*

(i) Assume that k is a given function which satisfies (H1). Then, as $\alpha \to \infty$,

$$\lambda(q,\alpha) \ge \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 + m_0 - r_{p,q} + o(1), \tag{1.7}$$

$$\lambda(q,\alpha) \le \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 + M_0 + o(1), \tag{1.8}$$

where

$$C_1 = \frac{p-1}{q}C(q), \tag{1.9}$$

$$C(q) := 2\int_0^1 \frac{1-s^q}{\sqrt{S(s)}}ds, \tag{1.10}$$

$$S(s) = 1 - s^2 - \frac{2}{p+1}(1-s^{p+1}), \tag{1.11}$$

$$a_0 = \frac{p-1}{2q}C(q)^2, \tag{1.12}$$

$$M_0 = \max_{0\le t\le 1/2}\Big|\frac{K''(t)}{K(t)}\Big| = \Big|\frac{K''(1/2)}{K(1/2)}\Big|, \tag{1.13}$$

$$m_0 = \min_{0\le t\le 1/2}\Big|\frac{K''(t)}{K(t)}\Big| = \Big|\frac{K''(0)}{K(0)}\Big|, \tag{1.14}$$

$$r_{p,q} = \frac{p-1}{q}C_2, \tag{1.15}$$

$$C_2 := 4M_1 w_{p,q}, \tag{1.16}$$

$$M_1 = \max_{0\le t\le 1/2}\Big|\frac{K'(t)}{K(t)}\Big| = \Big|\frac{K'(1/2)}{K(1/2)}\Big|. \tag{1.17}$$

$$w_{p,q} := \int_0^1 \frac{(1-s^q)\int_s^1\sqrt{S(\eta)}d\eta}{S(s)^{3/2}}ds. \tag{1.18}$$

(ii) Let $k(t) = k_b(t) = \cos^{1-p} bt$ with $0 \le b < \pi$. Then, as $\alpha \to \infty$,

$$\lambda(q,\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 + b^2 - \frac{4(p-1)}{q}w_{p,q}b\tan\frac{b}{2} + o(1). \tag{1.19}$$

As a corollary of Theorem 1.1, we obtain the following result.

Corollary 1.2. *Assume that k is a given function satisfying* (H1)*. Then, as $\alpha \to \infty$,*

$$\lambda(q,\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + O(1). \tag{1.20}$$

From (1.7) and (1.8), we see that the information about $k(t)$ is contained in the third term of $\lambda(q,\alpha)$. We also remark that the estimate of the third order in (1.20) is optimal by (1.7) and (1.8).

To solve (NIBP), we introduce the condition

(H2) $k(t)$ satisfies (H1) and $k(0) = 1$. Furthermore,
- (i) $k(t) \equiv 1$ on I, or
- (ii) If $k(t) \not\equiv 1$, then $k(t)$ satisfies

$$\frac{1}{R_1} \le \Big|\frac{K''(0)}{K(0)}\Big|, \quad \Big|\frac{K'(1/2)}{K(1/2)}\Big| \le R_2, \tag{1.21}$$

where $R_1, R_2 > 0$ are given constants.

The typical examples of $k(t)$ satisfying (H2) are as follows:

$$k(t) = (1-t^2)^{1-p},$$
$$k(t) = k_b(t) = \cos^{1-p} bt \quad (0 \le b < \pi, \text{ with } b \text{ a constant}).$$

Theorem 1.3. *Assume that the unknown function k satisfies* (H2). *Suppose that* (1.6) *holds for a constant $q \ge q_{p,R} \ge 1$, where $q_{p,R}$ is a constant that depends only on p and R_1, R_2. Then $k(t) \equiv 1$.*

Roughly speaking, if $k(t)$ is nearly flat at $t = \pm 1/2$, then $q_{p,R} = 1$. Now, let $k(t) = k_b(t) = \cos^{1-p} bt$ $(0 \le b < \pi)$. Then we have the following simple result.

Theorem 1.4. *Let $q \ge 1$ be fixed. Further, let $k(t) = k_b(t) = \cos^{1-p}(bt)$ with the unknown constant $0 \le b \ne b_{p,q}$, where $0 < b_{p,q} \le \pi$ is a constant determined explicitly by p and q. Assume* (1.6) *holds with fixed q. Then $b = 0$, namely, $k_b(t) \equiv 1$.*

Remark 1.5. (i) Assume that $k(t) \equiv 1$ in Theorem 1.1 (i). Then (1.7) and (1.8) imply that, as $\alpha \to \infty$,

$$\lambda_0(q,\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 + o(1). \tag{1.22}$$

We note that a more precise asymptotic formula for $\lambda_0(q,\alpha)$ as $\alpha \to \infty$ has been obtained in [16].

(ii) By (1.6), (1.22) and Theorem 1.1 (ii), we obtain

$$b^2 - 4w_{p,q} b \tan\frac{b}{2} = 2b\Big(\frac{b}{2} - \frac{2(p-1)}{q} w_{p,q}\tan\frac{b}{2}\Big) = 0. \tag{1.23}$$

Therefore, if $q/(2(p-1)) \le w_{p,q}$, then $b_{p,q} = \pi$. For example, let $(p,q) = (3,2)$. Then since $S(s) = \frac{1}{2}(1-s^2)^2$, by direct calculation, $w_{3,2} = 2(1/2 + \log 2)/3 \ge q/(2(p-1)) = 1/2$. Therefore, $b_{3,2} = \pi$.

If $w_{p,q} < q/(2(p-1))$, then there exists a unique constant $0 < b_{p,q} < \pi$ such that $b_{p,q}/2 = \frac{2(p-1)}{q} w_{p,q} \tan\frac{b_{p,q}}{2}$. So if $b \ne b_{p,q}$, then (1.6) fails. Therefore, $b = 0$ if (1.6) is valid.

(iii) If $k(t) = \cos^{1-p} bt$, then by direct calculation, we obtain $M_0 = m_0 = b^2$, $M_1 = b\tan(b/2)$. Then by the same argument as that to obtain Theorem 1.1 (i), we obtain Theorem 1.1 (ii).

The remainder of this article is organized as follows. In Section 2, we prove Theorems 1.1 (i) and 1.3 by accepting the key Proposition 2.2 without proof. In Section 3, we prove Proposition 2.2 by using the tools which are developed in [17]. For completeness, we give the proofs of the basic properties of the solutions in Section 4 (Appendix).

2. Proof of Theorems 1.1 and 1.3

In what follows, C denotes various positive constants independent of $\lambda \gg 1$. We put $v_\lambda(t) := v_{\lambda(q,\alpha)}(t) = K(t)^{-1}u_\alpha(t)$. Then by (1.1), we have

$$-v_\lambda''(t) - 2\frac{K'(t)}{K(t)}v_\lambda'(t) + v_\lambda(t)^p = \Big(\lambda + \frac{K''(t)}{K(t)}\Big)v_\lambda(t), \quad t \in I, \tag{2.1}$$

$$v_\lambda(t) > 0, \quad t \in I, \tag{2.2}$$

$$v_\lambda(\pm 1/2) = 0. \tag{2.3}$$

We begin with the fundamental properties of v_λ. Let $V_{\lambda,M_0}(t)$ be the unique solution of (1.1)–(1.3) with $k(t) \equiv 1$, and λ replaced by $\lambda - M_0$. Then we see that $V_{m_0}(t) \equiv (\lambda - m_0)^{1/(p-1)}$ and $V_{\lambda,M_0}(t)$ are super-solution and sub-solution of (2.1)–(2.3) with

$$V_{\lambda,M_0}(t) < (\lambda - m_0)^{1/(p-1)}, \quad t \in I.$$

Then by [13],

$$V_{\lambda,M_0}(t) \le v_\lambda(t) < (\lambda - m_0)^{1/(p-1)}. \tag{2.4}$$

In particular, from [16] for $\lambda \gg 1$, we obtain

$$(\lambda - M_0)^{1/(p-1)} - o(1) \le \|V_{\lambda,M_0}\|_\infty \le \|v_\lambda\|_\infty < (\lambda - m_0)^{1/(p-1)}. \tag{2.5}$$

By [14], we see that $u_\alpha(t)$ is symmetric with respect to $t = 0$. By this and (1.4),

$$v_\lambda(t) = v_\lambda(-t) \quad \text{for } t \in I. \tag{2.6}$$

It is easy to see that

$$\|v_\lambda\|_\infty = v_\lambda(0). \tag{2.7}$$

Further, for $0 \le t \le 1/2$,

$$v_\lambda'(t) \le 0. \tag{2.8}$$

For completeness, the proof of (2.7) and (2.8) will be given in the Appendix. By [14, Theorem 1.2] and (2.5), as $\lambda \to \infty$,

$$\Big|\frac{v_\lambda(t)}{\|v_\lambda\|_\infty} - 1\Big| = O(\lambda^{-1}) \tag{2.9}$$

uniformly on any compact interval in I. Multiply (2.1) by $v_\lambda'(t)$, we have

$$\Big[v_\lambda''(t) + 2\frac{K'(t)}{K(t)}v_\lambda'(t) - v_\lambda(t)^p + \big(\lambda + \frac{K''(t)}{K(t)}\big)v_\lambda(t)\Big]v_\lambda'(t) = 0.$$

By (2.7), for $0 \le t \le 1/2$,

$$\begin{aligned} &\frac{1}{2}v_\lambda'(t)^2 + \int_0^t \frac{2K'(s)}{K(s)}v_\lambda'(s)^2 ds - \frac{1}{p+1}v_\lambda(t)^{p+1} + \frac{1}{2}\lambda v_\lambda(t)^2 \\ &+ \int_0^t \frac{K''(s)}{K(s)}v_\lambda(s)v_\lambda'(s)ds = \text{constant} \\ &= \frac{1}{2}\lambda\|v_\lambda\|_\infty^2 - \frac{1}{p+1}\|v_\lambda\|_\infty^{p+1} \quad (\text{put } t = 0). \end{aligned} \tag{2.10}$$

This implies

$$v_\lambda'(t)^2 = A_\lambda(v_\lambda(t)) + B_\lambda(t) + D_\lambda(t). \tag{2.11}$$

Here,

$$A_\lambda(\theta) := \lambda(\|v_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|v_\lambda\|_\infty^{p+1} - \theta^{p+1}), \tag{2.12}$$

$$B_\lambda(t) := -4\int_0^t \frac{K'(s)}{K(s)}v_\lambda'(s)^2 ds \ge 0, \tag{2.13}$$

$$D_\lambda(t) := -2\int_0^t \frac{K''(s)}{K(s)}v_\lambda(s)v_\lambda'(s)ds \le 0 \tag{2.14}$$

for $0 \le 1 \le 1/2$. Then inequalities (2.13) and (2.14) follow from (H1).

Let $\mu := \lambda - m_0$. By (1.4), (2.8), (2.10) and (2.14), for $0 \le t \le 1/2$,

$$-v_\lambda'(t) = \sqrt{A_\lambda(v_\lambda(t)) + B_\lambda(t) + D_\lambda(t)} \le \sqrt{A_{0,\lambda}(v_\lambda(t)) + B_\lambda(t)}, \tag{2.15}$$

where

$$A_{0,\lambda}(\theta) := \mu(\|v_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|v_\lambda\|_\infty^{p+1} - \theta^{p+1}). \tag{2.16}$$

We know that $A_{0,\lambda}(\theta) > 0$ for $0 \le \theta < \|v_\lambda\|_\infty$. Therefore, $A_{0,\lambda}(v_\lambda(t)) + B_\lambda(t) > 0$ for $0 < t \le 1/2$. By (2.15),

$$\begin{aligned}
&\|v_\lambda\|_\infty^q - \|v_\lambda\|_q^q \\
&= 2\int_0^{1/2} (\|v_\lambda\|_\infty^q - v_\lambda(t)^q)\frac{-v_\lambda'(t)}{\sqrt{A_\lambda(v_\lambda(t)) + B_\lambda(t) + D_\lambda(t)}}dt \\
&\ge 2\int_0^{1/2} (\|v_\lambda\|_\infty^q - v_\lambda(t)^q)\frac{-v_\lambda'(t)}{\sqrt{A_{0,\lambda}(v_\lambda(t)) + B_\lambda(t)}}dt \\
&= 2\int_0^{1/2} (\|v_\lambda\|_\infty^q - v_\lambda(t)^q)\frac{-v_\lambda'(t)}{\sqrt{A_{0,\lambda}(v_\lambda(t))}}dt \\
&\quad + 2\int_0^{1/2} (\|v_\lambda\|_\infty^q - v_\lambda(t)^q)\Big(\frac{-v_\lambda'(t)}{\sqrt{A_{0,\lambda}(v_\lambda(t)) + B(t)}} + \frac{v_\lambda'(t)}{\sqrt{A_{0,\lambda}(v_\lambda(t))}}\Big)dt \\
&:= I + II.
\end{aligned} \tag{2.17}$$

We put

$$R_\lambda(s) := 1 - s^2 - \frac{2}{p+1}\frac{\|v_\lambda\|_\infty^{p-1}}{\mu}(1 - s^{p+1}), \tag{2.18}$$

$$U_\lambda := 2\int_0^1 \frac{(1-s^q)(S(s) - R_\lambda(s))}{\sqrt{R_\lambda(s)}\sqrt{S(s)}(\sqrt{R_\lambda(s)} + \sqrt{S(s)})}ds. \tag{2.19}$$

Lemma 2.1. *For $\lambda \gg 1$,*

$$I = \frac{\|v_\lambda\|_\infty^q}{\sqrt{\mu}}\left(C(q) + U_\lambda\right), \tag{2.20}$$

$$|U_\lambda| \le C\lambda^{-1}\log\lambda. \tag{2.21}$$

The proof of the above lemma is the variant of [16, Lemmas 3.1 and 3.2]. For completeness, it will be given in Appendix.

Proposition 2.2. *For $\lambda \gg 1$, the integral II defined by (2.17) satisfies*

$$II = -C_2\|v_\lambda\|_\infty^{q+1-p}(1 + o(1)). \tag{2.22}$$

The proof of this proposition will be given in Section 3. Meanwhile, we use for proving Theorem 1.1.

Proof of Theorem 1.1. We first prove (1.8). By (1.22) and (2.4), for $\lambda \gg 1$, we obtain

$$\begin{aligned}
\lambda - M_0 &= \|V_{\lambda,M_0}\|_q^{p-1} + C_1\|V_{\lambda,M_0}\|_q^{(p-1)/2} + a_0 + o(1) \\
&\le \|v_\lambda\|_q^{p-1} + C_1\|v_\lambda\|_q^{(p-1)/2} + a_0 + o(1).
\end{aligned}$$

Therefore, we obtain (1.8).

Next we show (1.7). By (2.5) and (2.21), we see that for $\lambda \gg 1$,

$$|U_\lambda| \le C\lambda^{-1}\log\lambda = o(\|v_\lambda\|_\infty^{(1-p)/2}). \tag{2.23}$$

By (2.17), (2.23), Lemma 2.1 and Proposition 2.2, we obtain

$$\|v_\lambda\|_\infty^q - \|v_\lambda\|_q^q \geq \frac{\|v_\lambda\|_\infty^q}{\sqrt{\mu}}(C(q) - C_2\|v_\lambda\|_\infty^{(1-p)/2}(1+o(1))). \tag{2.24}$$

That is,

$$\|v_\lambda\|_\infty^q\Big(1 - \frac{1}{\sqrt{\mu}}(C(q) - C_2\|v_\lambda\|_\infty^{(1-p)/2}(1+o(1)))\Big) \geq \alpha^q. \tag{2.25}$$

By this inequality, (1.8), (2.5) and the Taylor expansion, we obtain

$$\begin{aligned}
\mu = \lambda - m_0 &\geq \|v_\lambda\|_\infty^{p-1} \\
&\geq \alpha^{p-1}\Big(1 - \frac{1}{\sqrt{\mu}}(C(q) - C_2\|v_\lambda\|_\infty^{(1-p)/2}(1+o(1)))\Big)^{-(p-1)/q} \\
&= \alpha^{p-1}\big\{1 + \frac{1}{\sqrt{\mu}}\frac{p-1}{q}C(q)(1+o(1))\big\} \\
&= \alpha^{p-1} + \frac{p-1}{q}C(q)\alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}).
\end{aligned}$$

By this equality and (1.8), for $\alpha \gg 1$, we have

$$\mu = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + o(\alpha^{(p-1)/2}). \tag{2.26}$$

By (2.5), (2.25) and the Taylor expansion, for $\lambda \gg 1$, we have

$$\begin{aligned}
&\lambda - m_0 \geq \|v_\lambda\|_\infty^{p-1} \\
&\geq \alpha^{p-1}\big\{1 - \frac{1}{\sqrt{\mu}}(C(q) - C_2\|v_\lambda\|_\infty^{(1-p)/2}(1+o(1)))\big\}^{-(p-1)/q} \\
&\geq \alpha^{p-1}\Big\{1 + \frac{p-1}{q}\frac{1}{\alpha^{(p-1)/2}(1 + C_1\alpha^{(1-p)/2} + o(\alpha^{(1-p)/2}))^{1/2}} \\
&\quad \times (C(q) - C_2\alpha^{(1-p)/2} + o(\alpha^{(1-p)/2})) \\
&\quad + \frac{(p-1)(p+q-1)}{2q^2}\frac{1}{\alpha^{p-1}(1 + C_1\alpha^{(1-p)/2} + o(\alpha^{(1-p)/2}))} \\
&\quad \times (C(q) - C_2(1+o(1))\alpha^{(1-p)/2} + o(\alpha^{(1-p)/2}))^2(1+o(1))\Big\} \\
&= \alpha^{p-1} + \frac{p-1}{q}\alpha^{(p-1)/2}(1 - \frac{1}{2}C_1\alpha^{(1-p)/2} + o(\alpha^{(1-p)/2})) \\
&\quad \times (C(q) - C_2\alpha^{(1-p)/2} + o(\alpha^{(1-p)/2})) \\
&\quad + \frac{(p-1)(p+q-1)}{2q^2}(C(q)^2 - C_1C(q)^2\alpha^{(1-p)/2} \\
&\quad - 2C(q)C_2\alpha^{(1-p)/2} + o(\alpha^{(1-p)/2})) \\
&= \alpha^{p-1} + \frac{p-1}{q}C(q)\alpha^{(p-1)/2} - \frac{p-1}{2q}C_1C(q) + \frac{(p-1)(p+q-1)}{2q^2}C(q)^2 \\
&\quad - \frac{p-1}{q}C_2 + o(1) \\
&= \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 - \frac{p-1}{q}C_2 + o(1).
\end{aligned}$$

Thus we obtain (1.7). The proof of Theorem 1.1 is complete. $\square$

Proof of Theorem 1.3. By (1.6) and (1.7), we have

$$m_0 \le r_{p,q}. \tag{2.27}$$

We assume that (H2) (ii) is valid, and obtain a contradiction. Since $r_{p,q} = 4(p-1)w_{p,q}M_1/q$, for $0 < \epsilon \ll 1$, by (1.21) and (2.27), we have

$$\begin{aligned}
\frac{1}{R_1R_2} \le \frac{m_0}{M_1} &\le \frac{4(p-1)}{q} w_{p,q} \\
&= \frac{4(p-1)}{q}\int_0^{1-\epsilon} \frac{(1-s^q)\int_s^1 \sqrt{S(\eta)}d\eta}{S(s)^{3/2}} ds \\
&\quad + \frac{4(p-1)}{q}\int_{1-\epsilon}^1 \frac{(1-s^q)\int_s^1 \sqrt{S(\eta)}d\eta}{S(s)^{3/2}} ds \\
&:= L_1 + L_2.
\end{aligned} \tag{2.28}$$

We have to consider only the case where $q \gg 1$. If $q \gg 1$, then we have

$$L_1 \le \frac{C}{q} \ll 1. \tag{2.29}$$

Furthermore, for $1-\epsilon \le \eta \le 1$ and $0 < \delta \ll 1$, by Taylor expansion,

$$S_\lambda(\eta) \le (p-1)(1-\eta)^2, \tag{2.30}$$

$$S_\lambda(\eta) \ge (p-1-\delta)(1-\eta)^2. \tag{2.31}$$

By (2.30) and (2.31), for $q \gg 1$,

$$L_2 \le \frac{C}{q}\int_{1-\epsilon}^1 \frac{(1-s^q)\int_s^1(1-\eta)d\eta}{(1-s)^3} ds \le \frac{C}{q}\int_{1-\epsilon}^1 \frac{1-s^q}{1-s} ds \le Cq^{-1}\log q \ll 1. \tag{2.32}$$

This inequality and (2.29) contradict (2.28). Therefore, (H2) (i) holds. Thus the proof is complete. □

3. Proof of Proposition 2.2

Let an arbitrary $0 < \epsilon \ll 1$ be fixed. The integral II defined by (2.17) satisfies

$$\begin{aligned}
II &= 2\int_0^{1/2} (\|v_\lambda\|_\infty^q - v_\lambda(t)^q) \frac{B_\lambda(t)v_\lambda'(t)}{\sqrt{A_{0,\lambda}(v_\lambda(t)) + B_\lambda(t)}} \\
&\quad \times \frac{1}{\sqrt{A_{0,\lambda}(v_\lambda(t))}\big(\sqrt{A_{0,\lambda}(v_\lambda(t)) + B_\lambda(t)} + \sqrt{A_{0,\lambda}(v_\lambda(t))}\big)} dt \\
&= 2\Big(\int_0^{1/2-\epsilon} + \int_{1/2-\epsilon}^{1/2}\Big) := II_1 + II_2.
\end{aligned} \tag{3.1}$$

Lemma 3.1. *For $0 \le t \le 1/2$,*

$$B_\lambda(t) \le C\|v_\lambda\|_\infty^{(p+1)/2}(\|v_\lambda\|_\infty - v_\lambda(t)). \tag{3.2}$$

Proof. There exists $0 \le t_\lambda \le 1/2$ such that $\max_{0\le t\le 1/2}|v_\lambda'(t)| = |v_\lambda'(t_\lambda)|$. We first show that

$$\frac{v_\lambda'(t_\lambda)^2}{\|v_\lambda\|_\infty^{p+1}} \le C. \tag{3.3}$$

To prove this, we assume that there exists a subsequence of $\{\lambda\}$, which is denoted by $\{\lambda\}$ again, such that, as $\lambda \to \infty$,

$$\frac{v_\lambda'(t_\lambda)^2}{\|v_\lambda\|_\infty^{p+1}} \to \infty \tag{3.4}$$

and derive a contradiction. Since

$$\begin{aligned} \Big|\int_0^t \frac{K''(s)}{K(s)} v_\lambda(s) v_\lambda'(s) ds\Big| &\le C\Big|\int_0^t v_\lambda(s) v_\lambda'(s) ds\Big| \\ &= C(\|v_\lambda\|_\infty^2 - v_\lambda(t)^2) \le C\|v_\lambda\|_\infty^2, \end{aligned} \tag{3.5}$$

by putting $t = t_\lambda$ in (2.10), we obtain from (3.4) and (3.5) that

$$\begin{aligned} \frac{1}{2}(1 - o(1)) v_\lambda'(t_\lambda)^2 &= -2\int_0^{t_\lambda} \frac{K'(s)}{K(s)} v_\lambda'(s)^2 ds \\ &\le C|v_\lambda'(t_\lambda)| \int_0^{t_\lambda} -v_\lambda'(s) ds \\ &\le C|v_\lambda'(t_\lambda)| \|v_\lambda\|_\infty. \end{aligned}$$

This inequality implies $|v_\lambda'(t_\lambda)| \le C\|v_\lambda\|_\infty$, which contradicts (3.4). Therefore, we obtain (3.3). Then by (2.8), (2.13) and (3.3), for $0 \le t \le 1$,

$$\begin{aligned} B_\lambda(t) &\le C\int_0^t v_\lambda'(s)^2 ds \\ &\le C|v_\lambda'(t_\lambda)| \int_0^{1/2} (-v_\lambda'(s)) ds \\ &\le C\|v_\lambda\|_\infty^{(p+1)/2} (\|v_\lambda\|_\infty - v_\lambda(t)). \end{aligned}$$

Thus the proof is complete. □

Lemma 3.2. *For $0 \le t \le 1/2$,*

$$B_\lambda(t) \le C\|v_\lambda\|_\infty^{p/2} (\|v_\lambda\|_\infty - v_\lambda(t))^{3/2}. \tag{3.6}$$

Proof. Recall that $B(t)$ is increasing for $0 \le t \le 1/2$ by (H1) and (2.13). By (2.5), (2.13), (2.15) and Lemma 3.1,

$$\begin{aligned} B_\lambda(t) &= -4\int_0^t \frac{K'(s)}{K(s)} \sqrt{A_{0,\lambda}(v_\lambda(s)) + B_\lambda(s)} (-v_\lambda'(s)) ds \\ &\le 4M_1 \int_0^t \Big(\sqrt{A_{0,\lambda}(v_\lambda(s))} + \sqrt{B_\lambda(s)}\Big) (-v_\lambda'(s)) ds \\ &\le 4M_1 \int_{v_\lambda(t)}^{\|v_\lambda\|_\infty} \sqrt{A_{0,\lambda}(\theta)} d\theta + 4M_1 \max_{0 \le s \le t} \sqrt{B_\lambda(s)} \int_0^t (-v_\lambda'(s)) ds \\ &\le C\sqrt{\mu}\|v_\lambda\|_\infty^2 \int_{v_\lambda(t)/\|v_\lambda\|_\infty}^1 \sqrt{R_\lambda(s)} ds + C\sqrt{B_\lambda(t)} (\|v_\lambda\|_\infty - v_\lambda(t)). \\ &= C\sqrt{\mu}\|v_\lambda\|_\infty^2 \int_{v_\lambda(t)/\|v_\lambda\|_\infty}^1 \sqrt{1 - s^2} ds + C\sqrt{B_\lambda(t)} (\|v_\lambda\|_\infty - v_\lambda(t)) \\ &\le C\sqrt{\mu}\|v_\lambda\|_\infty^2 \Big(1 - \frac{v_\lambda(t)}{\|v_\lambda\|_\infty}\Big)^{3/2} + C\|v_\lambda\|_\infty^{(p+1)/4} (\|v_\lambda\|_\infty - v_\lambda(t))^{3/2} \\ &\le C\|v_\lambda\|_\infty^{p/2} (\|v_\lambda\|_\infty - v_\lambda(t))^{3/2}. \end{aligned}$$

Thus the proof is complete. □

Lemma 3.3. *For $\lambda \gg 1$,*

$$II_1 = o(\|v_\lambda\|_\infty^{q+1-p}). \tag{3.7}$$

Proof. By (2.9), (2.31), (3.1), Lemma 3.2 and putting $s = v_\lambda(t)/\|v_\lambda\|_\infty$, for $\lambda \gg 1$, we have

$$\begin{aligned}
|II_1| &\le C \int_0^{1/2-\epsilon} \frac{(\|v_\lambda\|_\infty^q - v_\lambda(t)^q)B_\lambda(t)(-v_\lambda'(t))}{2(A_{0,\lambda}(v_\lambda(t))^{3/2}} dt \\
&\le C \int_{v_\lambda(1/2-\epsilon)/\|v_\lambda\|_\infty}^1 \frac{\|v_\lambda\|_\infty^{q+1} \cdot (1-s^q)\|v_\lambda\|_\infty^{(p+3)/2}(1-s)^{3/2}}{\mu^{3/2}\|v_\lambda\|_\infty^3 S_\lambda(s)^{3/2}} ds \\
&\le C\|v_\lambda\|_\infty^{q+1-p} \int_{v_\lambda(1/2-\epsilon)/\|v_\lambda\|_\infty}^1 (1-s)^{-1/2} ds \\
&= C\|v_\lambda\|_\infty^{q+1-p} \left(1 - \frac{v_\lambda(1/2-\epsilon)}{\|v_\lambda\|_\infty}\right)^{1/2} = o(\|v_\lambda\|_\infty^{q+1-p}).
\end{aligned}$$

Thus the proof is complete. □

Now we estimate II_2.

Lemma 3.4. *For $\lambda \gg 1$,*

$$-II_2 \le C_2(1+o(1))\|v_\lambda\|_\infty^{q+1-p}. \tag{3.8}$$

Proof. Let $1/2 - \epsilon \le t \le 1/2$. Then by (1.17), (2.15), Lemmas 3.1 and 3.2,

$$\begin{aligned}
B_\lambda(t) &= -4\int_0^t \frac{K'(s)}{K(s)} v_\lambda'(s)^2 ds \le 4M_1 \int_0^t v_\lambda'(s)^2 ds \\
&\le 4M_1 \int_0^t (\sqrt{A_{0,\lambda}(v_\lambda(s)} + \sqrt{B_\lambda(s)})(-v_\lambda'(s)) ds \\
&\le 4M_1\sqrt{\mu}\|v_\lambda\|_\infty^2 \int_{v_\lambda(t)/\|v_\lambda\|_\infty}^1 \sqrt{R_\lambda(s)} ds \\
&\quad + C\|v_\lambda\|_\infty^{(p+1)/4} \int_0^t (\|v_\lambda\|_\infty - v_\lambda(s))^{1/2}(-v_\lambda'(s)) ds \\
&\le 4M_1(1+o(1))\sqrt{\mu}\|v_\lambda\|_\infty^2 \\
&\quad \times \int_{v_\lambda(t)/\|v_\lambda\|_\infty}^1 \sqrt{S_\lambda(s)} ds + C\|v_\lambda\|_\infty^{(p+1)/4}(\|v_\lambda\|_\infty - v_\lambda(t))^{3/2}.
\end{aligned}$$

By this inequality, (2.9) and (3.1), we have

$$\begin{aligned}
-II_2 &\le 4M_1(1+o(1))\sqrt{\mu}\|v_\lambda\|_\infty^2 \\
&\quad \times \int_{1/2-\epsilon}^{1/2} (\|v_\lambda\|_\infty^q - v_\lambda(t)^q) \frac{\int_{v_\lambda(t)/\|v_\lambda\|_\infty}^1 \sqrt{S_\lambda(\eta)} d\eta}{A_{0,\lambda}(v_\lambda(t))^{3/2}} (-v_\lambda'(t)) dt \\
&\quad + C\|v_\lambda\|_\infty^{(p+1)/4} \int_{1/2-\epsilon}^{1/2} \frac{(\|v_\lambda\|_\infty^q - v_\lambda(t)^q)(\|v_\lambda\|_\infty - v_\lambda(t))^{3/2}}{A_{0,\lambda}(v_\lambda(t))^{3/2}} (-v_\lambda'(t)) dt \\
&= 4M_1(1+o(1))\|v_\lambda\|_\infty^{q+1-p} \int_0^{v_\lambda(1/2-\epsilon)/\|v_\lambda\|_\infty} \frac{(1-s^q)\int_s^1 \sqrt{S_\lambda(\eta)} d\eta}{S_\lambda(s)^{3/2}} ds
\end{aligned}$$

$$
\begin{aligned}
&+ C\|v_\lambda\|_\infty^{q+1-p+(1-p)/4} \int_0^{v_\lambda(1/2-\epsilon)/\|v_\lambda\|_\infty} \frac{(1-s^q)(1-s)^{3/2}}{S_\lambda(s)^{3/2}} ds \\
&= C_2(1+o(1))\|v_\lambda\|_\infty^{q+1-p}.
\end{aligned}
$$

Thus the proof is complete. □

Lemma 3.5. *Let an arbitrary $0 < \delta \ll 1$ be fixed. Then for $\lambda \gg 1$,*

$$
- II_2 \ge C_2(1 - C\epsilon)\|v_\lambda\|_\infty^{q+1-p}. \tag{3.9}
$$

Proof. By (2.15), we have

$$
-v_\lambda'(t) = \sqrt{A_\lambda(v_\lambda(t)) + B_\lambda(t) + D_\lambda(t)} \ge \sqrt{A_\lambda(v_\lambda(t))} - \sqrt{B_\lambda(t)} - \sqrt{|D_\lambda(t)|}. \tag{3.10}
$$

Let $1/2 - \epsilon < t < 1/2$. Since ϵ is small enough, by (1.17) and (3.10), we have

$$
\begin{aligned}
B_\lambda(t) &\ge -4\int_{1/2-\epsilon}^t \frac{K'(s)}{K(s)} v_\lambda'(s)^2 ds \ge 4(M_1-\delta)\int_{1/2-\epsilon}^t v_\lambda'(s)^2 ds \\
&\ge 4(M_1 - 2\delta)\int_{1/2-\epsilon}^t \sqrt{A_\lambda(v_\lambda(s))}(-v_\lambda'(s))ds \\
&\quad - C\int_{1/2-\epsilon}^t \sqrt{B_\lambda(s)}(-v_\lambda'(s))ds - C\int_{1/2-\epsilon}^t \sqrt{|D_\lambda(s)|}(-v_\lambda'(s))ds.
\end{aligned} \tag{3.11}
$$

By (2.14), we obtain

$$
\begin{aligned}
\int_{1/2-\epsilon}^t \sqrt{|D_\lambda(s)|}(-v_\lambda'(s))ds &\le C\int_{1/2-\epsilon}^t \left(\|v_\lambda\|_\infty^2 - v_\lambda(t)^2\right)^{1/2}(-v_\lambda'(s))ds \\
&\le C\int_{v_\lambda(t)}^{v_\lambda(1/2-\epsilon)} (\|v_\lambda\|_\infty^2 - \theta^2)^{1/2} d\theta \\
&\le C\|v_\lambda\|_\infty^2 \int_{v_\lambda(t)/\|v_\lambda\|_\infty}^1 (1-s^2)^{1/2} ds \\
&\le C\|v_\lambda\|_\infty^{1/2}(\|v_\lambda\|_\infty - v_\lambda(t))^{3/2}.
\end{aligned} \tag{3.12}
$$

Then by (3.11), (3.12), Lemma 3.2 and the same argument as the one to obtain (2.21) and Lemma 3.4, we obtain (3.9). Thus the proof is complete. □

Since $0 < \delta \ll 1$ is arbitrary, by Lemmas 3.4 and 3.5, we have competed the proof of Proposition 2.2.

4. Appendix

Proof of (2.7). We assume that $\|v_\lambda\|_\infty > v_\lambda(0)$ and derive a contradiction. First, suppose that $v_\lambda''(0) \ge 0$. Let $0 < t_\lambda < 1/2$ satisfy $v_\lambda(t_\lambda) = \|v_\lambda\|_\infty$. Then by (2.1) and (2.5),

$$
0 \ge -v_\lambda''(0) = v_\lambda(0)\big(\lambda + \frac{K''(0)}{K(0)} - v_\lambda(0)^{p-1}\big) = v_\lambda(0)(\lambda - m_0 - v_\lambda(0)^{p-1}) > 0.
$$

This is a contradiction. Next, suppose that $v_\lambda''(0) < 0$. Then there exists $0 < s_\lambda < t_\lambda < 1/2$ such that $v_\lambda'(s_\lambda) = 0$ and $v_\lambda''(s_\lambda) \ge 0$. By this and the fact that $v_\lambda(s_\lambda) < v_\lambda(t_\lambda) = \|v_\lambda\|_\infty$, and (2.1), we have

$$
0 \ge -v_\lambda''(s_\lambda) = -v_\lambda(s_\lambda)^{p-1} + \Big(\lambda + \frac{K''(s_\lambda)}{K(s_\lambda)}\Big)v_\lambda(s_\lambda),
$$

$$0 \le -v_\lambda''(t_\lambda) = -v_\lambda(t_\lambda)^{p-1} + \Big(\lambda + \frac{K''(t_\lambda)}{K(t_\lambda)}\Big) v_\lambda(t_\lambda).$$

This implies

$$\Big(\lambda + \frac{K''(s_\lambda)}{K(s_\lambda)}\Big) \le v_\lambda(s_\lambda)^{p-1} < v_\lambda(t_\lambda)^{p-1} \le \Big(\lambda + \frac{K''(t_\lambda)}{K(t_\lambda)}\Big). \tag{4.1}$$

This contradicts to (H1). Thus we obtain (2.7). □

Proof of (2.8). By (2.7), we see that there exists a constant $0 < t_0 \le 1/2$ such that

$$t_0 := \sup\{s : 0 \le s \le 1 \text{ and } v_\lambda'(t) \le 0 \text{ for any } 0 \le t \le s\}. \tag{4.2}$$

If $t_0 = 1/2$, then the proof is complete. Assume that $t_0 < 1/2$. Then $v_\lambda'(t_0) = 0$. Assume that there exists a constant $0 < t_0 < s_0 < 1/2$ such that $v_\lambda(t_0) < v_\lambda(s_0)$. Then there exists $t_0 \le t_1 < s_1 < 1/2$ such that $v_\lambda(t_1) < v_\lambda(s_1)$ and $v_\lambda'(t_1) = 0, v_\lambda''(t_1) \ge 0$ and $v_\lambda'(s_1) = 0, v_\lambda''(s_1) \le 0$ Then by the same argument as the proof of (2.7) above, we obtain a contradiction. Therefore, $v_\lambda(t)$ is non-increasing for $t_0 \le t \le 1/2$. Thus the proof of (2.8) is complete. □

Proof of Proposition 2.2. We apply the same argument as that in [16, Lemma 3.2] to our situation. For $\lambda > \pi^2$ and $0 \le s \le 1$, we put

$$\begin{aligned} M_\lambda(\theta) &:= \mu(\|v_\lambda\|_\infty^2 - \theta^2) - \frac{2}{p+1}(\|v_\lambda\|_\infty^{p+1} - \theta^{p+1}), \\ Q_\lambda(s) &:= \mu\|v_\lambda\|_\infty^2(1 - s^2) - \frac{2}{p+1}\|v_\lambda\|_\infty^{p+1}(1 - s^{p+1}). \end{aligned}$$

By putting $\theta = u_\lambda(t)$ and $s = \theta/\|u_\lambda\|_\infty$, we obtain

$$\begin{aligned} I &= 2\int_0^1 (\|v_\lambda\|_\infty^q - v_\lambda^q(t)) \frac{-v_\lambda'(t)}{\sqrt{A_{0,\lambda}(v_\lambda(t))}} dt \\ &= 2\int_0^{\|v_\lambda\|_\infty} (\|v_\lambda\|_\infty^q - \theta^q) \frac{1}{\sqrt{M_\lambda(\theta)}} d\theta \\ &= 2\frac{\|v_\lambda\|_\infty^q}{\sqrt{\mu}} \int_0^1 \frac{1 - s^q}{\sqrt{Q_\lambda(s)/(\mu\|v_\lambda\|_\infty^2)}} ds \\ &= 2\frac{\|v_\lambda\|_\infty^q}{\sqrt{\mu}} \int_0^1 \frac{1 - s^q}{\sqrt{R_\lambda(s)}} ds \\ &= \frac{\|v_\lambda\|_\infty^q}{\sqrt{\mu}} \Big(2\int_0^1 \frac{1 - s^q}{\sqrt{S_\lambda(s)}} ds + U_\lambda\Big) \\ &= \frac{\|v_\lambda\|_\infty^q}{\sqrt{\mu}} \big(C(q) + U_\lambda\big). \end{aligned}$$

By (2.5), for $\lambda \gg 1$, we have

$$\xi_\lambda := \lambda - \|v_\lambda\|_\infty^{p-1} = O(1). \tag{4.3}$$

Let $0 < \epsilon \ll 1$ be fixed. Then by Taylor expansion, there exists a constant $0 < \delta \ll 1$ such that for $\lambda \gg 1$ and $1 - \epsilon \le s \le 1$

$$R_\lambda(s) \ge \frac{\xi_\lambda}{\lambda}(1 - s) + (p - 1 - \delta)(1 - s)^2. \tag{4.4}$$

By (2.19),

$$\begin{aligned}U_\lambda &= U_{1,\lambda} + U_{2,\lambda}\\ &:= 2\int_0^{1-\epsilon} \frac{(1-s^q)(S_\lambda(s)-R_\lambda(s))}{\sqrt{R_\lambda(s)}\sqrt{S_\lambda(s)}(\sqrt{R_\lambda(s)}+\sqrt{S_\lambda(s)})}ds\\ &\quad + 2\int_{1-\epsilon}^1 \frac{(1-s^q)(S_\lambda(s)-R_\lambda(s))}{\sqrt{R_\lambda(s)}\sqrt{S_\lambda(s)}(\sqrt{R_\lambda(s)}+\sqrt{S_\lambda(s)})}ds.\end{aligned} \tag{4.5}$$

By (2.18), (2.31) and (4.4), we have

$$\begin{aligned}|U_{2,\lambda}| &\le 2\int_{1-\epsilon}^1 \frac{(1-s^q)(1-s^{p+1})(1-\|v_\lambda\|_\infty^{p-1}/\lambda)}{R_\lambda(s)\sqrt{S_\lambda(s)}}ds\\ &\le C\frac{\xi_\lambda}{\lambda}\int_{1-\epsilon}^1 \frac{1}{(\xi_\lambda/\lambda)+(p-1-\delta)(1-s)}ds\\ &= C\frac{\xi_\lambda}{\lambda}\int_0^\epsilon \frac{1}{(\xi_\lambda/\lambda)+C\eta}d\eta\\ &\le C\frac{\xi_\lambda}{\lambda}\big|\log\big(\frac{\xi_\lambda}{\lambda}\big)\big|\\ &\le C\lambda^{-1}\log\lambda.\end{aligned} \tag{4.6}$$

Finally, it is clear that $S_\lambda(s) \ge C, R_\lambda(s) \ge C$ for $0 \le s \le 1-\epsilon$ and $\lambda \gg 1$. By (4.3) and (4.5),

$$|U_{1,\lambda}| \le C\int_0^{1-\epsilon}(1-s^q)(1-s^{p+1})(1-\|v_\lambda|_\infty^{p-1}/\lambda)ds \le C\frac{\xi_\lambda}{\lambda} \le C\lambda^{-1}.$$

By (4.5) and (4.6), we obtain Proposition 2.2. Thus the proof is complete. □

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