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# ASYMPTOTIC STABILITY OF TRAVELING FRONTS IN DELAYED REACTION-DIFFUSION MONOSTABLE EQUATIONS ON HIGHER-DIMENSIONAL LATTICES

HAI-QIN ZHAO

Abstract. This article is concerned with traveling wave fronts for spatially discrete delayed reaction-diffusion equations on higher-dimensional lattices. Under the monostable assumption and some reasonable conditions, we prove the globally asymptotic stability of traveling wave fronts in the sense of phase shift by using the comparison principle and the squeezing technique.

## 1. Introduction

In this article, we study the globally asymptotic stability of traveling fronts of the general delayed reaction-diffusion equation on higher-dimensional lattices,

$$u'_\eta(t) = D(\triangle_n u)_\eta + f\big(u_\eta(t), u_\eta(t-\tau)\big), \tag{1.1}$$

where $n \in \mathbb{Z}_+$, $\eta \in \mathbb{Z}^n$, $t > 0$, $u_\eta(t) \in \mathbb{R}$, $D > 0, \tau \geq 0$ are constants, $(\triangle_n u)_\eta$ is the standard $n$-dimensional discrete Laplacian; i.e.,

$$(\triangle_n u)_\eta = \sum_{\|\eta_1-\eta\|=1,\, \eta_1\in\mathbb{Z}^n} \left[u_{\eta_1}(t) - u_\eta(t)\right].$$

Here the reaction function $f$ satisfies the following assumptions:

(A1) $f \in C^2([0,K]^2, \mathbb{R})$, $f(0,0) = f(K,K) = 0$, $f(u,u) > 0$ for all $u \in (0,K)$, and $\partial_1 f(K,K) + \partial_2 f(K,K) < 0$, where $K > 0$ is a constant;

(A2) $\partial_2 f(u,v) \geq 0$, and $f(u,v) \leq \partial_1 f(0,0)u + \partial_2 f(0,0)v$ for $(u,v) \in [0,K]^2$;

(A3) For any $\delta \in (0,1)$, there exist $a = a(\delta) > 0$, $\alpha = \alpha(\delta) \geq 0$ and $\beta = \beta(\delta) \geq 0$ with $\alpha + \beta > 0$ such that for any $\varpi \in (0,\delta]$ and $(u,v) \in [0,K]^2$

$$(1-\varpi)f(u,v) - f\big((1-\varpi)u, (1-\varpi)v\big) \leq -a\varpi u^\alpha v^\beta.$$

From (A1) and (A2), we can see that (1.1) has two equilibria 0 and $K$, and $\partial_1 f(0,0) + \partial_2 f(0,0) \geq \frac{2}{K} f\left(\frac{K}{2}, \frac{K}{2}\right) > 0$. We would like to point out that (A1) is a standard monostable assumption, (A2) is a quasi-monotone and sub-tangential condition, and (A3) is not a more restrictive condition. Indeed, the assumption (A3) is a convex condition and in general, monostable nonlinearities satisfy it, see Section 4 for applications. We are interested in traveling wave solutions of (1.1) that connect

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the two equilibria 0 and $K$. Throughout this paper, a traveling wave solution always refers to a trinity $(U, c, \sigma)$, where $U = U(\cdot) : \mathbb{R} \to [0, K]$ is a function, $c > 0$ is a constant and $\sigma \in \mathbb{R}^n$ is a unit vector, such that $u_\eta(t) := U(\eta \cdot \sigma + ct)$, is a solution of (1.1), and

$$U(-\infty) := \lim_{\xi\to-\infty} U(\xi) = 0, \quad U(+\infty) := \lim_{\xi\to+\infty} U(\xi) = K. \tag{1.2}$$

The vector $\sigma$ represents the direction of the wave. We call $c$ the *wave speed* and $U$ the *wave profile*. Moreover, we say $U$ is a *traveling (wave) front* if $U(\cdot) : \mathbb{R} \to \mathbb{R}$ is monotone.

For some special cases of (1.1), many well-known results on the traveling wave fronts have been obtained under the monostable assumptions. Some of them can be summarized as follows:

(i) If $n = 1$ and $f(u, v) = -du + b(v)$, $d > 0$ is a constant and $b$ is a function, then (1.1) reduces to the 1-D lattice differential equation with delay

$$u_i'(t) = D[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)] - du_i(t) + b\big(u_i(t-\tau)\big), \quad i \in \mathbb{Z},\ t > 0. \tag{1.3}$$

Ma and Zou [10] proved the existence, uniqueness and stability of traveling wave fronts of (1.3) by considering a related continuum equation.

(ii) If $n = 2$ and $f(u, v) = -d_m u + \varpi b(v)$, then (1.1) reduces to the 2-D delayed lattice differential equation

$$\begin{aligned} u_{i,j}'(t) = D[&u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t) - 4u_{i,j}(t)] \\ &- du_{i,j}(t) + \varpi b\big(u_{i,j}(t-\tau)\big), \quad i, j \in \mathbb{Z},\ t > 0, \end{aligned} \tag{1.4}$$

which was derived for a single species in a 2-D patchy environment. Cheng et al [4] studied the existence of the minimal wave speed and spreading speed. In [5], they further proved that the traveling wave front of (1.4) with "large speed" is exponentially stable, when the initial perturbation around the wave is sufficiently small in a weighted norm. In particular, the following equation is a special case of (1.4):

$$u_{i,j}'(t) = D[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}] + g\big(u_{i,j}(t)\big), \tag{1.5}$$

for $i, j \in \mathbb{Z}$, $t > 0$. Guo and Wu [7] considered the existence, uniqueness, monotonicity and asymptotic behavior of traveling wave fronts of (1.5).

(iii) If $f(u, v) = (1 - u)v$, then (1.1) becomes

$$u_\eta'(t) = D(\triangle_n u)_\eta + u_\eta(t-\tau)\big[1 - u_\eta(t)\big], \quad \eta \in \mathbb{Z}^n, \tag{1.6}$$

which was derived from branching theory in [9]. Zou [16] established the existence of traveling wave fronts of (1.6) by constructing a pair of sub- and super-solutions.

It should be mentioned that the traveling wave solutions of delayed lattice differential equations has been studied by many researchers; see e.g., [4, 2, 3, 10, 11, 8, 14, 16, 18]. In [16], Wu and Zou studied the existence of traveling fronts of (1.1) with $n = 1$ and general delay. Zou [18] considered the existence of the traveling fronts of (1.1) with general dimension $n$. More precisely, he reduced the existence of the traveling fronts to that of an admissible pair of sub- and super-solutions by establishing a monotone iteration starting from a supersolution. Wu and Liu [17] further considered the monotonicity, uniqueness, periodicity, and symmetry of traveling wave fronts of (1.1).

However, to the best of our knowledge, there has been no results on the stability of traveling fronts of (1.1). Although the weighted energy method is efficient for solving the wave stability, there are some unavoidable shortcomings. In particular, as mentioned before, this method can only prove the stability of traveling fronts for large wave speeds and small perturbations. In the present paper, we shall use the comparison principle and squeezing technique to prove the globally asymptotic stability of traveling fronts of (1.1) with speed $c > c_*$ (Theorem 2.3), where $c_*$ is the minimal wave speed. We point out that although the technique used here are similar to these in [1, 2, 10, 12, 15], the technique details are different. For example, for the 1-D discrete equations, the proof of stability theorem of traveling wave fronts [10] is through a related continuum equation by extending the spatial variable from $j \in \mathbb{Z}$ to $x \in \mathbb{R}$. But, here we shall only use the original equation (1.1) to prove the stability result of traveling wave fronts.

The rest of this paper is organized as follows. In Section 2, we first introduce some known results on the existence of traveling fronts. Then, we state our main result on the globally asymptotic stability of traveling fronts of (1.1). The proof of the main result; i.e., Theorem 2.3, is given in Section 3. In Section 4, we apply our results to two specific biological models and obtain some new results which essentially improve and complement the results obtained in [5, 10, 17, 18].

## 2. Preliminaries and main results

Substituting $U(\xi)$, $\xi = \eta \cdot \sigma + ct$ into (1.1), we obtain the corresponding wave equation

$$cU'(\xi) = E[U](\xi) + f\big(U(\xi), U(\xi - c\tau)\big), \tag{2.1}$$

where

$$E[U](\xi) := D \sum_{k=1}^{n} [U(\xi + \sigma_k) + U(\xi - \sigma_k) - 2U(\xi)]. \tag{2.2}$$

For $c \geq 0$ and $\lambda \in \mathbb{C}$, we define

$$\triangle(c, \lambda) = c\lambda - D \sum_{k=1}^{n} \left[e^{\lambda\sigma_k} + e^{-\lambda\sigma_k} - 2\right] - \partial_1 f(0,0) - \partial_2 f(0,0) e^{-\lambda c\tau}.$$

The following observation is straightforward.

**Proposition 2.1.** *For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, there exist $\lambda_* := \lambda_*(\sigma) > 0$ and $c_* := c_*(\sigma) > 0$ such that*

$$\triangle(c_*, \lambda_*) = 0, \quad \frac{\partial}{\partial\lambda} \triangle(c_*, \lambda)\Big|_{\lambda = \lambda_*} = 0.$$

*Furthermore,*

- (i) *if $0 < c < c_*$ and $\lambda > 0$, then $\triangle(c, \lambda) < 0$;*
- (ii) *if $c > c_*$, then the equation $\triangle(c, \lambda) = 0$ has two positive real roots $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_1(c) < \lambda_* < \lambda_2(c)$ such that*

$$\triangle(c, \lambda) \begin{cases} < 0 & \text{for } \lambda \in \mathbb{R} \setminus (\lambda_1(c), \lambda_2(c)), \\ > 0 & \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

The following result can be found in [17].

**Proposition 2.2.** *Let $\sigma \in \mathbb{R}^n$ be a given unit vector. For every $c \geq c_*$, (1.1) has a traveling wave front $(U(\cdot), c, \sigma)$ with $U'(\xi) > 0$ for all $\xi \in \mathbb{R}$. Moreover, for any $c > c_*$, $U(\xi)$ satisfies*

$$\lim_{\xi \to -\infty} U(\xi) e^{-\lambda_1(c)\xi} = 1, \quad \lim_{\xi \to -\infty} U'(\xi) e^{-\lambda_1(c)\xi} = \lambda_1(c).$$

Now, we state our main result in this article.

**Theorem 2.3.** *Assume that* (A1)–(A3) *hold. For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, let $(U(\cdot), c, \sigma)$ be the traveling wave front of* (1.1) *with direction $\sigma$ and speed $c > c_*$ given in Proposition 2.2. If $\varphi = \{\varphi_\eta\}_{\eta \in \mathbb{Z}^n}$ with $\varphi_\eta \in C([-\tau, 0], [0, K])$ satisfies*

$$\liminf_{\eta\cdot\sigma \to +\infty} \varphi_\eta(0) > 0, \quad \liminf_{\eta\cdot\sigma \to -\infty} \max_{s \in [-\tau, 0]} \left|\varphi_\eta(s) e^{-\lambda_1(c)\eta\cdot\sigma} - \rho_0 e^{\lambda_1(c)cs}\right| = 0, \tag{2.3}$$

*then the unique solution $\{u_\eta(t)\}_{\eta \in \mathbb{Z}^n}$ of* (1.1) *with initial data $\varphi$ satisfies*

$$\lim_{t \to +\infty} \sup_{\eta \in \mathbb{Z}^n} \Big| \frac{u_\eta(t)}{U(\eta \cdot \sigma + ct + \xi_0)} - 1 \Big| = 0, \tag{2.4}$$

*where $\xi_0 = \ln(\rho_0)/\lambda_1(c)$.*

**Remark 2.4.** Zou [18] reduced the existence of the traveling fronts of (1.1) to that of an admissible pair of sub- and super-solutions. Wu and Liu [17] further considered the monotonicity and uniqueness of traveling wave fronts of (1.1). Here, we obtain the stability of the traveling wave fronts. So, Theorem 2.3 complements the results in [17, 18].

We also mention that Cheng et al [5] studied the stability of traveling wave fronts of (1.4) by using the weighted energy method. However, if not impossible, it is difficult to apply the weighted energy method to the n-dimensional delayed reaction-diffusion equations (1.1). On the other hand, the weighted energy method can only be used to prove the stability of traveling wave fronts for large wave speeds and small initial perturbations. Clearly, our result on the stability of traveling fronts in Theorem 2.3 is valid not only for large initial perturbations but also for small wave speeds. Thus, Theorem 2.3 essentially extends the results in Cheng et al [5].

## 3. Asymptotic stability of traveling fronts

3.1. **The initial value problem.** To study the stability of traveling fronts, Theorem 2.3, we first consider the initial-value problem

$$\begin{aligned} u'_\eta(t) &= F[u](\eta, t), \quad \eta \in \mathbb{Z}^n, \ t > 0, \\ u_\eta(s) &= \varphi_\eta(s), \quad \eta \in \mathbb{Z}^n, \ s \in [-\tau, 0]. \end{aligned} \tag{3.1}$$

Here and in the sequel, $\sum_{\|\eta_1 - \eta\| = 1}$ denotes the sum over $\eta_1 \in \mathbb{Z}^n$ with $\|\eta_1 - \eta\| = 1$, and

$$F[u](\eta, t) := D \sum_{\|\eta_1 - \eta\| = 1} \left[u_{\eta_1}(t) - u_\eta(t)\right] + f\big(u_\eta(t), u_\eta(t - \tau)\big), \quad \eta \in \mathbb{Z}^n, \ t > 0.$$

To take advantage of our estimates for sub- and super-solutions, we make the following extension for the function $f$. Define function $\widehat{f} : [0, K] \times [0, 2K] \to \mathbb{R}$ by

$$\widehat{f}(u, v) = \begin{cases} f(u, v), & \text{for } (u, v) \in [0, K]^2, \\ f(u, K) + (v - K)\partial_2 f(u, K), & \text{for } (u, v) \in [0, K] \times [K, 2K]. \end{cases}$$

Clearly, $\partial_1 \widehat{f}(u,v)$ is continuous on $[0,K]^2$, $\partial_2 \widehat{f}(u,v)$ is continuous on $[0,K]\times[0,2K]$. For convenience, we will denote $\widehat{f}$ by $f$ in the remainder of this paper. In the sequel, we also denote $L_i = \max_{(u,v)\in[0,K]^2} |\partial_i f(u,v)|$, $i=1,2$.

For the existence and positivity of solutions of (3.1), we have the following result.

**Lemma 3.1.** *For any $\varphi = \{\varphi_\eta\}_{\eta\in\mathbb{Z}^n}$ with $\varphi_\eta \in C([-\tau,0],[0,K])$, (3.1) admits a unique solution $u(t) = \{u_\eta(t)\}_{\eta\in\mathbb{Z}^n}$ on $[0,+\infty)$ satisfies $u_\eta \in C([-\tau,+\infty),[0,K])$. Moreover, for any $i\in\{1,\dots,n\}$,*

$$u_{\eta\pm jp_i}(t) \ge D^j \varphi_\eta(0) t^j e^{-(L_1+2nD)t}/j! \quad \forall \eta\in\mathbb{Z}^n,\ j\in\mathbb{N}\cup\{0\},\ t>0, \tag{3.2}$$

*where $p_i$ is the vector in $\mathbb{Z}^n$ whose $i-th$ component is 1 and all other components are 0. In what follows, we denote $p_i = (0,\dots,0,1_i,0,\dots,0)$.*

*Proof.* We define $F_1[u](\eta,t) = D\sum_{\|\eta_1-\eta\|=1} u_{\eta_1}(t) + L_1 u_\eta(t) + f\big(u_\eta(t), u_\eta(t-\tau)\big)$. Clearly, $F_1[u](\cdot) \ge F_1[v](\cdot)$ for $0 \le v(\cdot) \le u(\cdot) \le K$, and (3.1) is equivalent to

$$u_\eta(t) = \varphi_\eta(0) e^{-(L_1+2nD)t} + \int_0^t e^{(L_1+2nD)(s-t)} F_1[u](\eta,s)ds. \tag{3.3}$$

The existence of solutions then follows by Picard's iteration and the monotonicity of the operator $F_1$, see also Ma and Zou [11, Lemma 4.1].

Form (3.3), we have $u_\eta(t) \ge \varphi_\eta(0)e^{-(L_1+2nD)t}$ and for any $i\in\{1,\dots,n\}$,

$$\begin{aligned} u_\eta(t) &\ge \int_0^t e^{(L_1+2nD)(s-t)} F_1[u](\eta,s)ds \\ &\ge D\int_0^t e^{(L_1+2nD)(s-t)} [u_{\eta+p_i}(s) + u_{\eta-p_i}(s)]ds. \end{aligned}$$

Therefore, by an induction argument, (3.2) holds. The proof is complete. □

**Definition 3.2.** A sequence of continuous functions $\{v_\eta(t)\}_{\eta\in\mathbb{Z}^n}$, $t\in[-\tau,b)$, $b>0$, is called a supersolution (a subsolution) of (3.1) on $[0,b)$ if $v'_\eta(t) \ge (\le) F[v](\eta,t)$, a.e. for $t\in[0,b)$.

**Theorem 3.3** (Comparison Principle)**.** *Assume $\{u^+_\eta(t)\}_{\eta\in\mathbb{Z}^n}$ and $\{u^-_\eta(t)\}_{\eta\in\mathbb{Z}^n}$ are a pair of sub- and super-solutions of (3.1) on $[0,\infty)$ with $0\le u^-_\eta(t), u^+_\eta(t) \le K$ for $\eta\in\mathbb{Z}^n$ and $t\in[-\tau,\infty)$, and $u^+_\eta(s) \ge u^-_\eta(s)$ for $\eta\in\mathbb{Z}^n$ and $s\in[-\tau,0]$. Then the following hold:*

- (i) *$u^+_\eta(t) \ge u^-_\eta(t)$ for $\eta\in\mathbb{Z}^n$ and $t\ge 0$.*
- (ii) *If there exists $\eta_0\in\mathbb{Z}^n$ such that $u^+_{\eta_0}(0) > u^-_{\eta_0}(0)$, then $u^+_\eta(t) > u^-_\eta(t)$ for $\eta\in\mathbb{Z}^n$ and $t>0$.*

*Proof.* The conclusion (i) can be easily verified by using a method similar to that of [10, Lemma 4.2], so we omit it here.

Let $w_\eta(t) = u^+_\eta(t) - u^-_\eta(t)$, then $w_\eta(t) \ge 0$ for $\eta\in\mathbb{Z}^n$ and $t\ge 0$. By the definition of the sub- and super-solutions of (3.1), we have, for any $i\in\{1,\dots,n\}$,

$t > 0$,

$$\begin{aligned} w_\eta(t) &= w_\eta(0)e^{-(L_1+2nD)t} + \int_0^t e^{(L_1+2nD)(s-t)}\Big[D\sum_{\|\eta_1-\eta\|=1} w_{\eta_1}(s) \\ &\quad + L_1 w_\eta(s) + f\big(u_\eta^+(s), u_\eta^+(s-\tau) - f\big(u_\eta^-(s), u_\eta^-(s-\tau)\big)\Big]ds \\ &\geq w_\eta(0)e^{-(L_1+2nD)t} + D\int_0^t e^{(L_1+2nD)(s-t)}[w_{\eta+p_i}(s) + w_{\eta-p_i}(s)]ds \\ &\geq D\int_0^t e^{(L_1+2nD)(s-t)}[w_{\eta+p_i}(s) + w_{\eta-p_i}(s)]ds \geq 0, \end{aligned} \tag{3.4}$$

where $p_i = (0, \dots, 0, 1_i, 0, \dots, 0)$.

Suppose on the contrary that there exist $\eta' \in \mathbb{Z}^n$ and $t' > 0$ such that $w_{\eta'}(t') = u_{\eta'}^+(t') - u_{\eta'}^-(t') = 0$, then from (3.4), we have $w_{\eta' \pm jp_i}(s) = 0$ for any $s \in [0, t']$, $i \in \{1, \dots, n\}$ and $j \in \mathbb{N}$. Noting that $\eta_0 = \eta' + \Sigma(\pm p_i)$, we obtain $w_{\eta_0}(0) = 0$, which is a contradiction. Therefore, $u_\eta^+(t) > u_\eta^-(t)$ for $\eta \in \mathbb{Z}^n$, $t > 0$. This completes the proof. □

**Lemma 3.4.** *Let $\bar{K} = L_1 + L_2$ and $u^i(t) = \{u_\eta^i(t)\}_{\eta\in\mathbb{Z}^n}$, $i = 1, 2$, be two solutions of (3.1) with $u_\eta^1$, $u_\eta^2 \in C([-\tau, +\infty), [0, K])$. Then for $t \geq 0$,*

$$\sup_{\eta\in\mathbb{Z}^n}\big\{u_\eta^1(t) - u_\eta^2(t)\big\} \leq \sup_{\eta\in\mathbb{Z}^n,\, s\in[-\tau,0]}\Big\{\max\big\{u_\eta^1(s) - u_\eta^2(s), 0\big\}\Big\}e^{\bar{K}t}. \tag{3.5}$$

The proof of the above lemma is similar to that of [10, Lemma 4.4] and is omitted. Now, we construct a few of sub- and super-solutions for initial-value problem (3.1).

**Lemma 3.5.** *Assume that $\phi(\xi) : \mathbb{R} \to [0, K]$ is continuous. Then $w_\eta(t) = \phi(\eta \cdot \sigma + ct)$ is a super-solution (resp. a sub-solution) of (3.1) on $[0, \infty)$ if $H_c[\phi](\xi) \geq 0$ (resp. $\leq 0$) a.e. on $\mathbb{R}$, where*

$$H_c[\phi](\xi) = c\phi'(\xi) - D\sum_{k=1}^n[\phi(\xi + \sigma_k) + \phi(\xi - \sigma_k) - 2\phi(\xi)] - f\big(\phi(\xi), \phi(\xi - c\tau)\big).$$

**Lemma 3.6.** *For any $\delta \in (0, 1)$, there exist $\rho > 0$, $\gamma > 0$ such that for each $\epsilon \in (0, \delta]$ and for any $\xi^\pm \in \mathbb{R}$, the functions $\overline{u}(t) = \{\overline{u}_\eta(t)\}_{\eta\in\mathbb{Z}^n}$ and $\underline{u}(t) = \{\underline{u}_\eta(t)\}_{\eta\in\mathbb{Z}^n}$ defined by*

$$\overline{u}_\eta(t) = \min\{\big(1 + \epsilon e^{-\rho t}\big)U\left(\eta \cdot \sigma + ct + \xi^+ - \gamma\epsilon e^{-\rho t}\right), K\}$$

*and*

$$\underline{u}_\eta(t) = \big(1 - \epsilon e^{-\rho t}\big)U\left(\eta \cdot \sigma + ct + \xi^- + \gamma\epsilon e^{-\rho t}\right)$$

*are a sub- and super-solution of (3.1), respectively.*

*Proof.* It is easy to see that $0 < U(\cdot) < K$ and $U'(\cdot) > 0$ on $\mathbb{R}$. We first show that $\overline{u}_\eta(t)$ is a supersolution of (3.1). In view of

$$\begin{aligned} &\lim_{(u,v,r,s,\rho)\to(K^-,K^-,K^-,K,0)}\left[\partial_1 f(u,v) + e^{\rho\tau}\partial_2 f(r,s) + \rho\right] \\ &= \partial_1 f(K,K) + \partial_2 f(K,K) < 0, \end{aligned}$$

there exist $\rho_1 \in (0, 1)$ and $\theta \in \big(0, \frac{K}{2}\big)$ such that

$$\partial_1 f(u,v) + e^{\rho\tau}\partial_2 f(r,s) < -\rho, \tag{3.6}$$

for $\rho \in (0, \rho_1]$, and $(u, v, r, s) \in [K-\theta, K] \times [K-\theta, K] \times [K-\theta, K] \times [K-\theta, K+\theta]$. Since $\lim_{\xi \to -\infty} U(\xi) e^{-\lambda_1(c)\xi} = 1$, and $\lim_{\xi \to -\infty} U'(\xi) e^{-\lambda_1(c)\xi} = \lambda_1(c)$, we can take $\xi_1 \in \mathbb{R}$ such that for any $\xi \le \xi_1$,

$$\frac{1}{2} \le U(\xi) e^{-\lambda_1(c)\xi} \le \frac{3}{2}, \quad \frac{1}{2}\lambda_1(c) \le U'(\xi) e^{-\lambda_1(c)\xi} \le \frac{3}{2}\lambda_1(c). \tag{3.7}$$

Fixed $0 < \rho \le \rho_1$ such that $\delta K(e^{\rho\tau} - 1) < \theta$, $\delta e^{\rho\tau} < 1$, and

$$\rho K - aU^{\alpha}(\xi_1) U^{\beta}(\xi_1 - c\tau) + L_2(e^{\rho\tau} - 1)K \le 0, \tag{3.8}$$

where $a = a(\delta)$, $\alpha = \alpha(\delta)$ and $\beta = \beta(\delta)$ are determined in (A3).

Choose $\xi_2 \in \mathbb{R}$ sufficiently large such that

$$U(\xi) \ge K - \theta, \quad \forall \xi \ge \xi_2 - c\tau. \tag{3.9}$$

Set $\varrho = \min\{U'(\xi) : \xi_1 \le \xi \le \xi_2\} > 0$. We can take $\gamma > 0$ sufficiently large such that

$$-\frac{3}{2}\rho + \frac{1}{2}\gamma\rho\lambda_1(c) - \frac{3}{2}L_1 - \frac{3}{2}L_2 e^{\rho\tau} e^{-\lambda_1(c)c\tau} \ge 0, \tag{3.10}$$

$$-K\rho + \gamma\rho\varrho - L_1 K - L_2 e^{\rho\tau} K \ge 0, \tag{3.11}$$

$$\frac{3}{2}\rho - \frac{1-\delta}{4}\gamma\rho\lambda_1(c) + \frac{3}{2}L_2(e^{\rho\tau} - 1)e^{-\lambda_1(c)c\tau} \le 0. \tag{3.12}$$

If $\overline{u}_\eta(t) = K$, then it is easy to verify that $\overline{u}'_\eta(t) - F[\overline{u}](\eta, t) \ge 0$, so we only consider the case $\overline{u}_\eta(t) = \big(1 + \epsilon e^{-\rho t}\big) U\left(\eta \cdot \sigma + ct + \xi^+ - \gamma\epsilon e^{-\rho t}\right)$.

Let $\xi = \eta \cdot \sigma + ct + \xi^+ - \gamma\epsilon e^{-\rho t}$. Noting that $\overline{u}_\eta(t - \tau) \le 2K$, then

$$\begin{aligned}
&\overline{u}'_\eta(t) - F[\overline{u}](\eta, t) \\
&= -\rho\epsilon e^{-\rho t} U(\xi) + \big(1 + \epsilon e^{-\rho t}\big)\big(c + \gamma\epsilon\rho e^{-\rho t}\big) U'(\xi) \\
&\quad - D\big(1 + \epsilon e^{-\rho t}\big) \sum_{k=1}^{n} [U(\xi + \sigma_k) + U(\xi - \sigma_k) - 2U(\xi)] - f\big(\overline{u}_\eta(t), \overline{u}_\eta(t - \tau)\big) \\
&= -\rho\epsilon e^{-\rho t} U(\xi) + \gamma\epsilon\rho\big(1 + \epsilon e^{-\rho t}\big) e^{-\rho t} U'(\xi) \\
&\quad + \big(1 + \epsilon e^{-\rho t}\big) f\big(U(\xi), U(\xi - c\tau)\big) - f\big(\overline{u}_\eta(t), \overline{u}_\eta(t - \tau)\big) \\
&\ge -\rho\epsilon e^{-\rho t} U(\xi) + \gamma\epsilon\rho\big(1 + \epsilon e^{-\rho t}\big) e^{-\rho t} U'(\xi) + \big(1 + \epsilon e^{-\rho t}\big) f\big(U(\xi), U(\xi - c\tau)\big) \\
&\quad - f\left(\big(1 + \epsilon e^{-\rho t}\big) U(\xi), \big(1 + \epsilon e^{-\rho(t-\tau)}\big) U(\xi - c\tau)\right).
\end{aligned} \tag{3.13}$$

We distinguish among three cases.

**Case (i):** $\xi \ge \xi_2$. By (3.13), (3.6) and (3.9), we have

$$\begin{aligned}
&\overline{u}'_\eta(t) - F[\overline{u}](\eta, t) \\
&\ge -\rho\epsilon e^{-\rho t} U(\xi) + \gamma\epsilon\rho e^{-\rho t} U'(\xi) + f\big(U(\xi), U(\xi - c\tau)\big) \\
&\quad - f\big(\big(1 + \epsilon e^{-\rho t}\big) U(\xi), U(\xi - c\tau)\big) + f\left(\big(1 + \epsilon e^{-\rho t}\big) U(\xi), U(\xi - c\tau)\right) \\
&\quad - f\left(\big(1 + \epsilon e^{-\rho t}\big) U(\xi), \big(1 + \epsilon e^{-\rho(t-\tau)}\big) U(\xi - c\tau)\right) \\
&= -\rho\epsilon e^{-\rho t} U(\xi) + \gamma\epsilon\rho e^{-\rho t} U'(\xi) - \epsilon e^{-\rho t} \partial_1 f\big((1 + \theta_1 \epsilon e^{-\rho t}) U(\xi), U(\xi - c\tau)\big) U(\xi) \\
&\quad - \epsilon e^{-\rho(t-\tau)} \partial_2 f\left(\big(1 + \epsilon e^{-\rho t}\big) U(\xi), \big(1 + \theta_2 \epsilon e^{-\rho(t-\tau)}\big) U(\xi - c\tau)\right) U(\xi - c\tau) \\
&\ge \epsilon e^{-\rho t} \Big\{ -\rho U(\xi) + \gamma\rho U'(\xi) - \Big[\partial_1 f\big((1 + \theta_1 \epsilon e^{-\rho t}) U(\xi), U(\xi - c\tau)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ e^{\rho\tau}\partial_2 f\Big(\big(1+\epsilon e^{-\rho t}\big)U(\xi), \big(1+\theta_2\epsilon e^{-\rho(t-\tau)}\big)U(\xi-c\tau)\Big)\Big]U(\xi)\Big\}\\
&\ge \epsilon e^{-\rho t}\big[-\rho U(\xi)+\gamma\rho U'(\xi)+\rho U(\xi)\big]\ge 0,
\end{aligned}
$$

where $\theta_i\in(0,1)$, $i=1,2$, and we have used the estimate

$$
\big(1+\theta_2\epsilon e^{-\rho(t-\tau)}\big)U(\xi-c\tau)\le \big(1+\epsilon e^{-\rho t}\big)U(\xi)+\epsilon e^{-\rho t}(e^{\rho\tau}-1)U(\xi)\le K+\theta.
$$

**Case (ii):** $\xi<\xi_1$, it follows from (3.13), (3.7) and (3.10) that

$$
\begin{aligned}
&\overline{u}'_\eta(t)-F[\overline{u}](\eta,t)\\
&\ge -\rho\epsilon e^{-\rho t}U(\xi)+\gamma\epsilon\rho e^{-\rho t}U'(\xi)+f\big(U(\xi),U(\xi-c\tau)\big)\\
&\quad - f\Big(\big(1+\epsilon e^{-\rho t}\big)U(\xi),\big(1+\epsilon e^{-\rho(t-\tau)}\big)U(\xi-c\tau)\Big)\\
&\ge -\rho\epsilon e^{-\rho t}U(\xi)+\gamma\epsilon\rho e^{-\rho t}U'(\xi)-L_1\epsilon e^{-\rho t}U(\xi)-L_2\epsilon e^{-\rho(t-\tau)}U(\xi-c\tau)\\
&\ge \epsilon e^{-\rho t}e^{\lambda_1(c)\xi}\big[-\frac{3}{2}\rho+\frac{1}{2}\gamma\rho\lambda_1(c)-\frac{3}{2}L_1-\frac{3}{2}L_2e^{\rho\tau}e^{-\lambda_1(c)c\tau}\big]\ge 0.
\end{aligned}
$$

**Case (iii):** $\xi_1\le\xi\le\xi_2$, by (3.13) and (3.11), we obtain

$$
\begin{aligned}
&\overline{u}'_\eta(t)-F[\overline{u}](\eta,t)\\
&\ge -\rho\epsilon e^{-\rho t}U(\xi)+\gamma\epsilon\rho e^{-\rho t}U'(\xi)+f\big(U(\xi),U(\xi-c\tau)\big)\\
&\quad - f\Big(\big(1+\epsilon e^{-\rho t}\big)U(\xi),\big(1+\epsilon e^{-\rho(t-\tau)}\big)U(\xi-c\tau)\Big)\\
&\ge -\rho\epsilon e^{-\rho t}K+\gamma\epsilon\rho e^{-\rho t}U'(\xi)-L_1\epsilon e^{-\rho t}K-L_2\epsilon e^{-\rho(t-\tau)}K\\
&\ge \epsilon e^{-\rho t}[-K\rho+\gamma\rho\varrho-L_1K-L_2e^{\rho\tau}K]\ge 0.
\end{aligned}
$$

Therefore, $\overline{u}_\eta(t)$ is a supersolution of (3.1).

Similarly, by virtue of (3.12) and (3.8), we can show that $\underline{u}_\eta(t)$ is a subsolution of (3.1) under the assumption (A3). This completes the proof. □

3.2. **Proof of Theorem 2.3.** We need to establish several technical lemmas.

**Lemma 3.7.** *For any $\varepsilon>0$, there exists $\xi_1(\varepsilon)<0$ such that, for $\xi=\eta\cdot\sigma+ct\le\xi_1(\varepsilon)$,*

$$
U(\xi+\xi_0-2\varepsilon)<\inf_{t\ge-\tau}u_\eta(t)\le\sup_{t\ge-\tau}u_\eta(t)<U(\xi+\xi_0+2\varepsilon), \tag{3.14}
$$

*Proof.* Let $\varepsilon_1=\rho_0\big(e^{\lambda_1(c)\varepsilon}-1\big)e^{-\lambda_1(c)c\tau}>0$. Then by (2.3), there exists $\xi^+(\varepsilon)<0$ such that for $\eta\cdot\sigma\le\xi^+(\varepsilon)$ and $s\in[-\tau,0]$,

$$
\begin{aligned}
\varphi_\eta(s)e^{-\lambda_1(c)\eta\cdot\sigma}&<\rho_0e^{\lambda_1(c)cs}+\varepsilon_1\\
&\le \rho_0e^{\lambda_1(c)cs}+\rho_0\big(e^{\lambda_1(c)\varepsilon}-1\big)e^{\lambda_1(c)cs}=\rho_0e^{\lambda_1(c)cs}e^{\lambda_1(c)\varepsilon},
\end{aligned}
$$

it follows that $\varphi_\eta(s)<e^{\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0+\varepsilon)}$. Similarly, one can verify that there exists $\xi^-(\varepsilon)<0$ such that $\varphi_\eta(s)>e^{\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0-\varepsilon)}$ for $\eta\cdot\sigma\le\xi^-(\varepsilon)$ and $s\in[-\tau,0]$. Let $x_1(\varepsilon):=\min\{\xi^+(\varepsilon),\xi^-(\varepsilon)\}$. Then, for any $\eta\cdot\sigma\le x_1(\varepsilon)$ and $s\in[-\tau,0]$,

$$
e^{\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0-\varepsilon)}<\varphi_\eta(s)<e^{\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0+\varepsilon)}.
$$

Set

$$
\phi^-(\xi)=\max\{0,e^{\lambda_1(c)(\xi+\xi_0-\varepsilon)}-le^{\nu\lambda_1(c)(\xi+\xi_0-\varepsilon)}\},
$$

where $\nu=\frac{1}{2}\big(1+\min\big\{2,\frac{\lambda_2(c)}{\lambda_1(c)}\big\}\big)$ and $l\ge\max\big\{Q(c,\nu),e^{-(\nu-1)\lambda_1(c)(x_1(\varepsilon)+\xi_0-\varepsilon-c\tau)}\big\}$. Then, by Lemma 3.5, $\phi^-(\eta\cdot\sigma+ct)$ is a subsolution of (3.1).

Noting that $e^{\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0-\epsilon)} - le^{\nu\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0-\varepsilon)} < 0$ for $\eta\cdot\sigma > x_1(\varepsilon)$ and $s\in[-\tau,0]$, we have, for $\eta\in\mathbb{Z}^n$ and $s\in[-\tau,0]$,

$$\varphi_\eta(s) > \max\Big\{0, e^{\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0-\varepsilon)} - le^{\nu\lambda_1(c)(\eta\cdot\sigma+cs+\xi_0-\varepsilon)}\Big\}.$$

Thus, by Theorem 3.3, for $\eta\in\mathbb{Z}^n$ and $t\ge-\tau$,

$$u_\eta(t) \ge e^{\lambda_1(c)(\eta\cdot\sigma+ct+\xi_0-\varepsilon)} - le^{\nu\lambda_1(c)(\eta\cdot\sigma+ct+\xi_0-\varepsilon)}.$$

Since $\lim_{\xi\to-\infty} U(\xi)e^{-\lambda_1(c)\xi} = 1$, there exists $x_2(\varepsilon) < 0$ such that for $\xi \le x_2(\varepsilon)$,

$$e^{\lambda_1(c)(\xi+\xi_0-\varepsilon)} - le^{\nu\lambda_1(c)(\xi+\xi_0-\varepsilon)} > U(\xi+\xi_0-2\varepsilon).$$

Therefore, for $\xi = \eta\cdot\sigma + ct \le x_2(\varepsilon)$,

$$\inf_{t\ge-\tau} u_\eta(t) \ge e^{\lambda_1(c)(\xi+\xi_0-\varepsilon)} - le^{\nu\lambda_1(c)(\xi+\xi_0-\varepsilon)} > U(\xi+\xi_0-2\varepsilon).$$

Set $\phi^+(\xi) = \min\big\{K, e^{\lambda_1(c)(\xi+\xi_0+\varepsilon)} + le^{\nu\lambda_1(c)(\xi+\xi_0+\varepsilon)}\big\}$, we can similarly show that there exists $x_3(\varepsilon) < 0$ such that for $\xi = \eta\cdot\sigma + ct \le x_3(\varepsilon)$,

$$\sup_{t\ge-\tau} u_\eta(t) \le e^{\lambda_1(c)(\xi+\xi_0+\varepsilon)} + le^{\nu\lambda_1(c)(\xi+\xi_0+\varepsilon)} < U(\xi+\xi_0+2\varepsilon).$$

Take $\xi_1(\varepsilon) = \min\{x_2(\varepsilon), x_3(\varepsilon)\}$, then the assertion of the lemma follows. □

**Lemma 3.8.** *There exist $\delta\in(0,1)$, $\rho>0$, $\gamma>0$ and $z_0>0$ such that for $\eta\in\mathbb{Z}^n$ and $t\ge1$,*

$$\begin{aligned}&\big(1-\delta e^{-\rho(t-1-\tau)}\big)U\big(\eta\cdot\sigma+ct+\xi_0-z_0+\gamma\delta e^{-\rho(t-1-\tau)}\big)\\ &\le u_\eta(t) \le \min\Big\{\big(1+\delta e^{-\rho t}\big)U\big(\eta\cdot\sigma+ct+\xi_0+z_0-\gamma\delta e^{-\rho t}\big), K\Big\}.\end{aligned}\tag{3.15}$$

*Consequently, for $t\ge1$,*

$$1-\delta e^{-\rho(t-1-\tau)} \le \inf_{\xi\in\mathbb{R}} \frac{u_\eta(t)}{U(\xi+\xi_0-z_0)}, \ \sup_{\xi\in\mathbb{R}} \frac{u_\eta(t)}{U(\xi+\xi_0+z_0)} \le 1+\delta e^{-\rho t},\tag{3.16}$$

*where $\xi = \eta\cdot\sigma+ct$.*

*Proof.* In view of (3.14), we have $u_\eta(1+\tau+s) \ge U(\eta\cdot\sigma + c(1+\tau+s)+\xi_0-2)$ for $\eta\cdot\sigma \le \xi_1(1) - c(1+\tau)$ and $s\in[-\tau,0]$.

Since $\liminf_{\eta\cdot\sigma\to+\infty}\varphi_\eta(0) > 0$, there exists $\delta_1>0$ and $x_4>0$ such that $\varphi_\eta(0) > \delta_1$ for $\eta\cdot\sigma > x_4$. Fix a positive integer $N > [x_4 - \xi_1(1) + c(1+\tau)]/\sigma_{i_0}$, where $\sigma_{i_0} := \max_{i=1,\dots,n}\{\sigma_i\} > 0$. If $\eta\cdot\sigma > \xi_1(1) - c(1+\tau)$, then $(\eta+Np_{i_0})\cdot\sigma > x_4$, and hence, it follows from Lemma 3.1 that

$$\begin{aligned}u_\eta(1+\tau+s) &= u_{(\eta+Np_{i_0})-Np_{i_0}}(1+\tau+s)\\ &\ge D^N(1+\tau+s)^N\varphi_{\eta+Np_{i_0}}(0)e^{-(L_1+2nD)(1+\tau+s)}/N!\\ &\ge D^N\delta_1 e^{-(L_1+2nD)(1+\tau)}/N! \ge (1-\delta)K,\end{aligned}$$

for $\eta\cdot\sigma > \xi_1(1) - c(1+\tau)$, $s\in[-\tau,0]$ and some $\delta\in(0,1)$. Thus, for any $\rho>0$ and $\gamma>0$,

$$\begin{aligned}u_\eta(1+\tau+s) &\ge (1-\delta)U\big(\eta\cdot\sigma+c(1+\tau+s)+\xi_0-2\big)\\ &\ge (1-\delta e^{-\rho s})U\big(\eta\cdot\sigma+c(1+\tau+s)+\xi_0-2-\delta\gamma e^{\rho\tau}+\delta\gamma e^{-\rho s}\big),\end{aligned}$$

for $\eta \in \mathbb{Z}^n$ and $s \in [-\tau, 0]$. Choose $\rho$ small enough and $\gamma$ large enough, respectively, such that the conclusion of Lemma 3.6 holds. Consequently, we have, for $\eta \in \mathbb{Z}^n$ and $t \geq -\tau$,

$$u_\eta(1+\tau+t) \geq (1-\delta e^{-\rho t})U\big(\eta\cdot\sigma + c(1+\tau+t) + \xi_0 - 2 - \delta\gamma e^{\rho\tau} + \delta\gamma e^{-\rho t}\big),$$

Then, for $\eta \in \mathbb{Z}^n$ and $t \geq 1$,

$$u_\eta(t) \geq (1-\delta e^{-\rho(t-1-\tau)})U\big(\eta\cdot\sigma + ct + \xi_0 - (2+\delta\gamma e^{\rho\tau}) + \delta\gamma e^{-\rho(t-1-\tau)}\big). \quad (3.17)$$

Again, in view of (3.14), $u_\eta(s) < U(\eta\cdot\sigma + cs + \xi_0 + 2)$ for $\xi = \eta\cdot\sigma \leq \xi_1(1)$ and $s \in [-\tau, 0]$. Also, for $\delta$ given in the above estimate and sufficiently large $x_5 > 0$ satisfying $U(\xi_1(1) - c\tau + x_5 + \xi_0 + 2) \geq K/(1+\delta)$, we get for $\xi = \eta\cdot\sigma \geq \xi_1(1)$ and $s \in [-\tau, 0]$, $u_\eta(s) \leq K \leq (1+\delta)U(\eta\cdot\sigma + cs + x_5 + \xi_0 + 2)$. Thus, for $\eta \in \mathbb{Z}^n$ and $s \in [-\tau, 0]$, we have

$$\begin{aligned} u_\eta(s) &\leq (1+\delta)U\big(\eta\cdot\sigma + cs + x_5 + \xi_0 + 2\big) \\ &\leq (1+\delta e^{-\rho s})U\big(\eta\cdot\sigma + cs + x_5 + \xi_0 + 2 + \delta\gamma e^{\rho\tau} - \delta\gamma e^{-\rho s}\big). \end{aligned}$$

Hence, for $\eta \in \mathbb{Z}^n$ and $s \in [-\tau, 0]$,

$$u_\eta(s) \leq \min\{(1+\delta e^{-\rho s})U(\eta\cdot\sigma + cs + x_5 + \xi_0 + 2 + \delta\gamma e^{\rho\tau} - \delta\gamma e^{-\rho s}), K\}.$$

Using the comparison principle, we obtain, for all $\eta \in \mathbb{Z}^n$ and $t \geq -\tau$,

$$u_\eta(t) \leq \min\Big\{(1+\delta e^{-\rho t})U\big(\eta\cdot\sigma + ct + x_5 + \xi_0 + 2 + \delta\gamma e^{\rho\tau} - \delta\gamma e^{-\rho t}\big), K\Big\}. \quad (3.18)$$

Let $z_0 = x_5 + 2 + \delta\gamma e^{\rho\tau}$, then (3.15) follows from (3.17) and (3.18), and (3.16) is a direct consequence of (3.15). This completes the proof. □

**Lemma 3.9.** *There exists $M_0 > 0$ such that for $\epsilon \in (0, \delta]$ and $\xi \geq M_0 + \xi_0$,*

$$(1-\epsilon)U\big(\xi + 3\epsilon\gamma e^{\rho\tau}\big) \leq U(\xi) \leq (1+\epsilon)U\big(\xi - 3\epsilon\gamma e^{\rho\tau}\big). \quad (3.19)$$

The proof of the above lemma is similar to that of Ma and Zou [10, Lemma 5.3] and is omitted.

In the sequel, the constants $\delta, \rho, \gamma, z_0, M_0$ are fixed as in Lemmas 3.8 and 3.9. Consider the continuum version of (1.1) for the moment

$$u_t(x,t) = D\sum_{i=1}^{n}\big[u(x+p_i,t) + u(x-p_i,t) - 2u(x,t)\big] + f\big(u(x,t), u(x,t-\tau)\big), \quad (3.20)$$

where $x \in \mathbb{R}^n, t > 0$, and $p_i = (0, \dots, 0, 1_i, 0, \dots, 0)$. One sees that if $u(x,s)|_{x=\eta} = u_\eta(s)$, for $\eta \in \mathbb{Z}^n$ and $s \in [-\tau, 0]$, then $u(x,t)|_{x=\eta} = u_\eta(t)$, for $\eta \in \mathbb{Z}^n$ and $t > 0$. Also, the corresponding wave equation of (3.20) takes the form (2.1).

**Lemma 3.10.** *Let $z$, $M_1$ and $T$ be any given positive constants and $u^\pm(x,t;T)$ be solutions of* (3.20) *with initial values*

$$\begin{aligned} u^+(x,s;T) &= U\big(x\cdot\sigma + cs + cT + \xi_0 + z\big)\zeta\big(x\cdot\sigma + cs + cT + M_1\big) \\ &\quad + U\big(x\cdot\sigma + cs + cT + \xi_0 + 2z\big)\big[1 - \zeta(x\cdot\sigma + cs + cT + M_1)\big], \end{aligned} \quad (3.21)$$

$$\begin{aligned} u^-(x,s;T) &= U\big(x\cdot\sigma + cs + cT + \xi_0 - z\big)\zeta\big(x\cdot\sigma + cs + cT + M_1\big) \\ &\quad + U\big(x\cdot\sigma + cs + cT + \xi_0 - 2z\big)\big[1 - \zeta(x\cdot\sigma + cs + cT + M_1)\big], \end{aligned} \quad (3.22)$$

*for $x \in \mathbb{R}^n$ and $s \in [-\tau, 0]$, respectively, where $\zeta(y) = \min\{\max\{0, -y\}, 1\}$ for $y \in \mathbb{R}$. Then there exists $\epsilon \in \big(0, \min\{\delta/2, ze^{-\rho\tau}/(3\gamma)\}\big)$, depending on $z$ and $M_1$( independent of $T$), such that for any $x \cdot \sigma + cT \geq -M_1$ and $s \in [-\tau, 0]$,*

$$u^+(x, 1+\tau+s; T) \leq (1+\epsilon)U\big(x \cdot \sigma + c(T+1+\tau+s) + \xi_0 + 2z - 3\epsilon\gamma e^{\rho\tau}\big), \tag{3.23}$$

$$u^-(x, 1+\tau+s; T) \geq (1-\epsilon)U\big(x \cdot \sigma + c(T+1+\tau+s) + \xi_0 - 2z + 3\epsilon\gamma e^{\rho\tau}\big). \tag{3.24}$$

*Proof.* We consider only $u^+$, since the inequality for $u^-$ can be proved similarly. In view of $u^+(x, s; T) \leq U(x \cdot \sigma + cs + cT + \xi_0 + 2z)$ for $x \in \mathbb{R}^n$ and $s \in [-\tau, 0]$, and $u^+(x, s; T) = U(x \cdot \sigma + cs + cT + \xi_0 + z) < U(x \cdot \sigma + cs + cT + \xi_0 + 2z)$ for $x \cdot \sigma \in (-\infty, -M_1 - 1 - cT]$ and $s \in [-\tau, 0]$. Using a comparison principle for the continuum equation (3.20) (see e.g., [10, Lemma 4.3]), we obtain for $x \in \mathbb{R}^n$ and $t > 0$, $u^+(x, t; T) < U(x \cdot \sigma + ct + cT + \xi_0 + 2z)$, which implies that for $x \in \mathbb{R}^n$ and $s \in [-\tau, 0]$,

$$u^+(x, 1+\tau+s; T) < U(x \cdot \sigma + c(T+1+\tau+s) + \xi_0 + 2z).$$

**Case (i):** $T \in [0, c_0)$, where $c_0 := \frac{1}{c}$. Then, by the uniform continuity of $u^+$ and $U$, there exists $\epsilon \in \big(0, \min\{\delta/2, ze^{-\rho\tau}/(3\gamma)\}\big)$ such that for any $T \in [0, c_0)$,

$$u^+(x, 1+\tau+s; T) \leq U\big(x \cdot \sigma + c(T+1+\tau+s) + \xi_0 + 2z - 3\epsilon\gamma e^{\rho\tau}\big), \tag{3.25}$$

for $x \cdot \sigma + cT \in [-M_1, M_0 - 2z]$ and $s \in [-\tau, 0]$.
**Case (ii):** $T \geq c_0$. There exist $k \in \mathbb{N}$ and $T_0 \in [0, c_0)$ such that $T = T_0 + kc_0$. From (3.21), we obtain, for $x \in \mathbb{R}^n$ and $s \in [-\tau, 0]$,

$$\begin{aligned}
&u^+(x, s; T) \\
&= U\big((x + k\sigma) \cdot \sigma + cs + cT_0 + \xi_0 + z\big)\zeta\big((x + k\sigma) \cdot \sigma + cs + cT_0 + M_1\big) \\
&\quad + U\big((x + k\sigma) \cdot \sigma + cs + cT_0 + \xi_0 + 2z\big)\big[1 - \zeta((x + k\sigma) \cdot \sigma + cs + cT_0 + M_1)\big] \\
&= u^+(x + k\sigma, s; T_0).
\end{aligned} \tag{3.26}$$

Hence, $u^+(x, t; T) = u^+(x + k\sigma, t; T_0)$, for $x \in \mathbb{R}^n$ and $t > 0$. For $x \cdot \sigma + cT \in [-M_1, M_0 - 2z]$, let $x = x' - k\sigma$, then $x \cdot \sigma + cT = (x' - k\sigma) \cdot \sigma + c(T_0 + k\frac{1}{c}) = x' \cdot \sigma + cT_0 \in [-M_1, M_0 - 2z]$. Hence, by (3.8) and (3.25), we obtain

$$\begin{aligned}
u^+(x, 1+\tau+s; T) &= u^+(x', 1+\tau+s; T_0) \\
&\leq U\big(x' \cdot \sigma + c(T_0+1+\tau+s) + \xi_0 + 2z - 3\epsilon\gamma e^{\rho\tau}\big) \\
&= U\big(x \cdot \sigma + c(T+1+\tau+s) + \xi_0 + 2z - 3\epsilon\gamma e^{\rho\tau}\big).
\end{aligned}$$

Furthermore, from Lemma 3.9, for $x \cdot \sigma + cT \in [M_0 - 2z, +\infty)$ and $s \in [-\tau, 0]$,

$$\begin{aligned}
u^+(x, 1+\tau+s; T) &< U\big(x \cdot \sigma + c(T+1+\tau+s) + \xi_0 + 2z\big) \\
&\leq (1+\epsilon)U\big(x \cdot \sigma + c(T+1+\tau+s) + \xi_0 + 2z - 3\epsilon\gamma e^{\rho\tau}\big).
\end{aligned}$$

Therefore, (3.23) holds, and this completes the proof. □

The following result is a direct consequence of Lemma 3.10.

**Corollary 3.11.** *Let $z$, $M_1$ and $T$ be any given positive constants and $u_\eta^\pm(t;T)$ be solutions of* (1.1) *with initial values*

$$\begin{aligned} u_\eta^+(s;T) = U\big(\eta\cdot\sigma + cs + cT + \xi_0 + z\big)\zeta\big(\eta\cdot\sigma + cs + cT + M_1\big) \\ + U\big(\eta\cdot\sigma + cs + cT + \xi_0 + 2z\big)\big[1 - \zeta(\eta\cdot\sigma + cs + cT + M_1)\big], \end{aligned} \tag{3.27}$$

$$\begin{aligned} u_\eta^-(s;T) = U\big(\eta\cdot\sigma + cs + cT + \xi_0 - z\big)\zeta\big(\eta\cdot\sigma + cs + cT + M_1\big) \\ + U\big(\eta\cdot\sigma + cs + cT + \xi_0 - 2z\big)\big[1 - \zeta(\eta\cdot\sigma + cs + cT + M_1)\big], \end{aligned} \tag{3.28}$$

*for $\eta\in\mathbb{Z}^n$ and $s\in[-\tau,0]$, respectively, where $\zeta(y) = \min\{\max\{0,-y\},1\}$ for $y\in\mathbb{R}$. Then there exists $\epsilon\in\big(0,\min\{\delta/2, ze^{-\rho\tau}/(3\gamma)\}\big)$, depending on $z$ and $M_1$ (independent of $T$), such that for any $\eta\cdot\sigma + cT\ge -M_1$ and $s\in[-\tau,0]$,*

$$u_\eta^+(1+\tau+s;T) \le (1+\epsilon)U\big(\eta\cdot\sigma + c(T+1+\tau+s) + \xi_0 + 2z - 3\epsilon\gamma e^{\rho\tau}\big),$$
$$u_\eta^-(1+\tau+s;T) \ge (1-\epsilon)U\big(\eta\cdot\sigma + c(T+1+\tau+s) + \xi_0 - 2z + 3\epsilon\gamma e^{\rho\tau}\big).$$

*Proof of Theorem 2.3.* Define

$$z^+ = \inf\{z : z\in A^+\},\quad A^+ = \{z\ge 0 : \limsup_{t\to+\infty}\sup_{\xi\in\mathbb{R}}\frac{u_\eta(t)}{U(\xi+\xi_0+2z)} \le 1\},$$

$$z^- = \inf\{z : z\in A^-\},\quad A^- = \{z\ge 0 : \liminf_{t\to+\infty}\inf_{\xi\in\mathbb{R}}\frac{u_\eta(t)}{U(\xi+\xi_0-2z)} \ge 1\},$$

in which $\xi = \eta\cdot\sigma + ct$. By Lemma 3.8, we see that $\frac12 z_0\in A^\pm$, and hence $z^\pm$ are well defined and $z^\pm\in[0,\frac12 z_0]$. Furthermore, as $\lim_{\varepsilon\to 0}\frac{U(\cdot+\varepsilon)}{U(\cdot)} = 1$ uniformly on $\mathbb{R}$, we see that $z^\pm\in A^\pm$ and $A^\pm = [z^\pm,+\infty)$.

Thus, to complete the proof, it is sufficient to show that $z^+ = z^- = 0$. First, we prove that $z^+ = 0$ by a contradiction argument. On the contrary, suppose that $z^+ > 0$. We fix $z = z^+$ and $M_1 = -\xi_1(\frac{z^+}{2})$ and denote by $\epsilon$ the resulting constant in Corollary 3.11. Since $z^+\in A^+$, $\limsup_{t\to+\infty}\sup_{\xi\in\mathbb{R}}\frac{u_\eta(t)}{U(\xi+\xi_0+2z^+)}\le 1$. It then follows that there exists $T\ge 0$ such that for $s\in[-\tau,0]$,

$$\sup_{\xi\in\mathbb{R}}\frac{u_\eta(T+s)}{U(\xi+\xi_0+2z^+)} \le 1 + \widehat\epsilon/K,$$

where $\xi = \eta\cdot\sigma + c(T+s)$ and $\widehat\epsilon = \epsilon U\big(-M_1+\xi_0-3\epsilon\gamma e^{\rho\tau}\big)e^{-\bar K(1+\tau)}$, $\bar K = L_1 + L_2$. Thus, for any $\eta\in\mathbb{Z}^n$ and $s\in[-\tau,0]$, $u_\eta(T+s)\le U(\eta\cdot\sigma + c(T+s)+\xi_0+2z^+) + \widehat\epsilon$. From (3.27), we obtain $u_\eta^+(s;T) = U(\eta\cdot\sigma + c(T+s)+\xi_0+2z^+)$ for $\eta\cdot\sigma\in[-M_1-cs-cT,+\infty)$. Then, for $\eta\cdot\sigma\in[-M_1-cs-cT,+\infty)$,

$$u_\eta(T+s)\le U(\eta\cdot\sigma + c(T+s)+\xi_0+2z^+) + \widehat\epsilon = u_\eta^+(s;T)+\widehat\epsilon.$$

For $\eta\cdot\sigma\in(-\infty,-M_1-cs-cT] = \big(-\infty,\xi_1(\frac{z^+}{2})-cs-cT\big]$, by (3.14) and definition of $u_\eta^+(s;T)$, we have $u_\eta(T+s) < U\big(\eta\cdot\sigma + c(T+s)+\xi_0+z^+\big)\le u_\eta^+(s;T)$. Thus, for any $\eta\in\mathbb{Z}^n$ and $s\in[-\tau,0]$, $u_\eta(T+s)\le u_\eta^+(s;T)+\widehat\epsilon$. By Lemmas 3.4, we obtain $u_\eta(T+1+\tau+s)\le u_\eta^+(1+\tau+s;T)+\widehat\epsilon e^{\bar K(1+\tau)}$, which implies

$$\begin{aligned} u_\eta(T+1+\tau+s) &\le u_\eta^+(1+\tau+s;T) + \widehat\epsilon e^{\bar K(1+\tau)} \\ &= u_\eta^+(1+\tau+s;T) + \epsilon U\big(-M_1+\xi_0-3\epsilon\gamma e^{\rho\tau}\big). \end{aligned}$$

Then by Corollary 3.11, we obtain, for $\eta\cdot\sigma + cT\in[-M_1,+\infty)$ and $s\in[-\tau,0]$,

$$u_\eta(T+1+\tau+s)\le(1+\epsilon)U\big(\eta\cdot\sigma + c(T+1+\tau+s)+\xi_0+2z^+-3\epsilon\gamma e^{\rho\tau}\big)$$

$$
\begin{aligned}
&+\epsilon U\big(-M_1+\xi_0-3\epsilon\gamma e^{\rho\tau}\big)\\
&\le (1+2\epsilon)U\big(\eta\cdot\sigma+c(T+1+\tau+s)+\xi_0+2z^+-3\epsilon\gamma e^{\rho\tau}\big).
\end{aligned}
$$

Again by (3.14) and $3\gamma\epsilon e^{\rho\tau}<z^+$, we obtain, for $\eta\cdot\sigma+c(T+1+\tau)\le\xi_1(\frac{z^+}{2})=-M_1$ and $s\in[-\tau,0]$,

$$
\begin{aligned}
u_\eta(T+1+\tau+s)&\le U(\eta\cdot\sigma+c(T+1+\tau+s)+\xi_0+z^+)\\
&\le (1+2\epsilon)U\big(\eta\cdot\sigma+c(T+1+\tau+s)+\xi_0+2z^+-3\epsilon\gamma e^{\rho\tau}\big).
\end{aligned}
$$

Thus, for $\eta\in\mathbb{Z}^n$ and $s\in[-\tau,0]$,

$$
\begin{aligned}
u_\eta(T+1+\tau+s)&\le (1+2\epsilon)U\big(\eta\cdot\sigma+c(T+1+\tau+s)+2z^++\xi_0-3\epsilon\gamma e^{\rho\tau}\big)\\
&\le (1+2\epsilon e^{-\rho s})\\
&\quad\times U\big(\eta\cdot\sigma+c(T+1+\tau+s)+\xi_0+2z^+-\epsilon\gamma-2\epsilon\gamma e^{-\rho s}\big).
\end{aligned}
$$

Hence, for $\eta\in\mathbb{Z}^n$ and $s\in[-\tau,0]$,

$$
\begin{aligned}
&u_\eta(T+1+\tau+s)\\
&\le \min\{(1+2\epsilon e^{-\rho s})U\big(\eta\cdot\sigma+c(T+1+\tau+s)+\xi_0+2z^+-\epsilon\gamma-2\epsilon\gamma e^{-\rho s}\big),K\}.
\end{aligned}
$$

It then follows from Theorem 3.3 and Lemma 3.6 that, for all $\eta\in\mathbb{Z}^n$ and $t>0$,

$$
\begin{aligned}
u_\eta(T+1+\tau+t)\le \min\{(1+2\epsilon e^{-\rho t})U\big(&\eta\cdot\sigma+c(T+1+\tau+t)\\
&+\xi_0+2z^+-\epsilon\gamma-2\epsilon\gamma e^{-\rho t}\big),K\};
\end{aligned}
$$

that is,

$$
\begin{aligned}
u_\eta(t)\le \min\{\big(1+2\epsilon e^{-\rho(t-T-1-\tau)}\big)U\big(&\eta\cdot\sigma+ct+\xi_0\\
&+2z^+-\epsilon\gamma-2\epsilon\gamma e^{-\rho(t-T-1-\tau)}\big),K\}.
\end{aligned}
$$

which implies

$$
\limsup_{t\to+\infty}\sup_{\xi\in\mathbb{R}}\frac{u_\eta(t)}{U(\xi+\xi_0+2z^+-\epsilon\gamma)}\le 1,
$$

where $\xi=\eta\cdot\sigma+ct$. Thus, $z^+-\epsilon\gamma/2\in A^+$, which is a contradiction, and hence $z^+=0$. Similarly, we can show that $z^-=0$. This completes the proof. □

## 4. Applications

In this section, we apply our results developed in Sections 2 and 3 to the models (1.4) and (1.6).

**Example 4.1.** Consider the equation

$$
\begin{aligned}
u'_{i,j}(t)=D_m[&u_{i+1,j}(t)+u_{i-1,j}(t)+u_{i,j+1}(t)+u_{i,j-1}(t)-4u_{i,j}(t)]\\
&-d_mu_{i,j}(t)+\varpi b\big(u_{i,j}(t-\tau)\big).
\end{aligned}\tag{4.1}
$$

Assume that

- (B1) $b\in C^2([0,K],\mathbb{R})$, $b(0)=\varpi b(K)-d_mK=0$, and $b(v)>0$ for $v\in(0,K)$, $\varpi b'(K)<d_m$, $b'(v)\ge 0$ and $b(v)\le b'(0)v$ for $v\in[0,K]$, where $K>0$ is a constant;
- (B2) For any $\delta\in(0,1)$, there exist $a=a(\delta)>0$ and $\beta=\beta(\delta)>0$ such that for $\gamma\in(0,\delta]$ and $v\in[0,K]$, $(1-\gamma)\varpi b(v)-\varpi b((1-\gamma)v)\le -a\gamma v^\beta$.

Let $f(u,v)=-d_mu+\varpi b(v)$, one can easily verify that (A1)–(A3) hold.

From Theorem 2.3, the following result holds.

**Theorem 4.2.** *Assume that* (B1), (B2) *hold. For any fixed* $\theta \in \mathbb{R}$*, let* $U$ *be the unique traveling front of* (4.1) *with direction* $\theta$ *and speed* $c > c_*$*, where* $c_*$ *is the minimal wave speed. If* $\varphi = \{\varphi_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ *with* $\varphi_{i,j} \in C([-\tau, 0], [0, K])$ *satisfies*

$$\liminf_{i\cos\theta + j\sin\theta \to +\infty} \varphi_\eta(0) > 0$$

*and*

$$\liminf_{i\cos\theta + j\sin\theta \to -\infty} \max_{s\in[-\tau,0]} \left|\varphi_{i,j}(s)e^{-\lambda_1(c)(i\cos\theta + j\sin\theta)} - \rho_0 e^{\lambda_1(c)cs}\right| = 0,$$

*then the unique solution* $\{u_{i,j}(t)\}_{(i,j)\in\mathbb{Z}^2}$ *of* (4.1) *with initial data* $\varphi$ *satisfies*

$$\lim_{t\to+\infty} \sup_{(i,j)\in\mathbb{Z}^2} \Big|\frac{u_{i,j}(t)}{U(i\cos\theta + j\sin\theta + ct + \xi_0)} - 1\Big| = 0,$$

*where* $\xi_0 = \frac{1}{\lambda_1(c)} \ln \rho_0$ *and* $\lambda_1(c)$ *is the smallest root of the equation*

$$c\lambda - D_m\big[e^{\lambda\cos\theta} + e^{-\lambda\cos\theta} + e^{\lambda\sin\theta} + e^{-\lambda\sin\theta} - 4\big] + d_m - b'(0)e^{-\lambda c\tau} = 0.$$

**Remark 4.3.** Cheng et al [5] proved the stability of traveling fronts of (4.1) for large wave speeds and small initial perturbations. Clearly, our result on the stability of traveling fronts in Theorem 4.2 is valid not only for large initial perturbations but also for small wave speeds. Thus, Theorem 4.2 complements the result in Cheng et al [5].

**Example 4.4.** Consider the equation

$$u'_\eta(t) = D(\triangle_n u)_\eta + u_\eta(t-\tau)\big[1 - u_\eta(t)\big], \ \eta \in \mathbb{Z}^n. \tag{4.2}$$

The results in [18] show that for any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, there exists a number $c_*(\sigma) > 0$ such that for each $c > c_*(\sigma)$, (4.2) has a traveling front $(U(\xi), c, \sigma)$ connecting 0 and 1. Wu and Liu [17] further obtained the monotonicity and uniqueness of the traveling wave fronts. However, there has been no results on the stability of the traveling fronts of (4.2).

Let $f(u, v) = (1-u)v$, then (A1)–(A3) hold. Theorem 2.3 implies that the traveling front $(U(\xi), c, \sigma)$ with direction $\sigma$ and speed $c > c_*(\sigma)$ is globally asymptotically stable with phase shift. Obviously, this result complements the one established by [17, 18].

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Hai-Qin Zhao
College of Mathematics and Information Science, Xianyang Normal University, Xianyang, Shaanxi 712000, China
*E-mail address*: xinyi98_01@126.com